Polynomials

Key Concepts

1. An algebraic expression in which the variable involved has only non- negative integral powers is called a **Polynomial**.

2. Algebraic expressions such as $3xy, x^2+2x, x^3-x^2+4x+3, \pi r^2, ax+b$ etc. are called **Polynomials**.

Note: $x^{\frac{1}{2}}+3$ is not a polynomial because the first term $x^{\frac{1}{2}}$ is a term with an exponent that is not a non- negative integer. (i.e. $\frac{1}{2}$).

$2x^2-\frac{3}{x}+5$ is not polynomial because it can be written as $2x^2-3x^{-1}+5$. Here the second term $(3x^{-1})$ has a negative exponent (i.e., -1).

3. A variable is denoted by a symbol that can take any real value. We use the letter x,y,z etc. To denote variables. We have algebraic expressions such as $2x, 3x, -x, \frac{3}{4}x....$ all in one variable x.

4. Each term of the polynomial consists of the products of a constant, called the **coefficient** of the term and a finite number of variables raised to non- negative integral powers.

5. In the polynomial $3x^2+7x+5,$ each of the expressions $3x^2, 7x$ and 5 are terms. Each term of the polynomial has a coefficient, so in $3x^2+7x+5,$ the coefficient of x^2 is 3, the coefficient of x is 7 and 5 is the coefficient of x^0 (Remember $x^0=1$) .

6. The degree of a polynomial is the highest degree of its variable term.

7. A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

Where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants and $a_n \neq 0$.

In particulars, if $a_0 = a_1 = a_2 = a_3 = \ldots. = a_n = 0$ (i.e. all the coefficients are zero), we get the **zero polynomial,** which is denoted by '0'.

8. A polynomial may be a multinomial but every multinomial need not be a polynomial.

A linear polynomial with one variable may be a monomial or a binomial.

E.g. : 3x or 2x – 5.

9. Zero polynomial is a constant polynomial having many zeros.

10. **Types of Polynomials:** (Classification on the basis of number of terms)

i) **Monomial:** Polynomial containing one term is called monomial.

Example: $6x^2, \frac{-3}{2}xy, 5y, 8$

ii) **Binomial:** Polynomial containing two terms is called binomial.

Example: $4-3x, 5x^2+4z$

iii) **Trinomial:** Polynomial containing three terms is called trinomial.

Example: $x^2-3x+13,$

Order (**or**) Degree of monomial: Sum of the powers of variables of a monomial is known as degree of a monomial.

Example: $5x^2y^3$

Degree of monomial is 2+3=5.

11. **Polynomial of Various Degrees:**

(i) Linear polynomial: If the degree of a polynomial is one, then that polynomial is said to be linear polynomial.

Example: 2x+5

Note: The general form of a linear polynomial in one variable 'x' is ax+b, where a, b $\in R$ and $a \neq 0$.

(ii) **Quadratic Polynomial:** If the degree of a polynomial is two, then that polynomial is said to be quadratic polynomial.

Example: x^2+2x+6

Note: The general form of quadratic polynomial is ax^2+bx+c, *where* $a, b, c \in R$ *and* $a \neq 0$

iii) **Cubic Polynomial:** If the degree of polynomial is three, then that polynomial is said to be cubic polynomial.

Example: $3x^3 - 4x^2 + 3x + 6$

Note: The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where a, b, c, d $\in R \text{ and } a \neq 0$.

Order (or) Degree of a Polynomial: The greatest degree of the terms in the polynomial is the degree of a polynomial.

Example: Degree of $6x^2 - 3x + 4$ *is* 1.

Note: The degree of a constant polynomial is zero.

12. **Zero Polynomial :** If all the coefficients of the polynomial are zeroes, then the polynomial is called zero polynomial

Note: Degree of zero polynomial is undefined.

13. **Zero of a Polynomial**

The number for which the value of a polynomial is zero that value of a variable is called zero of the polynomial.

Example: Zero of the polynomial $5x + 6$ *is* $\frac{-6}{5}$.

Remainder Theorem: If f(x) is a polynomial in 'x' of degree≥1, 'a' is any real number if f(x) is divided by (x-a), then the remainder is f (a).

Division Algorithm: If f(x), g(x) are two non – zero polynomial, then there exist polynomials q(x) and r(x) uniquely such that f(x) = q(x) g(x) + r(x), where r(x) = 0 or deg r (x) < deg g (x).

14. The general form of a "first degree polynomial" in one variable x is ax +b where a and b are real numbers and $a \neq 0$.

