Intermediate Mathematics Model Paper-(A)

MATHEMATICS
Paper - I (A)

(English Version)

Time: 3 Hours **Max. Marks: 75**

Section - A

I. ***Very Short Answer Questions. Answer all questions. Each question carries "Two" marks.***

10 × 2 = 20 M

1. If $A = \{x; -1 \le x \le 1\}$, $f(x) = x^2$, $g(x) = x^3$, then which of the following are surjections?

i) $f : A \to A$ ii) $g : A \to A$

Sol. i) Given $A = \{x; -1 \le x \le 1\}$,

$f : A \to A$ and $f(x) = x^2$

If $x \in [-1, 1]$ then

$x^2 = f(x) \in [0, 1] = f(A)$

$f(A) \ne A$

$\therefore$ $f : A \to A$ is not a surjection.

ii) $-1 \le x \le 1$

$\Rightarrow (-1)^3 \le x^3 \le (1)^3$

$\Rightarrow -1 \le x^3 \le 1$

$g(A) = A$

$\therefore$ $g: A \to A$ is a surjection.

2. Find Domain of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

Sol. $f(x)$ is defined when $1 - x^2 > 0$

$\Rightarrow x^2 - 1 < 0$

$\Rightarrow (x + 1)(x - 1) < 0$

$\Rightarrow x \in (-1, 1)$

$(\because (x-a)(x-b) < 0 \Rightarrow x \in (a, b))$

$\therefore$ Domain of $f(x)$ is $(-1, 1)$

3. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{bmatrix}$ and $\det A = 45$, then find x.

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{bmatrix}$; $\det A = 45$

$\Rightarrow 1\begin{vmatrix} 3 & 4 \\ -6 & x \end{vmatrix} = 45 \Rightarrow 1(3x+24) = 45$

$\Rightarrow 3x + 24 = 45 \Rightarrow 3x = 21 \Rightarrow x = 7$

4. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then show that $AA^T = A^TA = I_2$

Sol. $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$,

$A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ —— (1)

$A^TA = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \sin\alpha\cos\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ —— (2)

$\therefore AA^T = A^TA = I_2$

5. If the vectors $-3\bar{i} + 4\bar{j} + \lambda\bar{k}$ and $\mu\bar{i} + 8\bar{j} + 6\bar{k}$ are collinear vectors, then find μ and λ.

Sol. Let $\bar{a} = -3\bar{i} + 4\bar{j} + \lambda\bar{k}$

$\bar{b} = \mu\bar{i} + 8\bar{j} + 6\bar{k}$

$\bar{a}, \bar{b}$ are collinear vectors

$\Rightarrow \bar{a} = t\bar{b}$ where 't' is a scalar

$\Rightarrow -3\bar{i} + 4\bar{j} + \lambda\bar{k}$

$= t\,(\mu\bar{i} + 8\bar{j} + 6\bar{k})$

comparing the cofficients of $\bar{i}, \bar{j}, \bar{k}$ terms on both sides, we get

$-3 = t\mu$, $4 = 8t$, $\lambda = 6t$

$\Rightarrow t = \dfrac{4}{8} = \dfrac{1}{2}$

$\Rightarrow \dfrac{\mu}{2} = -3, \lambda = \dfrac{6}{2}$

$\mu = -6$, $\lambda = 3$

$\therefore \boxed{\lambda = 3}\ \boxed{\mu = -6}$

6. If $\overline{OA} = \bar{i} + \bar{j} + \bar{k}$, $\overline{AB} = 3\bar{i} - 2\bar{j} + \bar{k}$, $\overline{BC} = \bar{i} + 2\bar{j} - 2\bar{k}$ and $\overline{CD} = 2\bar{i} + \bar{j} + 3\bar{k}$ then find the vector $\overline{OD}$.

