CIRCLES

1. A circle is a collection of all points in a plane which are at a constant distance from a fixed point. The fixed point is called the centre and the constant distance is known as the radius.
2. A line which intersects a circle in two distinct points is called a secant of the circle.
3. A tangent to a circle is a line that intersects the circle in exactly one point.
4. The common point of a tangent and the circle is called (i) Point of contact
5. A circle may have two parallel tangents.
6. A tangent to a circle intersects it in one point (s).
7. The angle between tangent at a point on a circle and the radius through the point is 90^0.
8. **Length of tangent** The length of the segment of the tangent between the point and the given point of contact with the circle is called the length of the tangent from the point to the circle.

Some Properties of Tangent to A Circle

Theorem 1 : A tangent to a circle is perpendicular to the radius through the point of contact.

Given A circle (O, r) and a tangent AB at a point p.

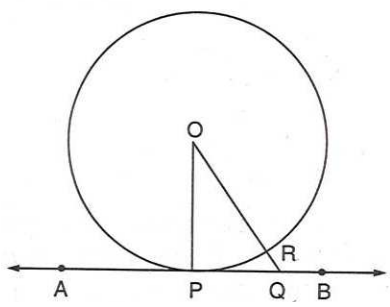
To Prove $OP \perp AB.$

Construction Take any point Q, other than P, on the tangent AB. Join $OQ.$ Suppose OQ meets the circle at R.

Proof We know that among all line segments joining the point O to a point on AB, the shortest one is perpendicular to AB. So, to prove that $OP \perp AB,$ it is sufficient to prove that OP is shorter than any other segment joining O to any point of AB.

Clearly, OP = OR (Radii of the same circle)

Now, $OQ = OR + RQ$



$\Rightarrow \quad OQ > OR$

$\Rightarrow \quad OQ > OP \qquad [\because OP = OR]$

$\Rightarrow \quad OP < OQ$

Thus, OP is shorter than any other segment joining O to any point of AB.

Hence, $OP \perp AB.$

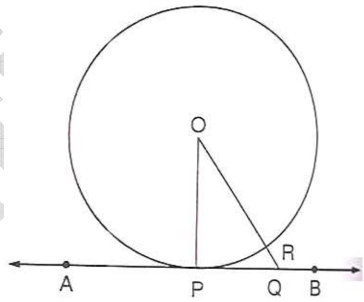
Theorem 2 A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

Given A radius OP of a circle C (O, r) and a line APB, perpendicular to OP.

To prove AB is a tangent to the circle at the point P.

Proof Take a point Q, different from P, on the line AB.

Now, $OP \perp AB.$



$\Rightarrow$ Among all the line segment joining O to a point on AB, OP is the shortest.

$\Rightarrow \quad OP < OQ$

$\Rightarrow \quad OQ > OP$

$\Rightarrow \quad Q$ lies outside the circle.

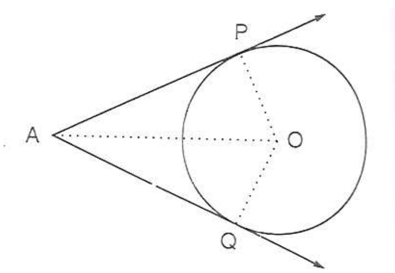
Thus, every point on AB, other than P, lies outside the circle. This shows that AB meets the circle only at the point P.

Hence, AB is a tangent to the circle at P.

Theorem 3 The lengths of two tangents drawn from an external point to a circle are equal.

Given AP and AQ are two tangents from a pint A to a circle C (O, r).

To Prove AP = AQ



Construction Join OP, OQ and OA.

Proof In order to prove that AP = AQ, we shall first prove that $\Delta OPA \cong OQA.$

Since a tangent at any point of a circle is perpendicular to the radius through the point of contact.

