Junior Inter Mathematics Model Paper

MATHEMATICS
Paper - I (B)

(English Version)

Time: 3 Hours **Max. Marks: 75**

Section - A

I. ***Very Short Answer Questions. Answer all questions. Each question carries "Two" marks.***

10 × 2 = 20 M

1. Find the angle which the straight line $y = \sqrt{3}x - 4$ makes with the Y - axis.

Sol: Given line is $y = \sqrt{3}x - 4$

Slope $m = \sqrt{3}$

$\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

Angle made by the line with

X-axis is $\frac{\pi}{3}$

Angle made by the line with

Y- axis is $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

2. Find the condition for the points (a, 0), (h, k), (0, b) to be collinear.

Sol: Let A(a, 0), B (h, k), C (0, b) are collinear

Slope of $\overleftrightarrow{AB}$ = Slope of $\overleftrightarrow{AC}$

$\Rightarrow \frac{k-0}{h-a} = \frac{b-0}{0-a} \Rightarrow \frac{k}{h-a} = \frac{b}{-a}$

$\Rightarrow -ak = bh - ab \Rightarrow bh + ak = ab$

$\Rightarrow \frac{h}{a} + \frac{k}{b} = 1$

3. Show that the points (2, 3, 5), (–1, 5, –1) and (4, –3, 2) form a right angled isosceles triangle.

Sol: Let A(2, 3, 5), B(–1, 5, – 1), C(4, –3, 2) are the given points.

$AB = \sqrt{(2+1)^2 + (3-5)^2 + (5+1)^2}$

$= \sqrt{9+4+36} = \sqrt{49} = 7$

$BC = \sqrt{(-1-4)^2 + (5+3)^2 + (-1-2)^2}$

$= \sqrt{25+64+9} = \sqrt{98} = 7\sqrt{2}$

$CA = \sqrt{(4-2)^2 + (-3-3)^2 + (2-5)^2}$

$= \sqrt{4+36+9} = \sqrt{49} = 7$

AB = CA and

$AB^2 + CA^2 = BC^2$

∴ ABC is a right angled isosceles triangle.

4. Find the equation of the plane if the foot of the perpendicular from origin to the plane is (1, 3, –5).

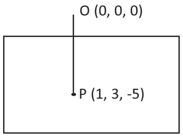
Sol: Let O(0, 0, 0), P(1, 3, –5)

OP is the normal to the plane,

d.r's of OP = (1 – 0, 3 – 0, –5– 0)

= (1, 3, –5) = (a, b, c)

The plane passes through P (1, 3, –5)



∴ Equation of the plane is

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

$\Rightarrow 1(x-1)+3(y-3) -5(z+5) = 0$

$\Rightarrow x - 1 + 3y - 9 - 5z - 25 = 0$

$\Rightarrow x + 3y - 5z - 35 = 0$

5. Find $\underset{x\to a}{\text{Lt}} \frac{\tan(x-a)}{x^2-a^2}$, $(a \neq 0)$

$= \underset{x\to a}{\text{Lt}} \frac{\tan(x-a)}{(x-a)(x+a)}$

$= \underset{x\to a}{\text{Lt}} \frac{\tan(x-a)}{(x-a)} . \underset{x\to a}{\text{Lt}} \frac{1}{(x+a)}$

$= 1.\frac{1}{(a+a)} = \frac{1}{2a}$

6. If $f(x) = -\sqrt{25-x^2}$ then find $\underset{x\to 1}{\text{Lt}} \frac{f(x)-f(1)}{x-1}$

Sol. $f(x) = -\sqrt{25-x^2}$

$f(1) = -\sqrt{25-1^2} = -\sqrt{24}$

$\underset{x\to 1}{\text{Lt}} \frac{f(x)-f(1)}{x-1}$

$= \underset{x\to 1}{\text{Lt}} \frac{-\sqrt{25-x^2}+\sqrt{24}}{x-1}$

$= \underset{x\to 1}{\text{Lt}} \frac{\sqrt{24}-\sqrt{25-x^2}}{x-1}$

$= \underset{x\to 1}{\text{Lt}} \frac{\sqrt{24}-\sqrt{25-x^2}}{x-1} \times \frac{\sqrt{24}+\sqrt{25-x^2}}{\sqrt{24}+\sqrt{25-x^2}}$

