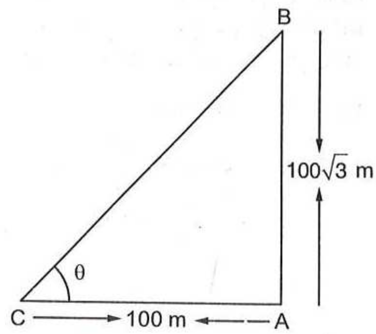
Heights and Distances

Example -1 A tower is $100\sqrt{3}$ meter high. Find the angle of elevation if it's top from a point 100 metres away from its foot.

Solution: Let AB be the tower of height $100\sqrt{3}$ metres, and let C be a point at a distance of 100 metres from the foot of the tower.

Let θ be the angle of elevation of the top of the tower from point C.

Clearly, in ΔCAB the lengths of base AC and perpendicular AB are known. So, we will use the trigonometric ratio containing base and perpendicular. Such a ratio is tangent. Taking tangent of angle $\angle ACB$ in ΔCAB, we have



$$\tan\ \theta = \frac{AB}{AC}$$

$$\Rightarrow \quad \tan\ \theta = \frac{100\sqrt{3}}{100} = \sqrt{3}$$

$$\Rightarrow \quad \tan\theta = \tan 60^0$$

$$\Rightarrow \quad \theta = 60^0$$

Hence, the angle of elevation of the top of the tower from a point 100 metres away from its foot is 60^0.

Example -2 The string of a kite is 100 metres long and it makes an angle of 60^0 with the horizontal. Find the height of the kite, assuming that there is no slack in the string.

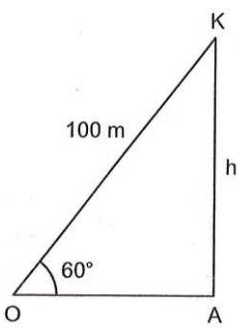
Solution: Let OA be the horizontal ground, and let K be the position of the kite at a height h above the ground. Then, AK = h.

It is given that OK=100 metres and $\angle AOK = 60^0$.

Thus, in ΔOAK, we have hypotenuse OK=100m and $\angle AOK = 60^0$ and we wish to find the perpendicular AK. So, we use the trigonometric ratio involving perpendicular and hypotenuse. Clearly, sine is such a ratio. So, we take the sine of $\angle AOK$ in ΔOAK.

In ΔAOK, we have

$$\sin 60^0 = \frac{AK}{OK}$$



$$\Rightarrow \quad \sin 60^0 = \frac{h}{100}$$

$$\Rightarrow \quad h = 100 \sin 60^0$$

$$\Rightarrow \quad h = 100\frac{\sqrt{3}}{2} = 50\sqrt{3} = 86.60 \; metres.$$

Hence, the height of the kite is 86.60 metres.

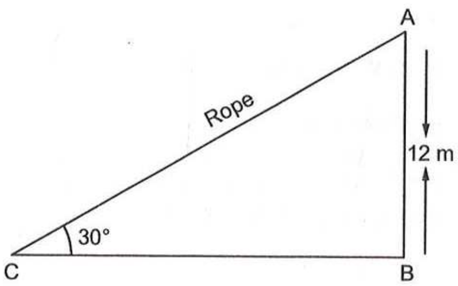
Example -3 **A circus artist is climbing from the ground along a rope stretched from the top of a vertical pole and tied at the ground. The height of the pole is 12m and the angle made by the rope with ground level is 30⁰. Calculate the distance covered by the artist in climbing to the top of the pole.**

Solution: Clearly, distance covered by the artist is equal to the length of the rope AC. Let AB be the vertical pole of height 12 m.

It is given that $\angle ACB = 30^0$.