Sol. Given

$\overline{OA} = \bar{i} + \bar{j} + \bar{k}$; $\overline{AB} = 3\bar{i} - 2\bar{j} + \bar{k}$

$\overline{BC} = \bar{i} + 2\bar{j} - 2\bar{k}$; $\overline{CD} = 2\bar{i} + \bar{j} + 3\bar{k}$

$\overline{OD} = \overline{OA} + \overline{AB} + \overline{BC} + \overline{CD}$

$\therefore \overline{OD} = 7\bar{i} + 2\bar{j} + 3\bar{k}$

7. If $\bar{a} = 2\bar{i} + 2\bar{j} - 3\bar{k}$, $\bar{b} = 3\bar{i} - \bar{j} + 2\bar{k}$, then find the angle between $2\bar{a} + \bar{b}$ and $\bar{a} + 2\bar{b}$

Sol. Let

$\bar{l} = 2\bar{a} + \bar{b} = 2(2\bar{i} + 2\bar{j} - 3\bar{k})$

$+ (3\bar{i} - \bar{j} + 2\bar{k}) = 7\bar{i} + 3\bar{j} - 4\bar{k}$

$\bar{m} = \bar{a} + 2\bar{b}$

$= (2\bar{i} + 2\bar{j} - 3\bar{k}) + 2(3\bar{i} - \bar{j} + 2\bar{k})$

$= 8\bar{i} + \bar{k}$

θ is angle between $\bar{l}$ and $\bar{m}$ then $\cos\theta = \dfrac{\bar{l}.\bar{m}}{|\bar{l}||\bar{m}|}$

$= \dfrac{(7\bar{i} + 3\bar{j} - 4\bar{k}).(8\bar{i} + \bar{k})}{|7\bar{i} + 3\bar{j} - 4\bar{k}||8\bar{i} + \bar{k}|}$

$= \dfrac{56 - 4}{\sqrt{49 + 9 + 16}\sqrt{64 + 1}} = \dfrac{52}{\sqrt{74}.\sqrt{65}}$

$\therefore$ Angle between $2\bar{a} + \bar{b}$ and $\bar{a} + 2\bar{b}$

is $\cos^{-1}\left(\dfrac{52}{\sqrt{74}\sqrt{65}}\right)$

8. If $3\sin A + 5\cos A = 5$ then prove that $5\sin A - 3\cos A = \pm 3$

sol: Given that

$3\sin A + 5\cos A = 5$ _____ (1)

and let $5\sin A - 3\cos A = k$ ____(2) say

$(1)^2 + (2)^2$

$\Rightarrow 34(\sin^2 A + \cos^2 A) = 25 + k^2$

$\Rightarrow k^2 = 34(1) - 25 = 9 \Rightarrow k = \pm 3$

$\therefore 5\sin A - 3\cos A = \pm 3$

9. Find the period of the following $\tan(x + 4x + 9x + ... + n^2 x)$.

Sol: $\tan(x + 4x + 9x + ... + n^2 x)$

$= \tan(1 + 4 + 9 + ... + n^2)x$

$= \tan(1^2 + 2^2 + 3^2 + ... + n^2)x$

$= \tan\left(\dfrac{n(n+1)(2n+1)}{6}\right)x$

$\therefore$ The period of $\tan\left[\dfrac{n(n+1)(2n+1)}{6}\right]x$ is

$\dfrac{\pi}{\frac{n(n+1)(2n+1)}{6}} = \dfrac{6\pi}{n(n+1)(2n+1)}$

10. Sinh(x)= ¾, find cosh($2x$) and sinh($2x$).

Sol. Given sinh$x$ = ¾,

We know that $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \cosh^2 x = 1 + \sinh^2 x$

$\Rightarrow \cosh^2 x = 1 + (3/4)^2$

$\Rightarrow \cosh^2 x = 1 + \dfrac{9}{16}$

$\Rightarrow \cosh^2 x = \dfrac{16+9}{16}$

$\Rightarrow \cosh^2 x = \dfrac{25}{16} \Rightarrow \sqrt{\dfrac{25}{16}} \Rightarrow \cosh x = \dfrac{5}{4}$

$\cosh(2x) = \cosh^2 x + \sinh^2 x$

$\Rightarrow \dfrac{25}{16} + \dfrac{9}{16} \Rightarrow \dfrac{25+9}{16} \Rightarrow \dfrac{34}{16}$

$\therefore \cosh(2x) = \dfrac{17}{8}$

sinh($2x$) = 2sin(hx).cos(hx)