$= \underset{x\to 1}{\text{Lt}} \frac{24-[25-x^2]}{(x-1)\left[\sqrt{24}+\sqrt{25-x^2}\right]}$

$= \underset{x\to 1}{\text{Lt}} \frac{(x-1)(x+1)}{(x-1)\left[\sqrt{24}+\sqrt{25-x^2}\right]} \Rightarrow \frac{1}{\sqrt{24}}$

7. Find the derivative functions $f(x)= (4 + x^2)e^{2x}$

Sol: Let $f(x) = (4+x^2)\, e^{2x}$

Differ entiating w.r.t. x

$f(x) = (4+x^2)\frac{d}{dx}e^{2x} + e^{2x}\frac{d}{dx}(4+x^2)$

$= (4+x^2)2e^{2x} + e^{2x}(2x)$

$= 2e^{2x}(4+x^2+x)$

$f(x) = 2e^{2x}(x^2 + x + 4)$

8. Find the derivative of the functions $\sin^{-1}(3x-4x^3)$

Sol: Let $y = \sin^{-1}(3x-4x^3)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$\Rightarrow y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$\Rightarrow y= \sin^{-1}(\sin3\theta) = 3\theta$

$\Rightarrow y = 3\sin^{-1}x$

Differentiating w.r.t. x

$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

9. Find the approximate value of $\sqrt{82}$.

Sol: Let $f(x) = \sqrt{x}$, $x = 81$, $\Delta x = 1$

Approximate value of

$f(x+\Delta x) = f(x) + f'(x).\Delta x$

$= \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x \quad = \sqrt{81} + \frac{1}{2\sqrt{81}}(1)$

$= 9 + \frac{1}{18} = 9 + (0.0555) = 9.056$

10. Verify Rolle's theorem for the functions x^2-1 on [–1, 1]

Sol: Let $f(x) = x^2-1$ is continuous on [–1, 1]

f(–1) = f(1) = 0

and 'f' is differentiable on [–1, 1]

∴ By Rolle's theorem $C \in (-1,1)$ such that $f^1(C) = 0$

$f^1(x) = 2x = 0$

$f^1(C) = 2C = 0 \Rightarrow C = 0$

∴ The point $C = 0 \in (-1,1)$

∴ Rolle's theorem is verified.

Section - B

II. *Short Answer Questions. Answer any 'Five' questions. Each question carries 'Four' marks.*

5 × 4 = 20 M

11. Find the equation of locus of P, if A = (4, 0), B = (–4, 0) and $|PA - PB| = 4$

Sol: Let P (x, y) be any point on the locus

A (4, 0), B = (– 4, 0) are the given points

Given Condition is $|PA - PB| = 4$

$PA - PB = \pm 4$

$PA = PB \pm 4$

$PA^2 = PB^2 + 16 \pm 8PB$

$\Rightarrow (x-4)^2 + (y-0)^2$

$= (x+4)^2 + (y-0)^2 + 16 \pm 8PB$

$\Rightarrow x^2 + 16 - 8x + y^2 =$

$x^2 + 16 + 8x + y^2 + 16 \pm 8PB$

$\Rightarrow -16x - 16 = \pm 8PB$

$\Rightarrow -16(x+1) = \pm 8PB$

$\Rightarrow -2(x+1) = \pm PB$

Squaring on both sides

$4(x+1)^2 = PB^2$

$\Rightarrow 4(x^2+1+2x) = (x+4)^2+(y-0)^2$

$\Rightarrow 4x^2+4+8x = x^2 + 16 + 8x + y^2$

$\Rightarrow 3x^2 - y^2 = 12$

∴ Equation to the locus of P is

$3x^2 - y^2 = 12$.

12. Find the angle through which the axes are to be rotated so as to remove the xy term in the equation

$x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$.

Sol: Given equation is

$x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$

It is in the form of $ax^2+2hxy+by^2+2gx+2fy+c = 0$

a = 1, h = 2, b = 1, g = –1,

f = 1, c = –6

Let θ is the angle of rotation of axes then

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{4}{1-1}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4}{0}\right)$$

$$= \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2}.\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

13. If the portion of a straight line intercepted between the axes of co-ordinates is bisected at (2p, 2q), write the equation of the straight line.

