Triangles

Concept of Similarity

1. Two geometric figures having the same shape and size are known as congruent figures.
2. Geometric figures having the same shape but different sizes are known as similar figures.
3. Two congruent figures are always similar but similar figures need not be congruent
4. Any two line segments are always similar but they need not be congruent. They are congruent, if their lengths are equal.

Similar Triangles and their properties

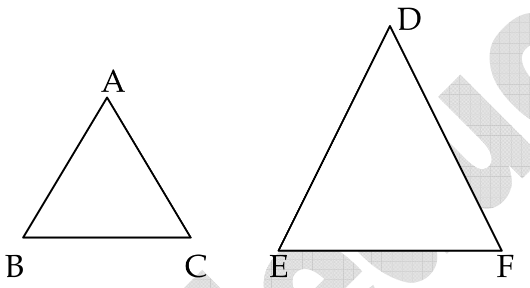
Definition Two triangles are said to be similar, if their

(i) Corresponding angles are equal and,

(ii) Corresponding sides are proportional.

It follow from this definition that two triangles ABC and DEF are similar, if

(i) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and (ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$



Note: In the later part of this chapter we shall show that the two conditions given in the above definition are not independent. In fact, if either of the two conditions holds, then the other holds automatically. So any one of the two conditions can be used to define similar triangles.

Theorem -1 **(Basic proportionality Theorem or Thales Theorem) If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.**

Given A triangle ABC in which DE $\parallel BC$, and intersects AB in D and AC in E.

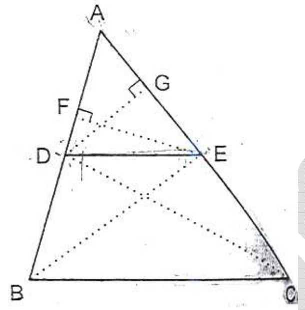
To prove $\frac{AD}{DB} = \frac{AE}{EC}$

Construction Join BE, CD and draw EF $\perp BA$ and $DG \perp CA$.

Proof: Since EF is perpendicular to AB. Therefore, EF is the height of triangles ADE and DBE.

Now, $\text{Area } (\Delta ADE) = \frac{1}{2}(base \times height) = \frac{1}{2}(AD.EF)$

and $\text{Area } (\Delta DBE) = \frac{1}{2}(base \times height) = \frac{1}{2}(DB.EF)$



$$\therefore \frac{Area\,(\Delta ADE)}{Area\,(\Delta DBE)} = \frac{\frac{1}{2}(AD.EF)}{\frac{1}{2}(DB.EF)} = \frac{AD}{DB} \;.....(i)$$

Similarly, we have

$$\frac{Area\,(\Delta ADE)}{Area\,(\Delta DEC)} = \frac{\frac{1}{2}(AE.DG)}{\frac{1}{2}(EC.DG)} = \frac{AE}{EC} \;.....(ii)$$

But, ΔDBE *and* ΔDEC are on the same base DE and between the same parallels DE and BC.

$$\therefore \; Area\,(\Delta DBE) = Area\,(\Delta DEC)$$

$$\Rightarrow \frac{1}{Area\,(\Delta DBE)} = \frac{1}{Area\,(\Delta DEC)}$$

$$\Rightarrow \frac{Area\,(\Delta ADE)}{Area\,(\Delta DBE)} = \frac{Area\,(\Delta ADE)}{Area\,(\Delta DEC)}$$

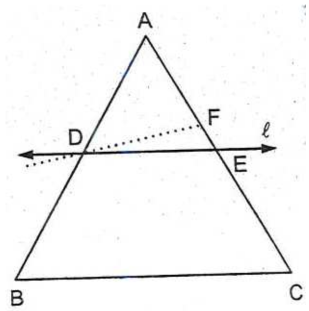
$$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$$

Theorem 2 **(Converse of Basic Proportionality Theorem) If a line divides any two sides of a triangle in the same ratio, then the line must be paralled to the third side.**

Given A ΔABC and a line l intersecting AB in D and AC in E, such that $\frac{AD}{DB} = \frac{AE}{EC}$ **To Prove** l : BC i.e. DE $\parallel BC$

Proof: If possible, let DE be not parallel to BC. Then, there must be another line parallel to BC. Let DF $\parallel BC$.

Since DF $\parallel BC$. Therefore, from Basic Proportionality Theorem, we get



$$\frac{AD}{DB} = \frac{AF}{FC} \qquad ...(i)$$

But, $\frac{AD}{DB} = \frac{AE}{EC}$ (*Given*) ...(*ii*)

From (i) and (ii), we get

$$\frac{AF}{FC} = \frac{AE}{EC}$$

$$\Rightarrow \frac{AF}{FC} + 1 = \frac{AE}{EC} + 1 \qquad [Adding\ 1\ on\ both\ sides]$$

$$\Rightarrow \frac{AF + FC}{FC} = \frac{AE + EC}{EC}$$

$$\Rightarrow \frac{AC}{FC} = \frac{AC}{EC}$$

$$\Rightarrow FC = EC$$

This is possible only when F and E coincide i.e. DF is the line l itself. But, DF $\parallel BC$.

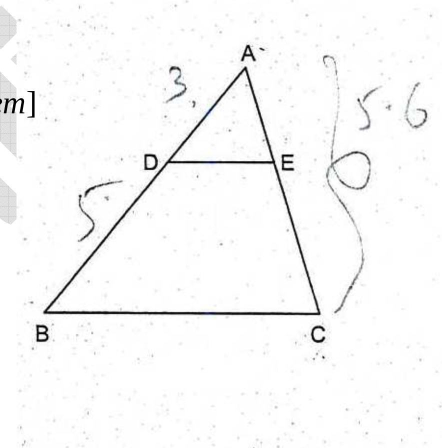
Hence, $\quad l \parallel BC$.

Example: 1 In a given $\Delta ABC, DE \parallel BC$ and $\frac{AD}{DB} = \frac{3}{5}$. If $AC = 5.6$, find AE.