Thus, in right – angled triangle ABC, we have

Perpendicular AB = 12m. $\angle ACB = 30^0$ and we wish to find hypotenuse AC.



$$\therefore \quad \sin 30^0 = \frac{AB}{AC}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{12}{AC}$$

$$\Rightarrow \quad AC = 24m$$

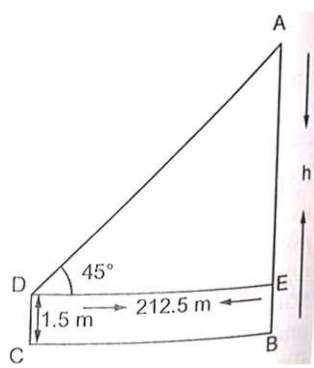
Hence, the distance covered by the circus artist is 24m.

Example -4 **An observer 1.5m tall is 28.5 m away from a tower. The angle of elevation of the top of the tower from her eyes is 45⁰. What is the height of the tower?**

Solution: Let AB be the tower of height h and CD be the observer of height 1.5 m at a distance of 28.5m from the tower AB.

In ΔAED, we have

$$\tan 45^0 = \frac{AE}{DE}$$



$$\Rightarrow \quad 1 = \frac{AE}{28.5}$$

$$\Rightarrow \quad AE = 28.5m$$

$$\therefore \quad h = AE + BE = AE + DC$$

$$= (28.5 + 1.5)m = 30m$$

Hence, the height of the tower is 30 m.

Example -5 From a point on the ground 40m away from the foot of a tower, the angle of elevation of the top of the tower is 30^0. The angle of elevation of the top of a water tank (on the top of the tower) is 45^0. Find the (i) height of the tower (ii) the depth of the tank.

Solution: Let BC be the tower of height h metre and CD be the water tank of height h_1 metre.

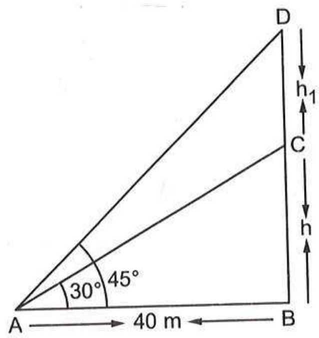
Let A be a point on the ground at a distance of 40m away from the foot B of the tower.

In ΔABD, we have

$$\tan 45^0 = \frac{BD}{AB}$$

$$\Rightarrow \quad 1 = \frac{h + h_1}{40} \quad \text{...(i)}$$

$$\Rightarrow \quad h + h_1 = 40m$$



In ΔABC, we have

$$\tan 30^0 = \frac{BC}{AB}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{40}$$

$$\Rightarrow \quad h = \frac{40}{\sqrt{3}}m = \frac{40\sqrt{3}}{3}m = 23.1m$$

Substituting the value of h in (i), we have

$$23.1 + h_1 = 40$$

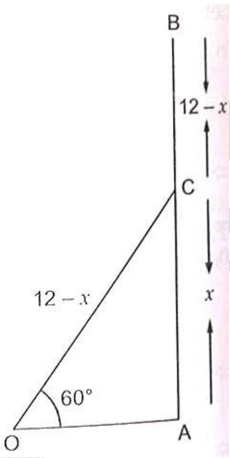
$$\Rightarrow \quad h_1 = (40 - 23.1)m = 16.9m$$

Hence, the height of the tower is h = 23.1 m and the depth of the tank is h_1 = 16.9m

Example -6 A tree 12m high, is broken by the wind in such a way that its top touches the ground and makes an angle 60^0 with the ground. At what height from the bottom the tree is broken by the wind?

Solution: Let AB be the tree of height 12 metres. Suppose the tree is broken by the wind at point C and the part CB assumes the position CO and meets the ground at O.