$= \dfrac{3}{4}.\dfrac{5}{4}$ $\therefore \sin(2x) = \dfrac{15}{8}$

Section - B

II. ***Short Answer Questions. Answer any 'Five' questions. Each question carries 'Four' marks.***

5 × 4 = 20 M

11. Show that

$$\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Sol. $\text{LHS} = \begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix}$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$$= \begin{vmatrix} bc & b+c & 1 \\ ca-bc & c+a-b-c & 0 \\ ab-bc & a+b-b-c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} bc & b+c & 1 \\ c(a-b) & a-b & 0 \\ b(a-c) & a-c & 0 \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} bc & b+c & 1 \\ c & 1 & 0 \\ b & 1 & 0 \end{vmatrix}$$

By expanding third column, we get

$= (a-b)(a-c)\{1(c-b)\}$

$= (a-b)(a-c)(c-b)$

$= (a-b)(b-c)(c-a)$

= RHS

$$\therefore \begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$$

12. In the two dimensional plane, prove by using vector method, the equation of line whose intercepts on the axes are 'a' and 'b' is $\dfrac{x}{a} + \dfrac{x}{b} = 1$

Sol. Let A = (a, 0) and B = (0, b)

$\bar{A} = a\bar{i}, B = b\bar{j}$

The equation of the line throuth the A,B is

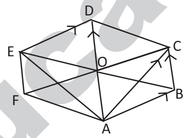
$\bar{r} = (1-t)a\bar{i} + t(b\bar{j})$

if $\bar{r} = x\bar{i} + y\bar{j}$,

then $x = (1 - t)a$ and $y = tb$

$\therefore \dfrac{x}{a} + \dfrac{y}{b} = 1 - t + t = 1$

13. Let ABCDEF be a regular hexagon with centre 'O' show that $\overline{AD} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\overline{AO}$



Sol. $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF}$

$= (\overline{AB} + \overline{AE}) + \overline{AD} + (\overline{AC} + \overline{AF})$

$= (\overline{AE} + \overline{ED}) + \overline{AD} + (\overline{AC} + \overline{CD})$

$(\because \overline{AB} = \overline{ED},\ \overline{AF} = \overline{CD})$

$= \overline{AD} + \overline{AD} + \overline{AD} = 3\overline{AD}$

$\because$ 'O' is the centre and $\overline{OD} = \overline{AO}$

$3\overline{AD} = 6\overline{AO}$

$\therefore \overline{AD} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\overline{AO}$

14. Prove that $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$

$\text{LHS} = \dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ}$

$= \dfrac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$= \dfrac{2\left(\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ\right)}{\sin 10^\circ \cos 10^\circ}$

$= \dfrac{2(\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ)}{\sin 10^\circ \cos 10^\circ}$

[$\because$ sinA cosB – cosA sinB = sin (A – B)]

$= \dfrac{2\sin(30^\circ - 10^\circ)}{\sin 10^\circ \cos 10^\circ}$

$= 2\dfrac{\sin 20^\circ}{\sin 10^\circ \cos 10^\circ}$

[$\because$ sin2A = 2 sinA cosA]

$= 2\dfrac{2.\sin 10^\circ \cos 10^\circ}{\sin 10^\circ \cos 10^\circ}$

= 4 = RHS

$\therefore \dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$

15. Prove that $\sin x + \sqrt{3}\cos x = \sqrt{2}$

Sol: $\sin x + \sqrt{3}\cos x = \sqrt{2}$

dividing with 2 on both sides

$\Rightarrow \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$

$\Rightarrow \sin\frac{\pi}{6}\sin x + \cos\frac{\pi}{6}\cos x = \frac{1}{\sqrt{2}}$

$= \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

General solution

$x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$

$x - \frac{\pi}{6} = 2n\pi + \frac{\pi}{4}$

or $x - \frac{\pi}{6} = 2n\pi - \frac{\pi}{4}$

$x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6}$

$x = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6}$

$= 2n\pi + \frac{5\pi}{12} = 2n\pi - \frac{\pi}{12}$

$\therefore$ solution set of the given equation is

$\left\{2n\pi + \frac{5\pi}{12}, n \in Z\right\} \cup \left\{2n\pi - \frac{\pi}{12}, n \in Z\right\}$

16. If $\cos^{-1}\left(\frac{p}{a}\right) + \cos^{-1}\frac{q}{b} = \alpha$ prove that $\frac{p^2}{a^2} - \frac{2pq}{ab}\cos\alpha + \frac{q^2}{b^2} = \sin^2\alpha$.