Solution In ΔABC, we have

$DE \parallel BC$

$$\frac{AD}{DB} = \frac{AE}{EC} \qquad [By\ Thale's\ Theorem]$$



$$\Rightarrow \frac{AD}{DB} = \frac{AE}{AC - AE}$$

$$\Rightarrow \frac{3}{5} = \frac{AE}{5.6 - AE} \qquad [\because AC = 5.6]$$

$$\Rightarrow 3(5.6 - AE) = 5AE$$

$$\Rightarrow 16.8 - 3AE = 5AE$$

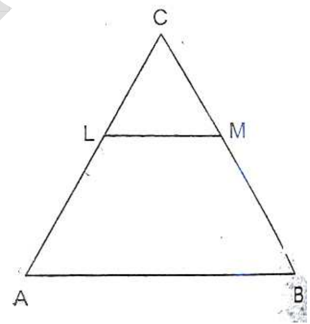
$$\Rightarrow 8AE = 16.8$$

$$\Rightarrow AE = \frac{16.8}{8} cm = 2.1 cm.$$

Example: 2 In Fig. 4.13, $LM \parallel AB$. IF AL= $x - 3$, $AC = 2x$, $BM = x - 2$ and BC= $2x + 3$, find the value of x.

Solution: In ΔABC, we have

$LM \parallel AB$



$\therefore \quad \frac{AL}{LC} = \frac{BM}{MC}$

$\Rightarrow \frac{AL}{AC - AL} = \frac{BM}{BC - BM}$

$\Rightarrow \frac{x-3}{2x-(x-3)} = \frac{x-2}{(2x+3)-(x-2)}$

$\Rightarrow \frac{x-3}{x+3} = \frac{x-2}{x+5}$

$\Rightarrow (x-3)(x+5) = (x-2)(x+3)$

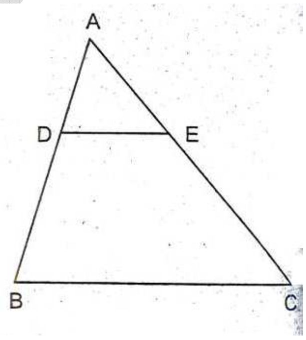
$\Rightarrow x^2 + 2x - 15 = x^2 + x - 6$

$\Rightarrow x = 9$

Example: 3 **D and E are respectively the points on the sides AB and AC of a ΔABC, such that AB=5.6cm, AD=1.4 cm, AC=7.2 cm and AE=1.8cm, show that $DE \parallel BC$**

Solution We have,

AB =5.6 cm, Ad=1.4 cm, AC=7.2 cm and AE=1.8 cm.



$\therefore \; BD = AB - AD = (5.6 - 1.4)cm = 4.2cm$

and,

$EC = AC - AE = (7.2 - 1.8)cm = 5.4cm$

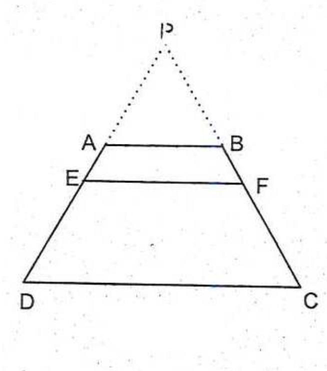
Now, $\frac{AD}{DB} = \frac{1.4}{4.2} = \frac{1}{3}$ and $\frac{AE}{EC} = \frac{1.8}{5.4} = \frac{1}{3}$

$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$

Thus, De divides sides AB and AC of ΔABC, in the same ratio. Therefore, by the converse of Basic Pro- portionality Theorem, we have

$DE \parallel BC$

Example: 4 **In Fig. 4.23, If** $EF \parallel DC \parallel AB$. *prove that* $\frac{AE}{ED} = \frac{BF}{FC}$.



Given $EF \parallel DC \parallel AB$ **in the given figure.**

To prove $\frac{AE}{ED} = \frac{BF}{FC}$.

Construction Produce DA and CB to meet at P (say).

Proof: In ΔPEF, We have

$AB \parallel EF$

$$\frac{PA}{AE}=\frac{PB}{BF} \qquad [By\ Thale's\ Theorem]$$

$$\Rightarrow \frac{PA}{AE}+1=\frac{PB}{BF}+1 \qquad [Adding\ 1\ on\ both\ sides]$$

$$\Rightarrow \frac{PA+AE}{AE}=\frac{PB+BF}{BF}$$

$$\Rightarrow \frac{PE}{AE}=\frac{PF}{BF} \qquad ...(i)$$

In ΔPDC, *we have*

$EF \parallel DC$

$$\frac{PE}{ED}=\frac{PF}{FC} \qquad [By\ Basic\ Proportionality\ Theorem] \qquad ...(ii)$$

On dividing equation (i) by equation (ii), we get

$$\frac{\frac{PF}{AE}}{\frac{PE}{ED}}=\frac{\frac{PF}{BF}}{\frac{PF}{FC}}$$

$$\Rightarrow \frac{ED}{AE}=\frac{FC}{BF}$$

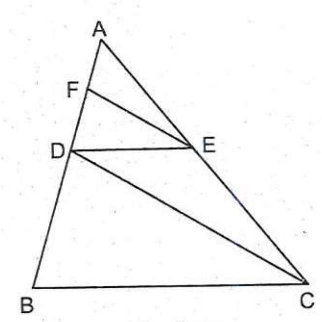
$$\Rightarrow \frac{AE}{ED}=\frac{BF}{FC}$$

Example: 5 In Fig. 4.27, $DE \parallel BC$ *and* $CD \parallel EF$. *prove that* $AD^2 = AB \times AF$.