Let AC= x. Then, CO = CB=12- x. It is given that $\angle AOC = 60^0$



$$\sin 60^0 = \frac{AC}{OC}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{x}{12-x}$$

$$\Rightarrow 12\sqrt{3} - \sqrt{3}x = 2x$$

$$\Rightarrow 12\sqrt{3} = x(2+\sqrt{3})$$

$$\Rightarrow x = \frac{12\sqrt{3}}{2+\sqrt{3}} = \frac{12\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = 12\sqrt{3}(2-\sqrt{3})$$

$$\Rightarrow x = 24\sqrt{3} - 36 = 5.569 \; metres$$

Hence, the tree is broken at a h eight of 5.569 metres from the ground.

Example -7 From the top of a hill, the angles of depression of two consecutive kilometer stones due east are found to be 30^0 and 45^0. Find the height of the hill.

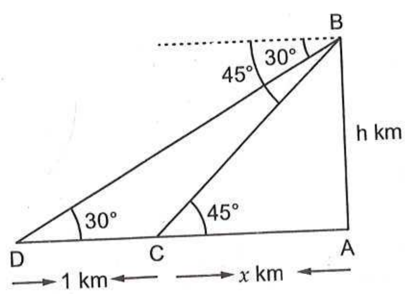
Solution: Let AB be the hill of height h km. Let C and D b e two stones due east of the hill at a distance of 1 k m from each other such that the angles of depression of C and D be 45^0 and 30^0 respectively. Let AC = x km.

In ΔCAB, we have

$$\tan 45^0 = \frac{AB}{AC}$$

$$\Rightarrow 1 = \frac{h}{x} \Rightarrow h = x \qquad ..(i)$$

In ΔDAB, we have



$$\tan 30^0 = \frac{AB}{AD}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{x+1}$$

$$\Rightarrow \sqrt{3}h = x+1$$

Substituting the value of x from equation (i) in equation (ii), we get

$$\sqrt{3}h = h+1$$

$$\Rightarrow h(\sqrt{3}-1) = 1$$

$$\Rightarrow h = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$\Rightarrow h = \frac{\sqrt{3}+1}{2} = \frac{2.73}{2} = 1.365\ Km$$

Hence, the height of the hill is 1.365 km.

Example - 8 Two pillars of equal height and on either side of a road, which is 100 m wide. The angles of elevation of the top of the pillars are 60^0 and 30^0 at a point on the road between the pillars. Find the position of the point between the pillars and the height of each pillar.

Solution: Let AB and CD be two pillars, each of height h metres. Let P be a point on the road such that AP = x metres. Then, CP = (100- x) metres. It is given that $\angle APB = 60^0$ *and* $\angle CPD = 30^0$.

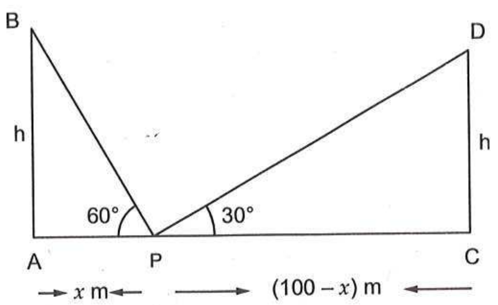
In ΔPAB, we have

$$\tan 60^0 = \frac{AB}{AP}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x} \qquad ..(i)$$

$$\Rightarrow h = \sqrt{3}x$$

In ΔPCD, we have



$$\tan 30^0 = \frac{CD}{PC}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{100-x} \qquad ...(ii)$$

$$\Rightarrow h\sqrt{3} = 100-x$$

Eliminating h between equation (i) and (ii), we get

$$3x = 100 - x \Rightarrow 4x = 100 \Rightarrow x = 25$$

Substituting $x = 25$ in equation (i), we get

$$h = 25\sqrt{3} = 25 \times 1.732 = 43.3\, m$$

Thus, the required point is at a distance of 25 metres from the first pillar and 75 metres from the second pillar. The height of the pillars is 43.3 metres.