Sol. $\cos^{-1}\left(\frac{p}{a}\right) = A$, $\cos^{-1}\left(\frac{q}{b}\right) = B$

$\Rightarrow \cos A = \left(\frac{p}{a}\right)$, $\cos B = \left(\frac{q}{b}\right)$

$\Rightarrow A + B = \alpha$

$\Rightarrow \cos(A + B) = \cos\alpha$

$\Rightarrow \cos(A + B)$

$= \cos A \cos B - \sin A \sin B$ $\qquad \left(\because \sin A = \sqrt{1-\cos^2 A}\right)$

$\Rightarrow \cos\alpha = \frac{p}{a}.\frac{q}{b} - \sqrt{1-\frac{p^2}{a^2}}\sqrt{1-\frac{q^2}{b^2}}$

$\Rightarrow \frac{pq}{ab} - \cos\alpha = \sqrt{1-\frac{p^2}{a^2}}\sqrt{1-\frac{q^2}{b^2}}$

$\Rightarrow \frac{(pq)^2}{(ab)^2} + \cos^2\alpha - \frac{2pq}{ab}\cos\alpha$

$= \left[1-\frac{p^2}{a^2}\right]\left[1-\frac{q^2}{b^2}\right]$

$\Rightarrow \frac{p^2q^2}{a^2b^2} + \cos^2\alpha - \frac{2pq}{ab}\cos\alpha$

$= 1 - \frac{p^2}{a^2} - \frac{q^2}{b} + \frac{p^2q^2}{ab}$

$\Rightarrow \frac{p^2}{a^2} + \frac{q^2}{b^2} - \frac{2pq}{ab}\cos\alpha = 1 - \cos^2\alpha$

$\Rightarrow \frac{q^2}{a^2} - \frac{2pq}{ab}\cos\alpha + \frac{q^2}{b^2} = \sin^2\alpha$

17. If $\sin\theta = \frac{a}{b+c}$ then show that

$\cos\theta = 2\frac{\sqrt{bc}}{b+c}\cos\frac{A}{2}$

Sol. $\sin\theta = \frac{a}{b+c}$

$\cos^2\theta = 1 - \sin^2\theta$

$= 1 - \frac{a^2}{(b+c)^2} \Rightarrow \frac{(b+c)^2 - a^2}{(b+c)^2}$

$\Rightarrow \frac{(b+c+a)(b+c-a)}{(b+c)^2}$

$= \frac{2s.2(s-a)}{(b+c)^2} \Rightarrow \frac{4s(s-a)}{(b+c)^2}$

$= \frac{4s(s-a)}{bc} \times \frac{bc}{(b+c)^2} = 4\cos^2\frac{A}{2}.\frac{bc}{(b+c)^2}$

$\cos\theta = 2\cos\frac{A}{2}\frac{\sqrt{bc}}{b+c} \Rightarrow \frac{2\sqrt{bc}}{b+c}.\cos\frac{A}{2}$

Section - C

III. ***Long Answer Questions. Answer any 'Five' questions. Each question carries 'Seven' marks.***

5 × 7 = 35 M

18. f: A → B and g: B → C are bijections, then prove that $(gof)^{-1} = f^{-1}og^{-1}$.

Proof:

f: A → B, g : B → C are bijections

⇒ gof : A → C is a bijection

⇒ $(gof)^{-1}$: C → A is a bijection.

f : A → B is bijection

⇒ f^{-1}: B → A is a bijection

g : B → C is bijections

⇒ g^{-1} : C → B is bijection

Also, g^{-1} : C → B, f^{-1}: B → A are bijections

⇒ $f^{-1}og^{-1}$: C → A is bijection

g: B → C is bijection

Let c∈C then there exist b∈B such that g(b) = c

⇒ b = g^{-1}(c)

f : A → B is bijection also there exist a ∈ A such that f(a) = b

⇒ a = f^{-1}(b)

Now, (gof)(a) = g(f(a))

= g(b)

= c

⇒ a = $(gof)^{-1}$(c) ___(1)

Also $(f^{-1}og^{-1})$(c) = $f^{-1}(g^{-1}(c))$

= f^{-1}(b)

= a ____ (2)

From (1) and (2)

$(gof)^{-1}(c) = (f^{-1}og^{-1})(c)$

∴ $(gof)^{-1} = f^{-1}og^{-1}$

19. Show that $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17.

Sol. Let S(n) be the statement that

$3.5^{2n+1} + 2^{3n+1}$ is divisible by 17.