Solution: In ΔABC, we have

$DE \parallel BC$

$$\Rightarrow \frac{AB}{AD}=\frac{AC}{AE} \qquad ...(i)$$



In ΔADC, *we have*

$FE \parallel DC$

$$\Rightarrow \frac{AD}{AF}=\frac{AC}{AE} \qquad ...(ii)$$

From (i) and (ii), we get

$$\frac{AB}{AD}=\frac{AD}{AF}$$

$$\Rightarrow AD^2 = AB \times AF$$

Example: 6 **Two triangles ABC and DBC lie on the same side of the base BC. From a point P on BC, $PQ \parallel BD$ and $PR \parallel BD$ are drawn. They meet AC in Q and DC in R respectively. Prove that $QR \parallel AD$**

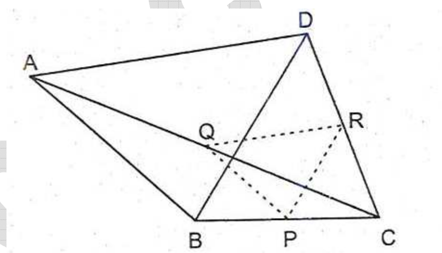
Given Two triangles ABC and DBC lie on the same side of the base BC. Points P,Q and R are points on BC, AC and CD respectively such that $PR \parallel BD$ and $PQ \parallel AB$. **To Prove** $QR \parallel AD$

Proof: In $\Delta\, ABC$, *we have*

$$PQ \parallel AB$$

$$\frac{CP}{PB} = \frac{CQ}{QA} \quad \text{...(i) [By Basic Proportionality Theorem]}$$

In ΔBCD, *we have*



$$PR \parallel BD$$

$$\frac{CP}{PB} = \frac{CR}{RD} \quad \text{..(ii) [By Thale's Theorem]}$$

From (i) and (ii), we have

$$\frac{CQ}{QA} = \frac{CR}{RD}$$

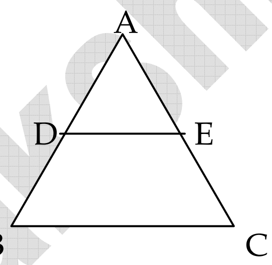
Thus, in ΔACD, Q *and* R are points on AC and CD respectively such that

$$\frac{CQ}{QA} = \frac{CR}{RD}$$

$\Rightarrow QR \parallel AD$

Example:7 **In ΔABC, D and E are points on the sides AB and AC respectively such that $DE \parallel BC$.**

(iv) If AD=4, AE=8,DB= $x-4$, and EC= $3x-19$, *find* x



$$\frac{4}{x=4} = \frac{8}{3x-19}$$

$12x - 76 = 8x - 32$

$4x = 44$

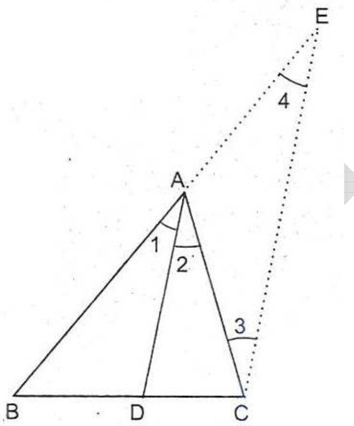
$x = 11cm$

Theorem 1 The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

Given A ΔABC in which AD is the internal bisector of $\angle A$ and meets BC in D.

To prove $\frac{BD}{DC} = \frac{AB}{AC}$

Construction Draw $CE \parallel DA$ to meet BA produced in E.



Proof Since $CE \parallel DA$ and AC cuts them.

$\angle 2 = \angle 3$ ...(*i*)

[*Alternate angles*]

and, $\angle 1 = \angle 4$ (*ii*) [*Corresponding angle*]

But, $\angle 1 = \angle 2$ [$\because$ *AD is the bisector of* $\angle A$]

From (i) and (ii),. We get

$\angle 3 = \angle 4$

Thus, in ΔACE, we have

$\angle 3 = \angle 4$

$\Rightarrow AE = AC$ ...(*iii*) [*Sides opposite to equal angles are equal*]

Now, in ΔBCE, we have

$DA \parallel CE$

$\Rightarrow \frac{BD}{DC} = \frac{BA}{AE}$ [Using Basic Proportionality Theorem]

$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC}$ [$\because BA = AB$ *and* $AE = AC$ (*From* (*iii*)]

Hence, $\frac{BD}{DC} = \frac{AB}{AC}$

In order to see whether the converse of the above theorem is true on not. Let us perform the following activity.

Theorem: 2 The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

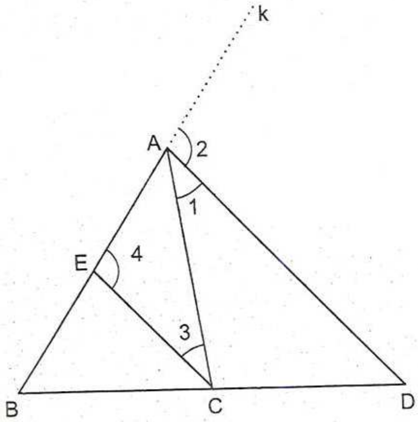
Given A ΔABC, in which AD is the bisector of the exterior of angle $\angle$A and intersects BC produced in D.

To Prove $\frac{BD}{CD} = \frac{AB}{AC}$

Construction Draw $CE \parallel DA$ meeting AB in E.

Proof Since $CE \parallel DA$ and AC intersects them.

$\therefore \quad \angle 1 = \angle 3$ ....(*i*)



Also, $CE \parallel DA$ and BK intersects them.

$\therefore \quad \angle 2 = \angle 4 \quad \quad ....(ii)$

$But, \quad \angle 1 = \angle 2 \quad \left[\begin{array}{l} \because AD \text{ is the bisector of} \\ \angle CAK \;\; \therefore \angle 1 = \angle 2 \end{array} \right]$

$\therefore \quad \angle 3 = \angle 4 \quad \quad [From(i) \text{ and } (ii)]$

Thus, in ΔACE, we have

$\angle 3 = \angle 4$

$\Rightarrow \quad AE = AC \quad$ [$\because$ Sides opposite to equal angles in a Δ are equal] ...(iii)

Now, in ΔBAD, we have

$EC \parallel AD$

$\therefore \quad \frac{BD}{CD} = \frac{BA}{EA}$ (Using corollary of Basic Proportionality Theorem]

$\Rightarrow \frac{BD}{CD} = \frac{AB}{AE} \quad [\because BA = AB \text{ and } EA = AE]$

$\Rightarrow \frac{BD}{CD} = \frac{AB}{AC} \quad [\because AE = AC, From(iii)]$

Example: 1 If the diagonal BD a quadrilateral ABCD bisects both $\angle$B and $\angle$D, show that $\frac{AB}{BC} = \frac{AD}{CD}$.