15. A "Quadratic polynomial" in x with real coefficients is of the form $ax^2 + bx + c$ where a, b, c are real numbers with $a \neq 0$.

16. A real number k is said to be *a* zero of *a* polynomial p(x), if p (k) = 0.

17. The zero of the linear polynomial ax + b is $\frac{-b}{a}$.

18. A linear polynomial ax+b, $(a \neq 0)$, the graph of y = ax + b is a "straight line" which intersects the X axis at exactly one point namely $\left(\frac{-b}{a}, 0\right)$.

19. i) A quadratic polynomial can have at most 2 zeroes.

20. ii) A cubic polynomial can have at most 3 zeroes.

21. If α *and* β are the zeroes of the quadratic polynomial $ax^2 + bx + c$ then

i) $\alpha + \beta = \frac{-b}{a} = \frac{-(coefficient\ of\ x)}{coefficient\ of\ x^2}$

ii) $\alpha\beta = \frac{c}{a} = \frac{constant\ term}{coefficient\ of\ x^2}$

22. For any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$. the graph of the corresponding equation $y = ax^2 + bx + c$ either opens upwards like this $\cup$ or open downwards like this $\cap$ depends on whether $a > 0$ *or* $a < 0$. These curves are called parabolas.

23. A polynomial p (x) of degree n has almost 'n' zeroes.

24. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d = 0$ then

i) $\alpha + \beta + \gamma = \frac{-b}{a}$

ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

iii) $\alpha\beta\gamma = \frac{-d}{a}$

25. **Division Algorithm:**

Dividend = Divisor x Quotient + Remainder.

Example -1 p(x) = x+2. Find p(1), p(2), p(-1) and p(-2). What are zeroes of the polynomial x + 2?

Solution: Let p(x) = x+2

Replace x by 1

p(1) = 1 +2 =3

Replace x by 2

p(2) = 2 +2 =4

Replace x by -1

p(-1) = -1 + 2 =1

Replace x by -2

p(-2) = -2 + 2 =0

Therefore, 1, 2, -1 are not the zeroes of the polynomial x+2, but -2 is the zero of the polynomial.

Example -2. Find a zero of the polynomial p(x) = 3x+1

Solution: Finding a zero of p(x), is same as solving the equation

p (x) =0

i.e. 3x+1 =0

3x= - 1

$$x = -\frac{1}{3}$$

So, $-\frac{1}{3}$ is a zero of the polynomial 3x+1.

Example -3. If p(x) = ax +b, $a \neq 0$. a linear polynomial, how will you find a zero of p(x)?

Solution: As we have seen to find zero of a polynomial p(x), we need to solve the polynomial equation p(x)=0

Which means ax+b=0, $a \neq 0$.

So ax $= -b$

i.e., $x = \frac{-b}{a}$

So, $x = \frac{-b}{a}$ is the only zero of the polynomial p(x) = ax+b i.e., **A linear polynomial in one variable has only one zero.**

Example - 4. Verify whether 2 and 1 are zeroes of the polynomial $x^2 - 3x + 2$

Solution: Let p(x) = $x^2 - 3x + 2$

p(2) =$(2)^2$-3(2)+2

= 4-6+2=0

p(1) = $(1)^2$-3(1)+2

= 1-3+2

= 0

Hence, both 2 and 1 are zeroes of the polynomial $x^2 - 3x + 2$.

Example – 5. What is the degree of the polynomial $x^2 - 3x + 2$? Is it a linear polynomial?

Solution: No, It is a quadratic polynomial. Hence, a quadratic polynomial has to zeroes.

Example-6. If 3 is a zero of the polynomial $x^2 + 2x - a$. Find a?

Solution: Let p(x) = $x^2 + 2x - a$

As the zero of this polynomial is 3, we know that p(3) =0.

$x^2 + 2x - a = 0$

Put x = 3, $(3)^2 + 2(3) - a = 0$

$9 + 6 - a = 0$

$15 - a = 0$

$-a = -15$

Or $a = 15$

Example -7. Divide the polynomial $2x^4 - 4x^3 - 3x - 1$ by (x-1) and verify the remainder with zero of the divisor.

Solution: Let $f(x) = 2x^4 - 4x^3 - 3x - 1$

First see how many times $2x^4$ is of x.