For n = 1,

$= 3.5^{2(1)+1} + 2^{3(1)+1}$

$= 3.5^3 + 2^4$

⇒ 375 + 16

= 391 is divisible by 17.

∴S(1) is true.

Assume that S(k) is true

$3.5^{2k+1} + 2^{3k+1} = 17m$ where

m ∈ N

⇒ $3.5^{2k+1} = 17m - 2^{3k+1}$

We show that S(k+1) is true

$= 3.5^{2(k+1)+1} + 2^{3(k+1)+1}$

$= 3.5^{2k+1}.5^2 + 2^{3k+1}.2^3$

$= (17m - 2^{3k+1})25 + 2^{3k+1}.8$

$= 17m.25 - 2^{3k+1}.25 + 2^{3k+1}.8$

$= 17m.25 - 17.2^{3k+1}$

$=17(25m - 2^{3k+1})$ is divisible by 17

∴S(k+1) is true.

∴By principle of Mathematical Induction S(n) is true for all

$n \in N$

20. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$$= \left(a^3+b^3+c^3-3abc\right)^2$$

Sol. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ba+c^2 & -ac+b^2+ca \\ -ba+c^2+ab & -b^2+ca+ac & -bc+cb+a^2 \\ -ca+ac+b^2 & -cb+a^2+bc & -c^2+ab+ba \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \_\_(1)$$

LHS: $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} =$

$$a\left(bc-a^2\right) - b\left(b^2-ca\right) + c\left(ba-c^2\right)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -a^3 - b^3 - c^3 + 3abc$$

$$= -\left(a^3+b^3+c^3-3abc\right)$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \left(a^3+b^3+c^3-3abc\right)^2 \_\_(2)$$

From (1) and (2) we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 =$$

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} =$$

$$\left(a^3+b^3+c^3-3abc\right)^2$$

21. Solve by Matrix inversion method:

$5x - 6y + 4z = 3$
$7x + 4y - 3z = 4$
$2x + y + 6z = 6$

Sol: Let $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Writting in AX = D form

$$A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$$

$$A_1 = \begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} = 24+3 = 27$$

$$B_1 = -\begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} = -(42+6) = -48$$

$$C_1 = \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = 7-8 = -1$$

$$A_2 = -\begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix} = -(-36-4) = 40$$

$$B_2 = \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 30-8 = 22$$

$$C_2 = -\begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} = -(5+12) = -17$$

$$A_3 = \begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix} = 18-16 = 2$$

$$B_3 = -\begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = -(-15-28) = 43$$

$$C_3 = \begin{vmatrix} 5 & -6 \\ 7 & -3 \end{vmatrix} = 20+42 = 62$$

$$Adj.A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$Adj.A = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

Det A = Δ = 419

$$A^{-1} = \frac{Adj.A}{Det\,A} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$X = A^{-1}D = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ 48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$= \frac{1}{149}\begin{bmatrix} 405+760+92 \\ -720+418+1978 \\ -15-323+2852 \end{bmatrix}$$

$$= \frac{1}{149}\begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

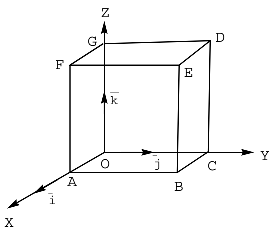
∴ Solution is $x = 3$, $y = 4$, $z = 6$

22. A line makes angles θ_1, θ_2, θ_3 and θ_4 with the diagonals of a cube.

Show that $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 = \frac{4}{3}$

Sol: Without loss of generality we may assume that the cube is a unit cube let $\overline{OA} = \overline{i}, \overline{OC} = \overline{j}, \overline{OG} = \overline{k}$

be coterminus edges of the cube.