Sol: Equation of the line in the

intercept form is $\frac{x}{a} + \frac{y}{b} = 1 \to 1$

A = (a, 0), B = (0, b)

Mid point of AB $\left(\frac{a+0}{2}, \frac{0+b}{2}\right)$

= (2p, 2q)

$\Rightarrow \left(\frac{a}{2}, \frac{b}{2}\right) = (2p, 2q)$

a = 4p, b = 4q

Equation of the required line is

$\Rightarrow \frac{x}{4p} + \frac{y}{4q} = 1 \Rightarrow \frac{x}{p} + \frac{y}{q} = 4$

14. If f is given by

$$f(x) = \begin{cases} k^2x - k & \text{if } x \ge 1 \\ 2 & \text{if } x < 1 \end{cases}$$

is a continuous function on R, then find the values of k.

Sol. LHL = $\underset{x \to 1-}{\text{Lt}}\, f(x)$

$= \underset{x \to 1-}{\text{Lt}}\,(2) = 2$

RHL = $\underset{x \to 1+}{\text{Lt}}\, f(x) = \underset{x \to 1+}{\text{Lt}}\,(k^2x - k)$

$= k^2(1) - k = k^2 - k$

Given that f(x) is continuous at

x = 1

⇒ LHL = RHL

$\Rightarrow 2 = k^2 - k \quad \Rightarrow k^2 - k - 2 = 0$

$\Rightarrow k^2 - 2k + k - 2 = 0$

$\Rightarrow k(k-2) + 1(k-2) = 0$

$\Rightarrow (k+1)(k-2) = 0$

$\Rightarrow k = -1$ or $k = 2$

15. Suppose we have a rectangular aquarium with dimensions of length 8m, width 4m and height 3m. Suppose we are filling the tank with water at the rate of 0.4 m^3/sec. How fast is the height of water changing when the water level is 2.5m?

Sol: Let 'l' be the length, 'b' be the width and 'h' be the height of cuboid.

∴ l = 8m; b = 4m; h = 3m,

Let 'V' be the volume,

V = lbh

$\frac{dV}{dt} = lb\frac{dh}{dt}$

$0.4 = 8 \times 4.\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{0.4}{32}$

$= \frac{1}{80}$ m/sec.

16. Find derivative by first principle cos ax.

Sol: Let f(x) = cosax

$$f(x) = \underset{h \to 0}{\text{Lt}}\frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{\text{Lt}}\frac{\cos a(x+h) - \cos ax}{h}$$

$$= \underset{h \to 0}{\text{Lt}}\frac{\cos(ax+ah) - \cos ax}{h}$$

$$= \underset{h \to 0}{\text{Lt}}\frac{-2\sin\left[\frac{ax+ah+ax}{2}\right]\sin\left[\frac{ax+ah-ax}{2}\right]}{h}$$

$$= \underset{h \to 0}{\text{Lt}}\frac{-2\sin\left(ax+\frac{ah}{2}\right)\sin\left(\frac{ah}{2}\right)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} - 2\sin\left(ax+\frac{ah}{2}\right)\underset{h \to 0}{\text{Lt}}\frac{\sin\left(\frac{ah}{2}\right)}{h}$$

$= -2\sin(ax+0)\frac{a}{2}$

$= -a\sin(ax)$

∴ $f^1(x) = -a\sin(ax)$

17. Find the length of Sub-tangent and subnormal at a point of the curve

$y = b\sin\left(\frac{x}{a}\right)$

Sol: Let P(x, y) be any point on the

curve $y = b\sin\left(\frac{x}{a}\right) \Rightarrow \frac{dy}{dx} = b\cos\frac{x}{a}.\frac{1}{a}$

$\left(\frac{dy}{dx}\right)_P = \frac{b}{a}\cos\left(\frac{x}{a}\right)$

Length of Sub tangent = $\left|\frac{y_1}{m}\right|$

$$\left|\frac{y}{\frac{b}{a}\cos\frac{x}{a}}\right| \Rightarrow \left|\frac{b\sin\left(\frac{x}{a}\right)}{\frac{b}{a}\cos\left(\frac{x}{a}\right)}\right| \Rightarrow \left|a\tan\frac{x}{a}\right|$$