$\frac{2x^4}{x} = 2x^3$

Now multiply $(x-1)(2x^3) = 2x^4 - 2x^3$

Then again see the first term of the remainder that

$$\begin{array}{r} 2x^3 - 2x^2 - 2x - 5 \\ x-1 \overline{) 2x^4 - 4x^3 - 3x - 1} \\ -2x^4 - 2x^3 \\ - \quad + \\ \hline -2x^3 - 3x - 1 \\ -2x^3 + 2x^2 \\ + \quad - \\ \hline -2x^2 - 3x - 1 \\ -2x^2 + 2x \\ + \quad - \\ \hline -5x - 1 \\ -5x + 5 \\ + \quad - \\ \hline -6 \\ \hline \end{array}$$

is - $2x^3$. Now do the same.

Here the quotient is $2x^3-2x^2-2x-5$ and the remainder is -6.

Now, the zero of the polynomial (x-1) is 1.

Put $x=1$ in $f(x), f(x)=2x^4-4x^3-3x-1$

$$f(1)=2(1)^4-4(1)^3-3(1)-1$$
$$=2(1)-4(1)-3(1)-1$$
$$=2-4-3-1$$
$$=-6$$

Example -8. Find the remainder when x^3+1 divided by (x+1)

Solution: Here $p(x)=x^3+1$

The zero of the linear polynomial x+1 is -1 [x+1 =0, x = -1]

$p(-1) = (-1)^3+1$

$=-1+1$

$=0.$

So, by Remainder Theorem, we know that (x^3+1) divided by (x+1) gives 0 as the remainder.

Example -9. Check whether (x-2) is a factor of x^3-2x^2-5x+4

Solution: Let p(x)= x^3-2x^2-5x+4

To check whether the linear polynomial (x-2) is a factor of the given polynomial,

Replace x, by the zero of (x-2) i.e. x-2=0 x=2.

$p(2)=(2)^3-2(2)^2-5(2)+4$

$=8-2(4)-10+4$

$=8-8-10+4$

$= - 6.$

As the remainder is not equal to zero, the polynomial (x-2) is not a factor of the given polynomial x^3-2x^2-5x+4.

Example -10. Check whether the polynomial $p(y) = 4y^3 + 4y^2 - y - 1$ is a multiple of $(2y+1)$.

Solution: $p\left(\frac{-1}{2}\right) = 4\left(\frac{-1}{2}\right)^3 + 4\left(\frac{-1}{2}\right)^2 - \left(\frac{-1}{2}\right) - 1$

$$= 4\left(\frac{-1}{8}\right) + 4\left(\frac{1}{4}\right) + \frac{1}{2} - 1$$

$$= \frac{-1}{2} + 1 + \frac{1}{2} - 1 \qquad = 0$$

So, (2y+1) is a factor of p(y). That is p(y) is a multiple of (2y+1).

Example – 11. If the polynomials $ax^3 + 3x^2 - 13$ *and* $2x^3 - 5x + a$ are divided by (x-2) leave the same remainder, find the value of a.

Solution: Let p(x)= $ax^3 + 3x^2 - 13$ *and* $q(x) = 2x^3 - 5x + a$

$\because p(x)$ *and* $q(x)$ When divided by x-2 leave same remainder.

$\therefore p(2) = q(2)$

$$a(2)^3 + 3(2)^2 - 13 = 2(2)^3 - 5(2) + a$$
$$8a + 12 - 13 = 16 - 10 + a$$
$$8a - 1 = a + 6$$
$$8a - a = 6 + 1$$
$$7a = 7$$
$$a = 1$$

Some More Important Problems

1. Write a quadratic polynomial and a cubic polynomial in variable x in the general form.

Sol. General form of a (in variable x)

i) Quadratic polynomial: $ax^2 + bx + c$

ii) Cubic polynomial: $ax^3 + bx^2 + cx + d$.

2. i) p(x) = $x^2 - 5x - 6$ find the value of p(1), p(2), p(3), p(0), p(-1), p(-2), p(-3).

Sol. P(x) = $x^2 - 5x - 6$

$p(1) = (1)^2 - 5(1) - 6 = 1 - 5 - 6$

$= 1 - 11 = -10$

$p(2) = (2)^2 - 5(2) - 6 = 4 - 10 - 6$

$= 4 - 16 = -12$

$p(3) = (3)^2 - 5(3) - 6 = 9 - 15 - 6$

$= 9 - 21 = -12$

$p(0) = (0)^2 - 5(0) - 6 = 0 - 0 - 6 = -6$

$p(-1) = (-1)^2 - 5(-1) - 6 = 1 + 5 - 6$

$= 6 - 6 = 0$

$p(-2) = (-2)^2 - 5(-2) - 6 = 4 + 10 - 6$

$= 14 - 6 = 8$

$p(-3) = (-3)^2 - 5(-3) - 6 = 9 + 15 - 6$

$= 24 - 6 = 18$

3. p(m)=m^2 - 3m +1, find the value of p(1) and p(-1).