Given A quadrilateral ABCD in which the diagonal BD bisects $\angle$B and $\angle$D.

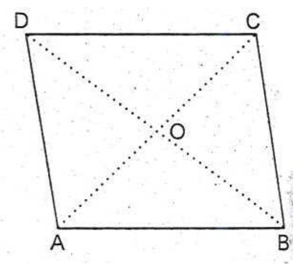
To Prove $\frac{AB}{BC} = \frac{AD}{CD}$.

Construction Join AC intersecting BD in O.

Proof: In ΔABC, BO is the bisector of $\angle$ B.

$\therefore \frac{AO}{OC} = \frac{BA}{BC}$

$\Rightarrow \frac{OA}{OC} = \frac{AB}{BC} \quad ....(i)$



In ΔADC, DO is the bisector of $\angle$D.

$\therefore \frac{AO}{OC} = \frac{DA}{DC}$

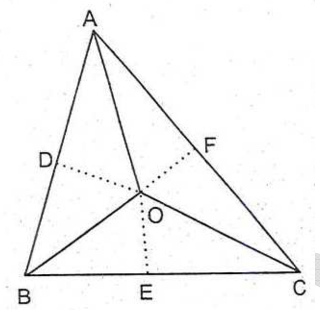
$\Rightarrow \frac{OA}{OC} = \frac{AD}{CD} \quad ....(ii)$

From (i) and (ii), we get $\frac{AB}{BC} = \frac{AD}{CD}$.

Example: 2 **O is any point inside a triangle ABC. The bisector of $\angle$AOB, $\angle$BOC and $\angle$ COA meet the sides AB, BC and CA in point D, E and F respectively. Show that**

$$AD \times BE \times CF = DB \times EC \times FA$$

Solution: In $\Delta AOB, OD$ is the bisector of $\angle$ AOB.



$$\therefore \quad \frac{OA}{OB} = \frac{AD}{DB} \qquad ....(i)$$

In $\Delta BOC, OE$ is the bisector of $\angle$ BOC.

$$\therefore \quad \frac{OB}{OC} = \frac{BE}{EC} \qquad ....(ii)$$

In $\Delta COA, OF$ is the bisector of $\angle$ COA.

$$\therefore \quad \frac{OC}{OA} = \frac{CF}{FA} \qquad ....(iii)$$

Multiplying the corresponding sides of (i), (ii) and (iii), we get

$$\frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA} = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow 1 = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow DB \times EC \times FA = AD \times BE \times CF$$

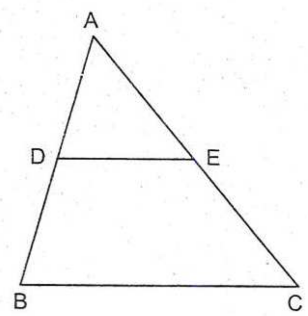
$$\Rightarrow AD \times BE \times CF = DB \times EC \times FA$$

Theorem 1: **The line drawn from the mid- point of one side of a triangle parallel to another side bisects the third side.**

Given A ΔABC in which D is the mid – point of side AB and the line DE is drawn parallel to BC, meeting AC in E.

To prove E is the mid – point of AC i.e., AE=EC.

Proof: In ΔABC, we have



$$DE \parallel BC$$

$$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC} \qquad \text{[By Thale's Theorem] .... (i)}$$

But, D is the mid – point of AB.

$$\Rightarrow AD = DB$$

$$\Rightarrow \frac{AD}{DB} = 1 \qquad (ii)$$

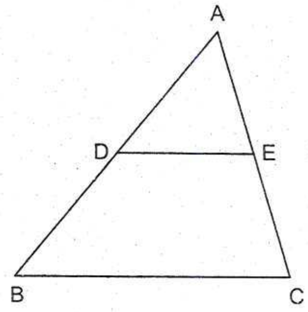
From (i) and (ii), we get

$$\frac{AE}{EC} = 1 \Rightarrow AE = EC.$$

Hence, E bisects AC.

Theorem: 2 **The line joining the mid – points of two sides of a triangle is parallel to the third side. Given A ΔABC in which D and E are mid – points of sides AB and AC respectively.**

To prove: $DE \parallel BC.$



Proof: Since D and E are mid – points of AB and Ac respectively.

AD = DB and AE = EC

$\Rightarrow \frac{AD}{DB} = 1 \, and \, \frac{AE}{EC} = 1$

$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$

Thus, the line DE divides the sides AB and AC of ΔABC in the same ratio. Therefore, by the converse of Basic Proportionality Theorem, we obtain $DE \parallel BC.$

Theorem: 3 **If the diagonals of a quadrilateral divide each other proportionally, then it is a trapezium.**

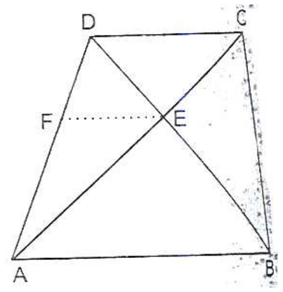
Given A quadrilateral ABCD whose diagonals AC and BD intersect at E such that $\frac{DE}{EB} = \frac{CE}{EA}$

To Prove Quadrilateral ABCD is a trapezium. For this it is sufficient to prove that $AB \parallel DC.$

Construction Draw $EF \parallel BA,$ meeting AD in F.