$\therefore \quad OP \perp AP \ \text{ and } OQ \perp AQ.$

$\Rightarrow \quad \angle OPA = \angle OQA = 90^0$

Now, in right triangles OPA and OQA, we have (Radii of a circle)

$OP = OQ$

$\angle OPA = \angle OQA$ (From (i))

And OA = OA

So, by RHS – criterion of congruence, we get (Common)

$\Delta OPA = \Delta OQA$

$\Rightarrow \quad AP = AQ$

Theorem 4 If two tangents are drawn to a circle from an external point, then:

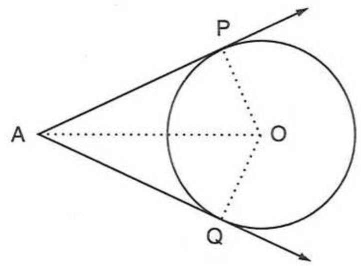
(i) They subtend equal angles at the centre,

(ii) They are equally inclined to the segment, joining the centre to that point.

Given A circle C (O, r) and a point A outside the circle such that AP and AQ are the tangents drawn to the circle from point A.

To Prove (i) $\angle AOP = \angle AOQ$

(ii) $\angle OAP = \angle OAQ$.



Proof In right triangles OAP and OAQ, we have

AP = AQ (Tangents from an external point are equal)

OP = OQ (Radii of a circle)

and, OA = OA (Common)

So, by SSS – criterion of congruence, we have

$\Delta OAP \cong \Delta OAQ$

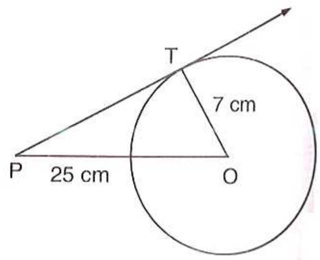
$\Rightarrow \quad \angle AOP = \angle AOQ \ \text{ and } \ \angle OAP = \angle OAQ.$

Example 1 Find the length of the tangent drawn from a point whose distance from the centre of a circle is 25 cm. Given that the radius of the circle is 7 cm.

Solution Let P be the given point, O be the centre of the circle and PT be the length of tangent from P. Then, OP = 25 cm and OT = 7 cm.

Since tangent to a circle is always perpendicular to the radius through the point of contact.

$\therefore \quad \angle OTP = 90^0$



In right triangle OTP, we have

$$OP^2 = OT^2 + PT^2$$

$\Rightarrow \quad 25^2 = 7^2 + PT^2$

$\Rightarrow \quad PT^2 = 25^2 - 7^2 = (25-7)(25+7) = 576$

$\Rightarrow \quad PT = 24\,cm.$

Hence, length of tangent from P = 24 cm.

Example 2 In if AB = AC, prove that BE = EC.

(or)

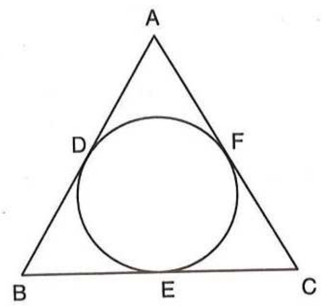
ABC is an isosceles triangle in which AB = AC, circumscribed about a circle, as shown in Prove that the base is bisected by the point of contact.

Solution Since tangents from an exterior point to a circle are equal in length.

AD = AF (Tangents from A) …(i)

BD = BE (Tangents from B) …(ii)

CE= CF (Tangents from C) …(iii)



Now,

AB= AC

$\Rightarrow \quad AB - AD = AC - AD$ (Subtracting AD from both sides)

$\Rightarrow \quad AB - AD = AC - AF$ (Using (i))

$\Rightarrow \quad BD = CF$

$\Rightarrow \quad BE = CF$ (Using (ii))

$\Rightarrow \quad BE = CE$ (Using (iii))

Example 3 In, the in circle of $\Delta\ ABC$ touches the sides BC, CA and AB at D,E and F respectively. Show that

$$AF + BD + CE = AE + BF + CD + \frac{1}{2}(Perimeter\ of\ \Delta ABC)$$

Solution Since lengths of the tangents from an exterior point to a circle are equal.