Sol. p(m) = m^2 - 3m+1

P(1) = $(1)^2$ -3 (1) + 1 = 1 – 3 +1

= 2 -3 = -1

P(-1) = $(-1)^2$ – 3 (-1) + 1 = 1 +3 +1 =5

4. Let p(x) = x^2 - 4x+3. Find the value of p(0), p(1), p(2), p(3) and obtain zeroes of the polynomial p(x).

Sol. $p(x) = x^2 – 4x + 3$

$p(0) = (0)^2 -4 (0) + 3 = 0 – 0 + 3 =3$

$p(1) = (1)^2 -4 (1) + 3= 1 - 4 +3 = 4 – 4 =0$

$p(2) = (2)^2 – 4 (2) + 3 = 4 – 8+3=7 – 8 = -1$

$p(3) = (3)^2 – 4 (3) + 3 =9 – 12 + 3 = 12 – 12 =0$

We see that p(1) = 0 and p(3) =0

These points x = 1 and x =3 are called zeroes of the polynomial $p(x) = x^2 – 4x + 3$.

5. Check whether -3 and 3 are the zeroes of the polynomial x^2 -9.

Sol. $p(x) = x^2 - 9 \Rightarrow p(-3) = (-3)^2 - 9 = 9 - 9 = 0$

$p(3) = (3)^2 - 9 = 9 - 9 = 0$

$p(-3) = 0 \text{ and } p(3) = 0$

-3 and 3 are the zeroes of the polynomial $p(x)=x^2 -9$.

6. If p(t) = t^3 -1, find the values of p(1), p(-1), p(0), p(2), p(-2).

Sol : $p(t) = t^3 -1$,

$p(1) = (1)^3 -1, = 1 – 1 = 0$

$p(-1) = (-1)^3 -1, = -1 – 1 = -2$

$p(0) = (0)^3 -1, = 0 – 1 = -1$

$p(2) = (2)^3 -1, = 8 – 1 = 7$

$p(-2) = (-2)^3 -1, = - 8 – 1 = -9$

7. Check whether -2 and 2 are the zeroes of the polynomial x^4 - 16.

Sol : $p(x) = x^4 - 16$.

$p(-2) = (-2)^4 -16 = 16 – 16 = 0$

$p(2) = (2)^4 -16 = 16 – 16 = 0$

yes, - 2 and 2 are zeroes of the polynomial x^2 - 16.

8. Check whether 3 and -2 are the zeroes of the polynomial p(x) when

$p(x) = x^2 - x - 6$.

Sol : $p(x) = x^2 - x - 6$.

$p(3) = (3)^2 - 3 - 6 = 9 - 3 - 6 = 9 - 9 = 0$

$p(-2) = (-2)^2 - (-2) - 6 = 4 + 2 - 6 = 6 - 6 = 0$

yes, 3 and – 2 are zeroes of the polynomial $p(x) = x^2 - x - 6$

9. Why are $\frac{1}{4}$ and -1 zeroes of the polynomials $p(x) = 4x^2 + 3x - 1$?

Sol : $p(x) = 4x^2 + 3x - 1$

$$p\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right) - 1$$

$$= 4 \times \frac{1}{16} + \frac{3}{4} - 1$$

$$= \frac{1}{4} + \frac{3}{4} - 1 = \frac{1+3-4}{4}$$

$$= \frac{4-4}{4} = \frac{0}{4} = 0$$

$$p\left(\frac{1}{4}\right) = 0$$

$$p(-1) = 4(-1)^2 + 3(-1) - 1$$

$$= 4 \times 1 - 3 - 1$$

$$= 4 - 3 - 1 = 4 - 4 = 0$$

$$p(-1) = 0$$

$$p\left(\frac{1}{4}\right) = 0 \ and \ p(-1) = 0$$

$\frac{1}{4}$ and -1 are zeroes of the polynomial $p(x)=4x^2 + 3x - 1$

10 Find the zeros of the quadratic polynomial $x^2+7x+12,$ and verify the relation between the zeros and its coefficients.

Solution:

The zeros of f(x) are given by f (x) = 0.