Proof: In ΔABC, we have

$EF \parallel BA,$



$\Rightarrow \frac{DF}{FA} = \frac{DE}{EB}$ [By Thale's Theorem] ...(i)

But, $\frac{DE}{EB} = \frac{CE}{EA}$ [Given] ... (ii)

From (i) and (ii), we get

$\frac{DF}{FA} = \frac{CE}{EA}$

Thus, in $\Delta DCA, E$ and F are points on CA and DA respectively such that

$\frac{DF}{FA} = \frac{CE}{EA}$

Therefore, by the converse of Basic Proportionality Theorem, we have

$FE \parallel DC,$

But, $FE \parallel BA,$ [By construction]

$\therefore DC \parallel BA \Rightarrow AB \parallel DC$

Hence, ABCD, is a trapezium.

Equiangular Triangles: Two triangles are said to be equiangular, if their corresponding angles are equal.

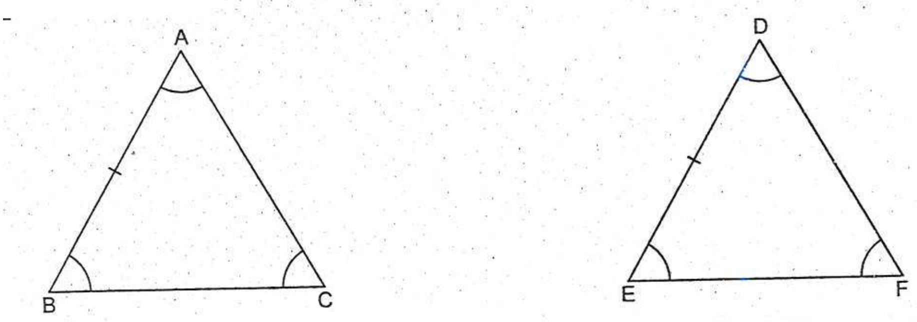
Theorem 1 (AAA Similarity Criterion) If two triangles are equiangular, then they are similar.

Given Two triangles ABC and DEF such that $\angle A = \angle D, \angle B = \angle E \text{ and } \angle C = \angle F$.

To Prove $\Delta ABC \sim \Delta DEF$

Proof: Recall that two triangles are similar iff their corresponding angles are equal and the corresponding sides are proportional. Since corresponding angles are given equal, we must prove that the corresponding sides are proportional i.e.,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$



Corollary (AA Similarity) If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

Note:

I) Two triangles are

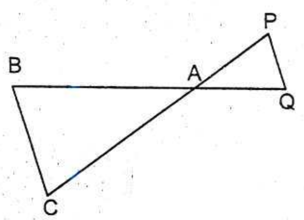
i) Similar if their corresponding angles are equal

ii) Two triangles are similar if their corresponding sides are proportional.

II. (SAS Similarity Criterion) If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

Example: 1. $\Delta ACB \sim \Delta APQ$. If BC=8cm, PQ=4cm, BA=6.5cm, AP=2.8cm, find CA and AQ.

Solution: We have,



$\Delta ACB \sim \Delta APQ$

$$\Rightarrow \frac{AC}{AP} = \frac{CB}{PQ} = \frac{AB}{AQ}$$

$$\Rightarrow \frac{AC}{AP} = \frac{CB}{PQ} \text{ and } \frac{CB}{PQ} = \frac{AB}{AQ}$$

$$\Rightarrow \frac{AC}{2.8} = \frac{8}{4} \text{ and } \frac{8}{4} = \frac{6.5}{AQ}$$

$$\Rightarrow \frac{AC}{2.8} = 2 \text{ and } \frac{6.5}{AQ} = 2 \Rightarrow AC = (2 \times 2.8)cm = 5.6cm \text{ and } AQ = \frac{6.5}{2}cm = 3.25cm$$

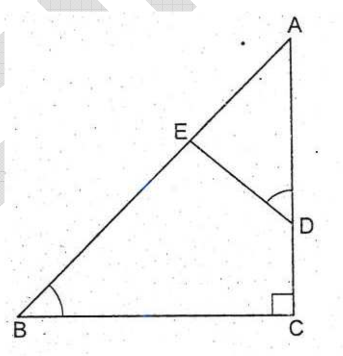
Example: 2 **If $\angle ADE = \angle B$ show that $\Delta ADE \sim \Delta ABC$. If AD=3.8 cm, AE=3.6cm, BE=2.1 cm and BC=4.2 cm, find DE.**

Solution: In triangles ADE and ABC, we have

$\angle ADE = \angle B$ (Given) and $\angle A = \angle A$ (Common)

So, by AA- criterion of similarity, we have

$\Delta ADE \sim \Delta ABC$



$$\Rightarrow \frac{AD}{AB} = \frac{DE}{BC}$$

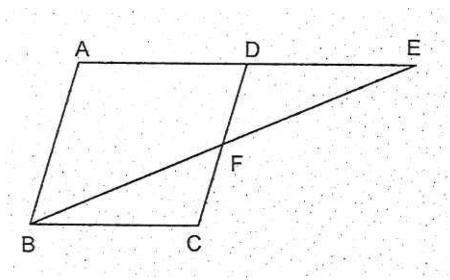
$$\Rightarrow \frac{AD}{AE + EB} = \frac{DE}{BC}$$

$$\Rightarrow \frac{3.8}{3.6 + 2.1} = \frac{DE}{4.2}$$

$$\Rightarrow DE = \frac{3.8 \times 4.2}{3.6 + 2.1}cm = 2.8cm$$

Hence, DE=2.8cm

Example 3 : **E is a point on side AD produced of a parallelogram ABCD and BE intersects CD at F. Prove that D ABE ~D CFB.**



Solution: In Δ's ABE and CFB, we have

$\angle AEB = \angle CBF$

$\angle A = \angle C$

Thus, by AA- criterion of similarity, we have

$\Delta ABE \sim \Delta CFB$.

Example 4 : **The perimeters of two similar triangles are 30 cm and 20 cm respectively. If one side of the first triangle is 12cm, determine the corresponding side of the second triangle.**

Solution: Let ΔABC and ΔDEF be two similar triangles of perimeters P_1 and P_2 respectively. Also, let AB=12cm, P_1= 30 cm and P_2 = 20cm. Then,

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}=\frac{p_1}{p_2}$$ [∵ Ratio of corresponding sides of similar triangles is equal to the ratio of their perimeters]

$$\Rightarrow \frac{AB}{DE}=\frac{P_1}{P_2}$$

$$\Rightarrow \frac{12}{DE}=\frac{30}{20}$$

$$\Rightarrow DE=\frac{12\times 20}{30}cm=8cm$$

Hence, the corresponding side of the second triangle is 8 cm.