Length of Sub normal = $|y_1 m|$

$$=\left|y.\frac{b}{a}\cos\frac{x}{a}\right| = \left|\left(b\sin\frac{x}{a}\right)\left(\frac{b}{a}\cos\frac{x}{a}\right)\right|$$

$$=\left|\frac{b^2}{2a}\sin\frac{2x}{a}\right|$$

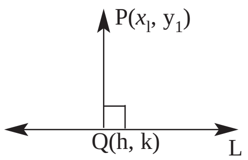
Section - C

III. ***Long Answer Questions. Answer any 'Five' questions. Each question carries 'Seven' marks.***

5 × 7 = 35 M

18. If Q(h, k) is the foot of the perpendicular from $P(x_1,y_1)$ on the line $ax + by + c = 0$ then

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$$



If Q(h, k) lies an $ax+bk+c=0$ then

$ah+bk+c = 0 \Rightarrow ah+bk = -c$

Given the L= $ax+by+c=0$

$PQ \perp L$

$\Rightarrow$ Slope of PQ × Slope of L= –1

$$\Rightarrow \left[\frac{k-y_1}{h-x_1}\right]\left[\frac{-a}{b}\right] = -1$$

$\Rightarrow a(k-y_1) = b(h-x_1)$

By the law of multipliers in ratio and proportion.

$$\Rightarrow \frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{a(h-x_1)+b(k-y_1)}{a^2+b^2}$$

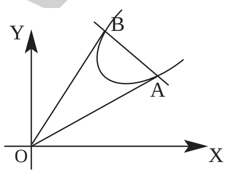
$$= \frac{ah-ax_1+bk-by_1}{a^2+b^2}$$

$$= \frac{-ax_1-by_1+ah+bk}{a^2+b^2}$$

$$= \frac{-ax_1-by_1-c}{a^2+b^2} \Rightarrow \frac{-(ax_1+by_1+c)}{a^2+b^2}$$

$$\therefore \frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$$

19. Find the values of k, if the lines joining the origin to the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line $x+2y=k$ are mutually perpendicular.



Sol: Given equation of the curve is

$S \equiv 2x^2+2xy+3y^2+2x-y-1=0$_(1)

Equation of the AB is $x+2y = k$

$$\Rightarrow \frac{x+2y}{k} = 1 \text{——}(2)$$

Homogenizing,

1) With the help of eq —(2)

2) Combined equation of OA, OB is

$2x^2-2xy+3y^2+2x.1 - y.1 - 1^2 = 0$

$$2x^2-2xy+3y^2+2x\frac{(x+2y)}{k}$$

$$-y\frac{(x+2y)}{k} - \frac{(x+2y)^2}{k^2} = 0$$

Multiplying with k^2

$2k^2x^2 - 2k^2xy + 3k^2y^2 + 2kx(x+2y)$

$- ky(x+2y) - (x+2y)^2 = 0$

$\Rightarrow 2k^2x^2 - 2k^2xy + 3k^2y^2 + 2kx^2$

$+4kxy + 4kxy - kxy - 2ky^2 - x^2 - 4xy - 4y^2 = 0$

$\Rightarrow (2k^2+2k-1)x^2 + (-2k^2+3k-4)$

$xy + (3k^2-2k-4)y^2 = 0$

Since OA, OB are perpendicular

Co-efficient of x^2 + Co-efficient of $y^2 = 0$

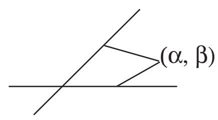
$\Rightarrow 2k^2+2k-1+3k^2-2k-4 = 0$

$\Rightarrow 5k^2 = 5 \Rightarrow k^2 = 1$

$\therefore k = \pm 1$

20. Theorem: The product of the perpendiculars from (α, β) to the pair of lines $ax^2+2hxy+by^2=0$ is

$$\frac{|a\alpha^2+2h\alpha\beta+b\beta^2|}{\sqrt{(a-b)^2+4h^2}}$$



Proof: Let $ax^2+2hxy+by^2=0$ represents the lines

$l_1x+m_1y = 0$ ___(1)