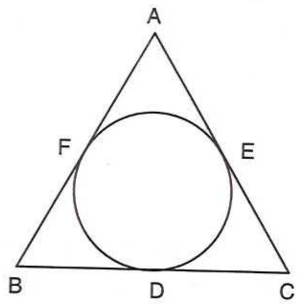
$\therefore \quad AF = AE$ (From A) …(i)

BD= BF (From B) … (ii)

and, CE = CD (From C) …. (iii)

adding equations (i), (ii) and (iii), we get

$$AF + BD + CE = AE + BF + CD$$



Now,

Perimeter of $\Delta ABC = AB + BC + AC$

$\Rightarrow$ Perimeter of $\Delta ABC = (AF + FB) + (BD + CD) + (AE + EC)$

$\Rightarrow$ Perimeter of $\Delta ABC = (AF + AE) + (BF + BD) + (CD + CE)$

$\Rightarrow$ Perimeter of $\Delta ABC = 2AF + 2BD + 2CE$

$\Rightarrow$ Perimeter of $\Delta ABC = 2(AF + BD + CE)$ $\left[\begin{array}{l} From\ (i),(ii)\ and\ (iii)\ ,we\ get \\ AE = AF, BD = BF\ and\ CD = CE \end{array}\right]$

$\Rightarrow AF + BD + CE = \frac{1}{2}(Perimeter\ of\ \Delta ABC)$

Hence, $AF + BD + CE = AE + BF + CD = \frac{1}{2}\ (Perimeter\ of\ \Delta ABC)$

Example 4 **A circle touches all the four sides of a quadrilateral ABCD. Prove that :**

AB+ CD = BC + DA.

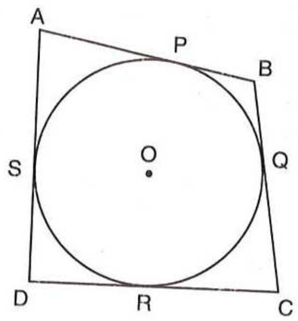
Solution Since tangents drawn from an exterior point to a circle are equal in length.

$\therefore$ $AP = AS$ (From A) ....(i)

BP = BQ (From B) ... (ii)

CR= CQ (From C) ... (iii)

and, DR= DS (From D) ... (iv)



adding (i), (ii), (iii) and (iv), we get

AP + BP + CR + DR = AS + BQ + CQ + DS

$\Rightarrow$ (AP+BP) + (CR+ DR) = (AS+ DS) + (BQ+CQ)

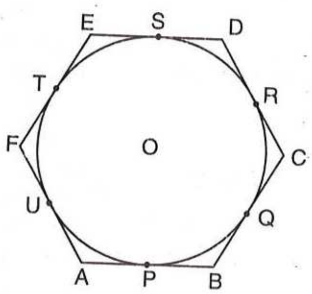
$\Rightarrow$ AB + CD = AD + BC

Hence, AB + CD = BC + DA

Example 5 **If a hexagon ABCDEF circumscribes a circle, prove that**

AB+ CD+ EF = BC + DE + FA.

Solution Let O be the centre of the circle touching sides AB, BC, CD, DE, EF and FA at P,Q, R, S, T and U respectively. The lengths of tangents drawn from an external point to a circle are equal.



$\therefore \quad AP = AU, BP = BQ, CQ = CR, DR = DS, ES = ET \text{ and } FU = FT$

Now,

AB+ CD+EF

= (AP+PB) + (CR+DR) + (ET+TF)

= (AU+ PB) + (CQ+DS) + (ES+ FU) $\left[\begin{array}{l} \because \; AP = AU, PB = BQ, CR = CQ, \\ DR = DS, ET = ES, FT = FU \end{array}\right]$

= (AU+ FU)+(BQ+CQ) + (DS+ ES)

= AF + BC+DE= BC + DE+ FA

Example 6 **Let s denote the semi perimeter of a triangle ABC in which BC = a, CA= b and AB= c. If a circle touches the sides BC, CA, AB at D, E, F respectively, prove that AF= AE = s –a, BD = BF = s – b and CD = CE = s – c.**