Example 5 : **The perimeters of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. IF PQ = 10cm, find AB.**

Solution: Since the ratio of the corresponding sides of similar triangles is same as the ratio of their perimeters.

$$\therefore \quad \Delta ABC \sim \Delta PQR$$

$$\Rightarrow \frac{AB}{PQ}=\frac{BC}{QR}\frac{AC}{PR}=\frac{36}{24}$$

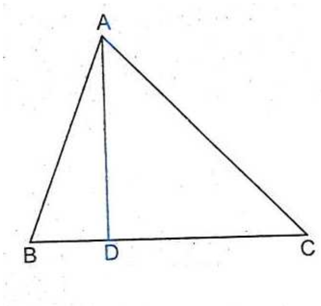
$$\Rightarrow \frac{AB}{PQ}=\frac{36}{24}$$

$$\Rightarrow \frac{AB}{10}=\frac{36}{24}$$

$$\Rightarrow AB=\frac{36\times 10}{24}cm=15cm$$

Example 6 : **If $\angle BAC=90^0$ and segment AD ⊥ BC. Prove that AD^2 = BD x DC.**

Solution: In ΔADB and ΔACD, We have



$\angle$ADB = $\angle$ADC [Each equal to 90^0]

and, $\angle$DBA= $\angle$DAC [Each equal to complement of $\angle$BAD i.e. 90^0 - $\angle$BAD]

Therefore, by AA- criterion of similarity, we have

$\Delta DBA \sim \Delta DAC$ [$\therefore \angle D \leftrightarrow \angle D, \angle B \leftrightarrow \angle DAC$

and $\angle BAD \leftrightarrow \angle DCA]$

$\Rightarrow \frac{DB}{DA} = \frac{DA}{DC}$ [In similar triangles corresponding sides are proportional]

$\Rightarrow \frac{BD}{AD} = \frac{AD}{DC}$

$\Rightarrow AD^2 = BD \times DC$

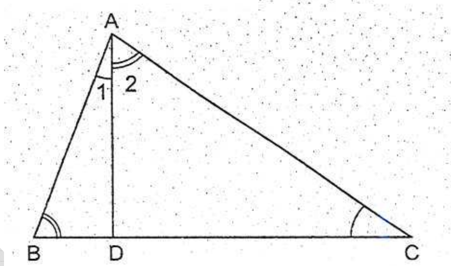
Example 7: In ΔABC, if AD $\perp$ BC and AD^2= BD x DC, prove that $\angle$ BAC=90^0

Solution: We have,

$AD^2 = BD \times DC$

$\Rightarrow AD \times AD = BD \times DC$

$\Rightarrow \frac{AD}{DC} = \frac{BD}{AD}$



Thus, in ΔABD and ΔACD, we have

$\frac{AD}{DC} = \frac{BD}{AD}$

and, $\angle BDA = \angle CDA$

So, by SAS – criterion of similarity, we get [Each equal to 90^0]

$\Delta DBA \sim \Delta DAC$

$\Rightarrow \Delta DBA$ and ΔDAC are equiangular

$\Rightarrow \angle 1 = \angle C$ and $\angle 2 = \angle B$

$\Rightarrow \angle 1 + \angle 2 = \angle B + \angle C$

$\Rightarrow \angle A = \angle B + \angle C$ $[\therefore \angle 1 + \angle 2 = \angle A]$

But, $\angle A + \angle B + \angle C = 180^0$

$\therefore \angle A + \angle A = 180^0$ $[\therefore \angle B + \angle C = \angle A]$

$\Rightarrow 2\angle A = 180^0 \Rightarrow \angle A = 90^0$

Hence, $\angle BAC = 90^0$

Example 8 : ABC is a triangle in which AB=AC and D is a point on AC such that BC^2= AC x CD. Prove that BD= BC.

Given: ΔABC in which AB=AC and D is a pint on the side AC such that

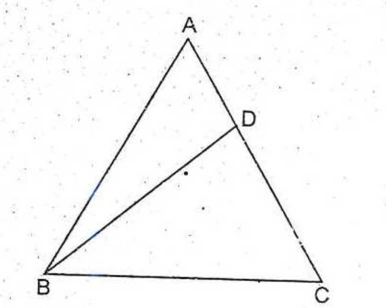
BC^2=AC x CD

To Prove BD = BC

Construction Join BD

Proof: We have,

BC^2=AC x CD



$\Rightarrow \dfrac{BC}{CD} = \dfrac{AC}{BC}$ ...(i)

Thus, in ΔABC and ΔBDC, we have

$\dfrac{AC}{BC} = \dfrac{BC}{CD}$ [Form (i)]

and, $\angle C, \angle C$ [Common]

$\therefore \Delta ABC \sim \Delta BDC$ [By SAS criterion of similarity]

$\Rightarrow \dfrac{AB}{BD} = \dfrac{BC}{DC}$

$\Rightarrow \dfrac{AC}{BD} = \dfrac{BC}{CD}$ [$\because AB = AC$]

$\Rightarrow \dfrac{AC}{BC} = \dfrac{BD}{CD}$ ..(ii)

From (i) and (ii), we get

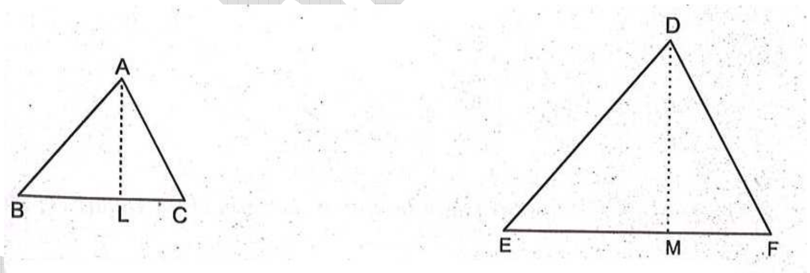
$\dfrac{BC}{CD} = \dfrac{BD}{CD} \Rightarrow BD = BC$

Areas of Two Similar Triangles

Theorem: 1 The ratio of the areas of two similar triangle are equal to the ratio of the squares of any two corresponding sides.