Diagonals are

$$\overline{OE} = \overline{i} + \overline{j} + \overline{k}$$

$$\overline{BG} = -\overline{i} - \overline{j} + \overline{k}$$

$$\overline{AD} = -\overline{i} + \overline{j} + \overline{k}$$

$$\overline{CF} = \overline{i} - \overline{j} + \overline{k}$$

Let $\overline{r} = l\overline{i} + m\overline{j} + n\overline{k}$ a unit vector is parallel to line which makes θ_1, θ_2, θ_3, θ_4 angles with four diagonals of a cube, then

$$\cos\theta_1 = \frac{\overline{r}.\overline{OE}}{|\overline{r}||\overline{OE}|}$$

$$= \frac{(l\overline{i}+m\overline{j}+n\overline{k}).(\overline{i}+\overline{j}+\overline{k})}{|l\overline{i}+m\overline{j}+n\overline{k}||\overline{i}+\overline{j}+\overline{k}|} = \frac{l+m+n}{1.\sqrt{3}}$$

$\cos\theta_2 = \frac{\bar{r}.\overline{BG}}{|\bar{r}||\overline{BG}|} = \frac{-l-m+n}{1.\sqrt{3}}$

$\cos\theta_3 = \frac{\bar{r}.\overline{AD}}{|\bar{r}||\overline{AD}|} = \frac{-l+m+n}{\sqrt{3}}$

$\cos\theta_4 = \frac{\bar{r}.\overline{CF}}{|\bar{r}||\overline{CF}|} = \frac{l-m+n}{\sqrt{3}}$

Now, $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4$

$= \frac{1}{3}((l + m + n)^2 + (-l - m + n)^2 + (-l + m + n)^2 + (l - m + n)^2)$

$= \frac{1}{3}.4(l^2 + m^2 + n^2)$

$= \frac{4}{3}(1) = \frac{4}{3}.$

23. In a triangle ABC prove that

$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$

$= 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$

Sol. RHS:

$= 4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$

$= 2\left[2\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\right]\cos\left(\frac{\pi-C}{4}\right)$

$= 2\left[\cos\left(\frac{2\pi-(A+B)}{4}\right) + \cos\left(\frac{A-B}{4}\right)\right].$

$\cos\left(\frac{\pi-C}{4}\right)$

$= 2\left[\cos\left(\frac{\pi+C}{4}\right) + \cos\left(\frac{A-B}{4}\right)\right].$

$\cos\left(\frac{\pi-C}{4}\right)$

$= 2\left[\cos\left(\frac{\pi+C}{4}\right)\cdot\cos\left(\frac{\pi-C}{4}\right)\right]$

$+2\left[\cos\left(\frac{A-B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)\right]$

$= \cos\left(\frac{\pi+C+\pi-C}{4}\right) + \cos\left(\frac{\pi+C-\pi+C}{4}\right)$

$+\cos\left(\frac{A-B+\pi-C}{4}\right) + \cos\left(\frac{A-B-\pi+C}{4}\right)$

$= \cos\frac{\pi}{2} + \cos\frac{C}{2} + \cos\frac{A}{2} + \cos\frac{B}{2}$

$= \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = L.H.S$

∴ Hence Proved.

24. Show that

$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + \frac{r}{2R}.$

Sol. LHS: $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$

$= \cos^2\frac{A}{2} + 1 - \sin^2\frac{B}{2} + \cos^2\frac{C}{2}$

$= 1 + \cos^2\frac{A}{2} - \sin^2\frac{B}{2} + \cos^2\frac{C}{2}$

$= 1 + \left[\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right] + \cos^2\frac{C}{2}$

$= 1 + \cos\left(\frac{\pi-C}{2}\right)\cos\left(\frac{A-B}{2}\right) +$

$1 - 2\sin^2\frac{C}{2}$

$= 2 + \sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin^2\frac{C}{2}$

$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$

$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\pi - \frac{(A+B)}{2}\right)\right]$

$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$

$= 2 + \sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$

$= 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= 2\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$

$= 2\left[1 + \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R}\right]$

$= 2\left[1 + \frac{r}{4R}\right]$

$= 2 + \frac{r}{2R} = RHS$

LHS = RHS

∴ Hence proved.