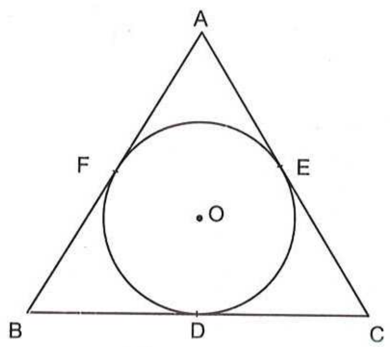
Solution we have,

$$s = \frac{AB + BC + CA}{2} = \frac{a+b+c}{2}$$

$\Rightarrow \; a + b + c = 2s$

$\Rightarrow \; b + c = 2s - a, c + a = 2s - b \text{ and } a + b = 2s - c$

$\Rightarrow \; b + c - a = 2(s - a), c + a - b = 2(s - b) \text{ and } a + b - c = 2(s - c)$



The lengths of tangents drawn from an external point to a circle are equal.

$\therefore \quad AF = AE, BD = BF \text{ and } CD = CE$

Now,

2s = BC+ CA+ BA

$$\Rightarrow \quad 2s = (BD + DC) + (CE + AE) + (AF + BF)$$
$$\Rightarrow \quad 2s = (BD + DC) + (CD + AF) + (AF + BD)$$
$$\Rightarrow \quad 2s = 2(BD + DC) - 2AF$$
$$\Rightarrow \quad 2s = 2BC + 2AF$$
$$\Rightarrow \quad 2s = 2a + 2AF$$
$$\Rightarrow \quad AF = s - a \Rightarrow AF = AE = s - a$$

Again,

2s= BC= CA+ AB

$$\Rightarrow \quad 2s = (BD + CD) + (CE + AE) + (AF + FB)$$
$$\Rightarrow \quad 2s = (BF + CE) + (CE + AE) + (AE + FB)$$
$$\Rightarrow \quad 2s = 2BF + 2(AE + CE)$$
$$\Rightarrow \quad 2s = 2BF + 2AC$$
$$\Rightarrow \quad 2s = 2BF + 2b$$
$$\Rightarrow \quad BF = s - b$$
$$\Rightarrow \quad BD = BF = s - b$$

Similarly, we can prove that CD= CE = s-c.

Example 7 If all the side of a parallelogram touch a circle, show that the parallelogram is a rhombus.

(or)

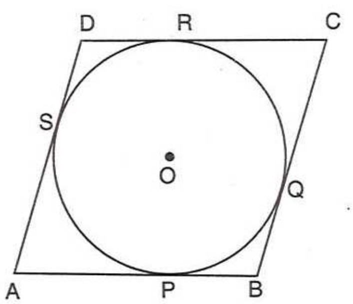
Prove that a parallelogram circumscribing a circle is a rhombus.

Solution Let ABCD be a parallelogram such that its

Sides touch a circle with centre O.

We know that the tangents to a circle from an exterior point

are equal in length.



$\therefore \quad AP = AS$ (From A) ….. (i)

BP = BQ (From B) …. (ii)

CR = CQ (From C) …. (iii)

and, DR = DS (From D) … (iv)

adding (i), (ii), (iii) and (iv), we get

AP + BP + CR + DR = AS + BQ + CQ+ DS

$\Rightarrow$ (AP + BP) + (CR+ DR) = (AS + DS) + (BQ + CQ)

$\Rightarrow$ AB + CD = AD + BC

$\Rightarrow$ 2AB = 2 BC $\quad [\because$ *ABCD is a parallelogram* $\therefore AB = CD$ *and* $BC = AD]$

$\Rightarrow$ AB = BC

Thus, AB = BC = CD = AD

Hence, ABCD is a rhombus.

Example 8 From an external point P, two tangents PA and PB are drawn to the circle with centre O. prove that OP is the perpendicular bisector of AB.

Solution Suppose OP intersects AB at C.