Given Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$.

To Prove: $\dfrac{Area\ (\Delta ABC)}{Area\ (\Delta\ DEF)} = \dfrac{AB^2}{DE^2} = \dfrac{BC^2}{EF^2} = \dfrac{AC^2}{DF^2}$



Construction Draw AL ⊥ BC and DM ⊥ EF.

Proof: Since similar triangles are equiangular and their corresponding sides are proportional. Therefore,

$\Delta ABC \sim \Delta DEF$.

$\Rightarrow \angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$ ....(i)

Thus, in ΔALB *and* ΔDME, we have

$\Rightarrow \quad \angle ALB = \angle DME$ [Each equal to 90^0]

and, $\quad \angle B = \angle E$ [From (i)]

So, by AA – criterion of similarity, we have

$\Delta ALB \sim \Delta DME$

$$\Rightarrow \frac{AL}{DM} = \frac{AB}{DE} \qquad \text{(ii)}$$

From (i) and (ii), we get

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AL}{DM}$$

Now,

$$\frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{\frac{1}{2}(BC \times AL)}{\frac{1}{2}(EF \times DM)} \qquad \text{(iii)}$$

$$\Rightarrow \frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{BC}{EF} \times \frac{AL}{DM}$$

$$\Rightarrow \frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{BC}{EF} \times \frac{BC}{EF} \qquad \left[From\,(iii), \frac{BC}{EF} = \frac{AL}{DM}\right]$$

$$\Rightarrow \frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{BC^2}{EF^2}$$

$$But, \quad \frac{BC}{EF} = \frac{AB}{DE} = \frac{AC}{DF}$$

$$\Rightarrow \frac{BC^2}{EF^2} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2}$$

$$Hence, \frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$

Theorem 2 **The areas of two similar triangles are in the ratio of the squares of the corresponding altitudes.**

Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$ *and* $AL \perp BC$, $DM \perp EF$.

$$\frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{AL^2}{DM^2}$$

Theorem 3 **The areas of two similar triangles are in the ratio of the squares of the corresponding medians.**

Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$ and AP, DQ are their medians.

$$\frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{AP^2}{DQ^2}$$

Theorem 4 **If the areas of two similar triangles are equal, then the triangles are congruent i.e. equal and similar triangles are congruent.**

Two triangles ABC and DEF such that $\Delta ABC \sim \Delta DEF$ and Area (ΔABC) = Area (ΔDEF).

$\Delta ABC \cong \Delta DEF$

Example 1 : **If $\Delta ABC \sim \Delta DEF$ such that area of ΔABC is 9 cm² and the are of ΔDEF is 16 cm² and BC = 2.1 cm. Find the length of EF.**

Solution: We have,

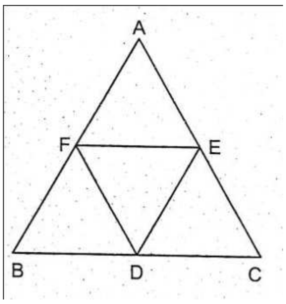
$$\frac{Area\,(\Delta ABC)}{Area\,(\Delta DEF)} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \frac{9}{16} = \frac{(2.1)^2}{EF^2} \Rightarrow \frac{3}{4} = \frac{2.1}{EF} \Rightarrow EF = \frac{4 \times 2.1}{3} cm = 2.8cm$$

Example 2 : **D, E, F are the mid – points of the sides BC, CA and AB respectively of a ΔABC. Determine the ratio of the areas of ΔDEF and ΔABC.**

Solution: Since D and E are the mid- points of the sides BC and AB respectively of ΔABC.

$\therefore \quad DE \| BA \Rightarrow DE \| FA$



Since D and F are mid – points of the sides BC and AB respectively of ΔABC. Therefore,

$DF \| CA \Rightarrow DF \| AE$ ...(ii)

From (i), and (ii), we conclude that AFDE is a parallelogram.

Similarly, BDEF is a parallelogram.

In ΔDEF and ΔABC, we have

$\angle FDE = \angle A$ [Opposite angles of parallelogram AFDE] and, $\angle DEF = \angle B$ [Opposite angles of parallelogram BDEF]

So, by AA- similarity criterion, we have

$\Delta DEF \sim \Delta ABC$

$$\Rightarrow \frac{Area\,(\Delta DEF)}{Area\,(\Delta ABC)} = \frac{DE^2}{AB^2} = \frac{(1/2AB)^2}{AB^2} = \frac{1}{4} \qquad \left[\because\ DE = \frac{1}{2}AB\right]$$

Hence, Area (ΔDEF) : Area (ΔABC)= 1:4

4.10 Pythagoras Theorem

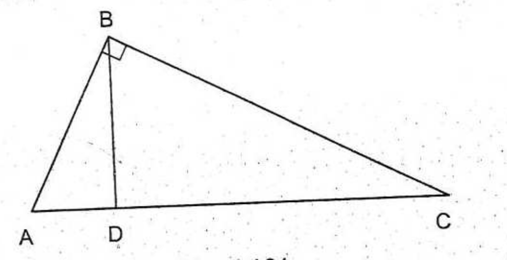
In this section, we shall prove an important theorem known as Pythagoras Theorem. This Theorem is also known as Baudhayan Theorem.

Theorem 1 In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given A right – angled triangle ABC in which $\angle B = 90^0$

To Prove (Hypotenuse)^2 = (Base)^2 + (Perpendicular)^2 i.e. $AC^2 = AB^2 + BC^2$.

Construction From B draw BD $\perp$ AC.