Example -9 **The angle of elevation of a jet plane from a point A on the ground is 60^0. After a fight of 30 seconds, the angle of elevation changes to 30^0. If the jet plane is flying at a constant height of $3600\sqrt{3}$ m, find the speed of the jet plane.**

Solution: Let P and Q be the two positions of the plane and let A be the point of observation. Let ABC be the horizontal line through A. it is given that angles of elevation of the plane in two positions P and Q from a point A are 60^0 and 30^0 respectively.

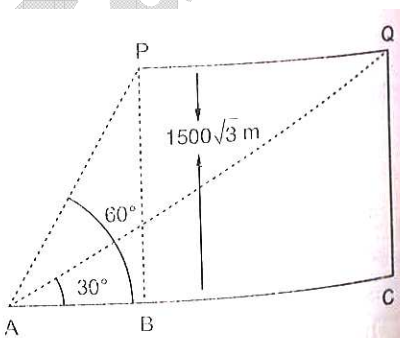
$\therefore\ \angle PAB = 60^0, \angle QAB = 30^0$. It is also given that PB = $3600\sqrt{3}$ metres

In ΔABP, we have

$$\tan 60^0 = \frac{BP}{AB}$$

$$\Rightarrow \sqrt{3} = \frac{3600\sqrt{3}}{AB}$$

$$\Rightarrow AB = 3600m$$



In ΔACQ, we have

$$\tan 30^0 = \frac{CQ}{AC}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{3600\sqrt{3}}{AC}$$

$$\Rightarrow AC = 3600 \times 3 = 10800m$$

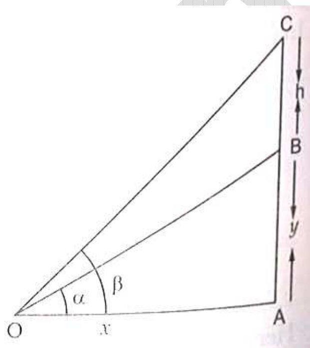
$$\therefore\ PQ = BC = AC - AB = 10800 - 3600 = 7200m$$

Thus, the plane travels 7200 m in 30 seconds.

Hence, Speed of plane $= \frac{7200}{30} = 240\, m/\sec = \frac{240}{1000} \times 60 \times 60 = 864\, km/hr$

Example -10 A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff of height h. At a point on the plane, the angles of elevation of the bottom and the top of the flag-staff are α and β respectively. Prove that the height of the tower is $\frac{h\tan\alpha}{\tan\beta-\tan\alpha}$.

Solution: Let AB be the tower and BC be the flag-staff. Let O be a point on the plane containing the foot of the tower such that the angles of elevation of t he bottom B and top C of the flag-staff at O are α and β respectively. Let OA = x metres, AB = y metres and BC = h metres.



In ΔOAB, we have

$$\tan\alpha=\frac{AB}{OA}$$

$$\Rightarrow \tan\alpha=\frac{y}{x}$$

$$\Rightarrow x=\frac{y}{\tan\alpha} \qquad ...(i)$$

$$\Rightarrow x=y\cot\alpha$$

In ΔOAC, we have

$$\tan\beta=\frac{y+h}{x}$$

$$\Rightarrow x=\frac{y+h}{\tan\beta}$$

$$\Rightarrow x=(y+h)\cot\beta$$

On equating the values of x given in equations (i) and (ii), we get ..(ii)

$$\Rightarrow y\cot\alpha=(y+h)\cot\beta$$

$$\Rightarrow (y\cot\alpha-y\cot\beta)=h\cot\beta$$

$$\Rightarrow y(\cot\alpha-\cot\beta)=h\cot\beta$$

$$\Rightarrow y=\frac{h\cot\beta}{\cot\alpha-\cot\beta}=\frac{\frac{h}{\tan\beta}}{\frac{1}{\tan\alpha}-\frac{1}{\tan\beta}}=\frac{h\tan\alpha}{\tan\beta-\tan\alpha}$$

Hence, the height of the tower is $\frac{h\tan\alpha}{\tan\beta-\tan\alpha}$