In triangles PAC and PBC, we have

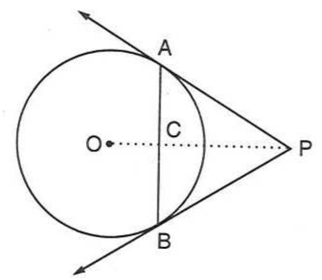
PA = PB $\quad [\because$ *Tangents from an external point are equal*$]$

$\angle APC = \angle BPC \quad [\because$ *PA and PB are equally inclined to OP*$]$

and, PC = PC (Common)

So, by SAS – criterion of similarity, we obtain

$\Delta PAC \cong \Delta PBC$



$\Rightarrow$ AC = BC and $\angle ACP = \angle BCP$

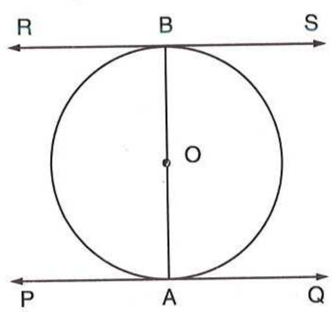
But, $\angle ACP + \angle BCP = 180^0$

$\therefore \angle ACP + \angle BCP = 90^0$

Hence, $OP \perp AB$.

Example 9 Show that tangent lines at the end points of a diameter of a circle are parallel.

Solution Let AB be a diameter of a given circle, and let PQ and RS be the tangent lines drawn to the circle at points A and B respectively. Since tangent at a point to a circle is perpendicular to the radius through t he point.



$\therefore\ AB \perp PQ$ and $AB \perp RS$

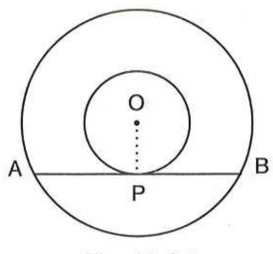
$\Rightarrow\ \angle PAB = 90^0$ and $\angle ABS = 90^0$

$\Rightarrow\ \angle PAB = \angle ABS$

$\Rightarrow\ PQ \parallel RS$ $\qquad [\because \angle PAB$ and $\angle ABS$ are alternate angles$]$

Example 10 In two concentric circles, prove that a chord of larger circle which is tangent to smaller circle is bisected at the point of contact.

Solution Let O be the common centre of two con- centric circles, and let AB be a chord of the larger circle touching the smaller circle at P.



Join OP.

Since OP is the radius of t he smaller circle and AB is a tangent to this circle at a point P.

$\therefore\ OP \perp AB$

We know that the perpendicular drawn from the centre of a circle to any chord of the circle, bi sects the chord. So,

$OP \perp AB$

$\Rightarrow$ AP = BP

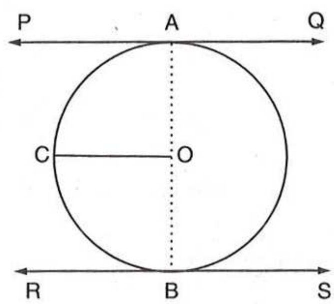
Hence, AB is bisected at P.

Example 11 Prove that the segment joining the points of contact of two parallel tangents passes through the centre.

Solution Let PAQ and RBS be two parallel tangents to a circle with centre O.

Join OA and OB. Draw $OC \parallel PQ$.

Now, $\qquad PA \parallel CO$



$\Rightarrow \quad \angle PAO + \angle COA = 180^0$ $\left[\begin{array}{l} \textit{Sum of the angles on the same} \\ \textit{side of a transversal is } 180^0 \end{array}\right]$

$\Rightarrow 90^0 + \angle COA = 180^0 \quad [\because \angle PAO = \textit{angle between a tangent and radius} = 90^0]$

$\Rightarrow \quad \angle COA = 90^0$

Similarly, $\angle COB = 90^0$

$\therefore \qquad \angle COA + \angle COB = 90^0 + 90^0 = 180^0$

Hence, AOB is a straight line passing through O.