$l_2x+m_2y = 0$ ___(2)

$\therefore (l_1x+m_1y)(l_2x+m_2y)$

$= ax^2 + 2hxy + by^2$ comparing

$l_1l_2 = a$, $m_1m_2 = b$,

$l_1m_2 + l_2m_1 = 2h$

The length of the perpendicular from (α, β) to lines are

$$\frac{|l_1\alpha+m_1\beta|}{\sqrt{l_1^2+m_1^2}} \text{—(3)}; \frac{|l_2\alpha+m_2\beta|}{\sqrt{l_2^2+m_2^2}} \text{—(4)}$$

The product of the perpendicular is..

$$\left|\frac{|l_1\alpha+m_1\beta|}{\sqrt{l_1^2+m_1^2}}\right|\left|\frac{|l_2\alpha+m_2\beta|}{\sqrt{l_2^2+m_2^2}}\right|$$

$$=\frac{(l_1\alpha+m_1\beta)(l_2\alpha+m_2\beta)}{\sqrt{(l_1^2+m_1^2)(l_2^2+m_2^2)}}$$

$$=\frac{|l_1l_2\alpha^2+(l_1m_2+l_2m_1)\alpha\beta+m_1m_2\beta^2|}{\sqrt{(l_1l_2-m_1m_2)^2+2l_1l_2m_1m_2 + (l_1m_2+l_2m_1)^2-2l_1m_2l_2m}}$$

$$=\frac{|a\alpha^2+2h\alpha\beta+b\beta^2|}{\sqrt{(a-b)^2+4h^2}}$$

21. Show that the lines whose direction cosines are given by $l+m+n=0$, $2mn+3nl-5lm = 0$ are perpendicular to each other.

Sol. Given $l + m + n = 0$ ___(1)

$l = -m - n$

$l = -(m+n)$

$2mn + 3nl - 5lm = 0$ ___ (2)

$2mn - 3n(m+n) + 5m(m+n) = 0$

$2mn - 3mn - 3n^2 + 5m^2 + 5mn = 0$

$5m^2 + 4mn - 3n^2 = 0$

$$5\left(\frac{m}{n}\right)^2 + 4\left(\frac{m}{n}\right) - 3 = 0$$

$$\frac{m_1m_2}{n_1n_2} = \frac{-3}{5} \quad \frac{m_1m_2}{-3} = \frac{n_1n_2}{5} \quad \text{\_\_\_\_(3)}$$

From (1); $m = -(l+n)$

Substituting in __ (2)

$\Rightarrow -2n(l+n)+3nl+5l(l+n) = 0$

$\Rightarrow -2ln-2n^2+3ln+5l^2+5ln = 0$

$\Rightarrow 5l^2 + 6ln - 2n^2 = 0$

$$5\left(\frac{l}{n}\right)^2 + 6\left(\frac{l}{n}\right) - 2 = 0$$

$$\frac{l_1l_2}{n_1n_2} = \frac{-2}{5} \Rightarrow \frac{l_1l_2}{-2} = \frac{n_1n_2}{5} \quad \text{\_\_(4)}$$

From (3) and (4) we get

$$\frac{l_1l_2}{-2} = \frac{m_1m_2}{-3} = \frac{n_1n_2}{5} = k \ (say)$$

$l_1l_2 = -2k$, $m_1m_2 = -3k$,

$n_1n_2 = 5k$

$l_1l_2 + m_1m_2 + n_1n_2$

$= -2k - 3k + 5k = 0$

∴ The two lines are perpendicular to each other.

22. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

for $0 < |x| < 1$. Find $\frac{dy}{dx}$

Sol: $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Put $x^2 = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}(x^2)$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= \frac{\pi}{4}+\theta \quad y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = 0 + \frac{1}{2}\frac{-1}{\sqrt{1-(x^2)^2}}.2x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^4}}}$$

23. If the tangent at any point P on the curve $x^m y^n = a^{(m+n)}$ $(mn \neq 0)$ meets the coordinate axes in A, B then show that AP : PB is a constant.