Proof In triangles ADB and ABC, we have

$\angle ADB = \angle ABC$ [Each equal to 90^0]

and, $\angle A = \angle A$ [Common]

So, by AA- similarity criterion, we have

$\Delta ADB \sim \Delta ABC$

$$\Rightarrow \frac{AD}{AB} = \frac{AB}{AC} \qquad [\because \text{In similar triangles corresponding sides are proportional}]$$

$$\Rightarrow AB^2 = AD \times AC \qquad ..(i)$$

In triangles BDC and ABC, we have

$\angle CDB = \angle ABC$ [Each equal to 90^0]

and, $\angle C = \angle C$ [Common]

So, by AA – Similarity criterion, we have

$\Delta BDC \sim \Delta ABC$

$\Rightarrow \frac{DC}{BC} = \frac{BC}{AC}$ [∵ In similar triangles corresponding sides are proportional]

$\Rightarrow BC^2 = AC \times DC$ (ii)

Adding equations (i) and (ii), we get

$$AB^2 + BC^2 = AD \times AC + AC \times DC$$

$\Rightarrow AB^2 + BC^2 = AC(AD + DC)$

$\Rightarrow AB^2 + BC^2 = AC \times AC$

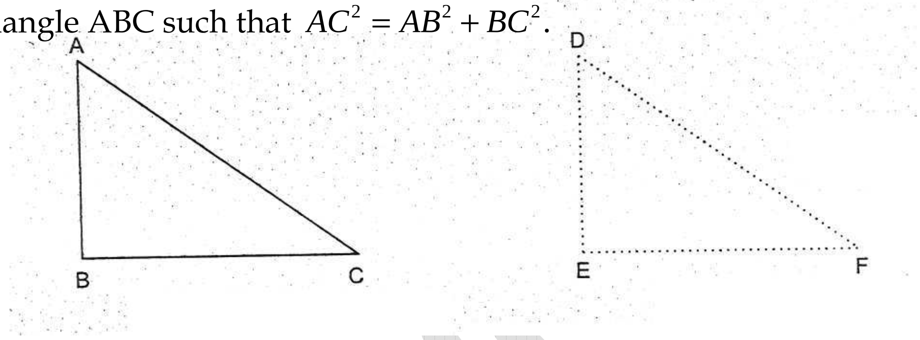
$\Rightarrow AB^2 + BC^2 = AC^2$

Hence, $AC^2 = AB^2 + BC^2$

Theorem 2 (Converse of Pythagoras Theorem) In a triangle, If the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the side is a right angle.

Given

A triangle ABC such that $AC^2 = AB^2 + BC^2$.



Example 1: The hypotenuse of a right triangle is 6m more than the twice of the shortest side. If the third side is 2m less than the hypotenuse, find the sides of the triangle.

Solution: Let the shortest side be x metres in length. Then, Hypotenuse = $(2x+6)$ m and , Third side = $(2x+4)$ m

By Pythagoras theorem, we have

$$(2x+6)^2 = x^2 + (2x+4)^2$$

$\Rightarrow 4x^2 + 24x + 36 = x^2 + 4x^2 + 16x + 16$

$\Rightarrow x^2 + 8x - 20 = 0$

$\Rightarrow (x-10)(x+2) = 0$

$\Rightarrow x = 10 \text{ or}, x = -2$

$\Rightarrow x = 10$ [∵ *x cannot be negative*]

Hence, the sides of the triangle are 10, 26m and 24m.

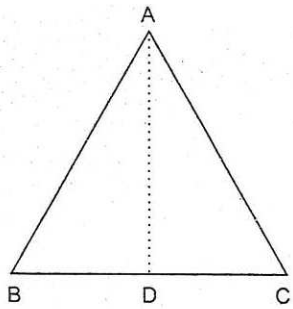
Example 2 : In an equilateral triangle with side a, prove that

(i) Altitude = $\frac{a\sqrt{3}}{2}$ (ii) Area = $\frac{\sqrt{3}}{4}a^2$

Solution: Let ABC be an equilateral triangle the length of whose each side is a units. Draw AD $\perp$ BC. Then, D is the mid – point of BC.

$$\Rightarrow AB = a, BD = \frac{1}{2}BC = \frac{a}{2}$$

Since ΔABD is a right triangle right – angled at D.



$$\therefore AB^2 = AD^2 + BD^2$$

$$\Rightarrow a^2 = AD^2 + \left(\frac{a}{2}\right)^2$$

$$\Rightarrow AD^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\Rightarrow AD = \frac{\sqrt{3a}}{2}$$

$$\therefore Altitude\ \frac{\sqrt{3}}{2}a$$

Now,

Area of $\Delta ABC = (1/2)$ (Base x Height)

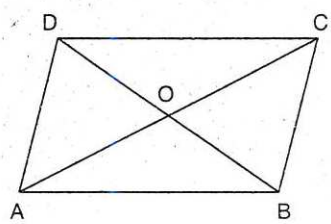
$$\Rightarrow Area\ of\ \Delta ABC = \frac{1}{2}(BC \times AD) = \frac{1}{2} \times a \times \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2$$

Example 3 : Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

Solution: We know that if AD is a median of ΔABC, then

$$AB^2 + AC^2 = 2AD^2 + \frac{1}{2}BC^2$$ [See Example 24(iii)]

Since diagonals of a parallelogram bisect each other. Therefore, BO and DO are medians of triangles ABC and ADC respectively.



$$\therefore AB^2 + BC^2 = 2\,BO^2 + \frac{1}{2}AC^2 \quad ...(i)$$

$$and,\quad AD^2 + CD^2 = 2DO^2 + \frac{1}{2}AC^2 \quad ...(ii)$$

Adding (i) and (ii), we have

$$AB^2 + BC^2 + CD^2 + AD^2 = 2(BO^2 + DO^2) + AC^2$$

$$\Rightarrow AB^2 + BC^2 + CD^2 + AD^2 = 2\left(\frac{1}{4}BD^2 + \frac{1}{4}BD^2\right) + AC^2 \qquad \left[\because DO = \frac{1}{2}BD\right]$$

$$\Rightarrow AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$$