Sol. Let $P(x_1, y_1)$ be any point on the curve $x^m y^n = a^{(m+n)}$

Taking logarithm on both sides

$\log(x^m y^n) = \log(a^{m+n})$

$m\log x + n\log y = (m+n)\log a$

Differentiating w.r.t. x

$$\Rightarrow m\frac{1}{x} + n\frac{1}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{n}{y}\frac{dy}{dx} = \frac{-m}{x} \Rightarrow \frac{dy}{dx} = \frac{-my}{nx} \qquad \Rightarrow \left[\frac{dy}{dx}\right]_{P(x_1,y_1)} = \frac{-my_1}{nx_1}$$

∴ Equation of the tangent at $P(x_1, y_1)$ is

$$\Rightarrow y - y_1 = \frac{-my_1}{nx_1}(x - x_1)$$

$\Rightarrow nx_1y - nx_1y_1 = -my_1x + mx_1y_1$

$\Rightarrow nx_1y + my_1x = mx_1y_1 + nx_1y_1$

$\Rightarrow nx_1y + my_1x = (m+n)x_1y_1$

$$\Rightarrow \frac{ny}{(m+n)y_1} + \frac{mx}{(m+n)x_1} = 1 \qquad \Rightarrow \frac{x}{\frac{(m+n)x_1}{m}} + \frac{y}{\frac{(m+n)y_1}{n}} = 1$$

The tangent at P meets the coordinate axes in A, B

$$\therefore A = \left(\frac{(m+n)x_1}{m}, 0\right) \qquad B = \left(0, \frac{(m+n)y_1}{n}\right)$$

P divides $\overline{AB}$ in the ratio AP : PB

$$\Rightarrow AP : PB = \frac{(m+n)x_1}{m} - x_1 : x_1 - 0 \qquad \Rightarrow \frac{mx_1 + nx_1 - mx_1}{m} : x_1$$

$\Rightarrow nx_1 : mx_1 = n : m$

AP : PB = n : m

24. A wire of length is l is cut into two parts which are bent respectively in the form of a square and a circle. What are the lengths of the piece of the wire so that the sum of the areas is the least?

Sol. The length of the wire l is bent in such a way that

The perimeter of the square

$4y = l - x$

$$\Rightarrow y = \frac{l-x}{4}$$

The Perimeter of the circle

$2\pi r = x$

$$\Rightarrow r = \frac{x}{2\pi}$$

The sum of the areas of the square and a circle is $y^2 + \pi r^2$

$$= \left(\frac{l-x}{4}\right)^2 + \pi\left(\frac{x}{2\pi}\right)^2$$

$$= \frac{(l-x)^2}{16} + \frac{x^2}{4\pi}$$

Differentiating w.r.t. 'x'

$$\Rightarrow \frac{dA}{dx} = \frac{-2(l-x)}{16} + \frac{2x}{4\pi} \qquad \text{\_\_\_\_(1)}$$

Again differentiating w.r.t. 'x'

$$\Rightarrow \frac{d^2A}{dx^2} = \frac{2}{16} + \frac{2}{4\pi} > 0 \qquad \text{\_\_\_\_ (2)}$$

∴ A is minimum

For minimum value

$$\frac{dA}{dx} = 0 \quad \Rightarrow \frac{-(l-x)}{8} + \frac{x}{2\pi} = 0$$

$$\Rightarrow \frac{x}{2\pi} = \frac{l-x}{8}$$

$\Rightarrow 4x = \pi l - \pi x$

$\Rightarrow x(\pi + 4) = \pi l$

$$\Rightarrow x = \frac{\pi l}{\pi + 4}$$

$$\Rightarrow l - x = l - \frac{\pi l}{\pi + 4}$$

$$\Rightarrow l - x = \frac{\pi l + 4l - \pi l}{\pi + 4}$$

$$\Rightarrow l - x = \frac{4l}{\pi + 4} \Rightarrow y = \frac{4l}{\pi + 4}$$

∴ The side of the Square is

$\frac{4l}{\pi + 4}$ and

the radius of the circle is

$$\frac{l}{2(\pi + 4)}$$