Now, f(x) =0

$$\Rightarrow \; x^2+7x+12=0$$
$$\Rightarrow \; (x+4)(x+3)=0$$
$$\Rightarrow \; x+4=0 \; or, \; x+3=0$$
$$\Rightarrow \; x=-4 \; or, \; x=-3.$$

Thus, the zeros of $f(x)=x^2+7x+12$ $are \; \alpha=-4 \; and \; \beta=-3.$

Now,

Sum of the zeros = $\alpha+\beta=(-4)+(-3)=-7 \; and, -\frac{Coefficient \; of \; x}{Coefficient \; of \; x^2}=-\frac{7}{1}=-7=-\frac{b}{a}$

Product of the zeros $=\alpha\beta=(-4)\times(-3)=12 \; and, \; \frac{Constant \; term}{Coefficient \; of \; x^2}=\frac{12}{1}=12=\frac{c}{a}$

11. Find the zeros of the quadratic polynomial $f(x)=6x^2-3,$ and verify the relationship between the zeros and its coefficients:

Solution:

The zeros of f(x) are given by f(x)=0.

Now, f(x)=0

$$6x^2-3=0 \Rightarrow (\sqrt{6}x)^2-(\sqrt{3})^2$$
$$\Rightarrow (\sqrt{6}x-\sqrt{3})(\sqrt{6}x+\sqrt{3})=0$$
$$\Rightarrow \sqrt{6}x-\sqrt{3}=0 \; or, \; \sqrt{6}x+\sqrt{3}=0$$

$$\Rightarrow x=\frac{\sqrt{3}}{\sqrt{6}} \; or, \; x=\frac{-\sqrt{3}}{\sqrt{6}}$$
$$x=\frac{1}{\sqrt{2}} \; or, \; x=-\frac{1}{\sqrt{2}}$$

Hence, the zeros of $f(x)=6x^2-3 \; are: \alpha=\frac{1}{\sqrt{2}} \; and \; \beta=-\frac{1}{\sqrt{2}}$

Now,

Sum of the zeros = $\alpha+\beta=\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)=0 \; and, \; \frac{Coefficient \; of \; x}{Coefficient \; of \; x^2}=-\frac{0}{6}=0$

Also,

$$\text{Product of the zeros} = \alpha\beta = \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = \frac{-1}{2} \text{ and, } \frac{Constant\ term}{Coefficient\ of\ x^2} = \frac{-3}{6} = \frac{-1}{2}$$

12. Find the zeros of the polynomial $f(u) = 4u^2 + 8u$, and verify the relationship between the zeros and its coefficients.

Solution

$$f(u) = 4u^2 + 8u,$$

$$\Rightarrow \quad f(u) = 4u\,(u+2)$$

The zeros of f(u) are given by f(u) =0.

Now, f(u) =0

$$\Rightarrow 4u(u+2) = 0$$
$$\Rightarrow u = 0 \text{ or, } u+2 = 0$$
$$\Rightarrow u = 0 \text{ or, } u = -2$$

Hence, the zeros of f(u) are : $\alpha = 0$ *and* $\beta = -2$

Now,

$$\alpha + \beta = 0 + (-2) = -2 \text{ and } \alpha\beta = 0 \times -2 = 0$$

$\therefore$ Sum of the zeros = $-\dfrac{Coefficient\ of\ u}{Coefficient\ of\ u^2}$ and, Product of the zeros $\dfrac{Constant\ term}{Coefficient\ of\ u^2}$

13. Find the zeros of the polynomial $f(x) = 4\sqrt{3}x^2 + 5x - 2\sqrt{3}$, and verify the relationship between the zeros and its coefficients.

Solution: We have,

$$f(x) = 4\sqrt{3}x^2 + 5x - 2\sqrt{3},$$

$$\Rightarrow \quad f(x) = 4\sqrt{3}x^2 + 8x - 3x - 2\sqrt{3}$$
$$\Rightarrow \quad f(x) = 4x(\sqrt{3}x+2) - \sqrt{3}(\sqrt{3}x+2)$$
$$\Rightarrow \quad f(x) = (\sqrt{3}x+2)(4x-\sqrt{3})$$

$4\sqrt{3} \times -2\sqrt{3} = -24$
$+8 \times -3 = -24$
$+8 + (-3) = +5$

The zeros of f(x) are given by f(x) = 0.

Now, f(x)=0

$$\Rightarrow (\sqrt{3}x+2)(4x-\sqrt{3}) = 0$$
$$\Rightarrow \sqrt{3}x + 2 = 0 \text{ or, } 4x - \sqrt{3} = 0$$
$$\Rightarrow x = -\frac{2}{\sqrt{3}} \text{ or, } x = \frac{\sqrt{3}}{4}$$

Hence, the zeros of f(x) are: $\alpha = -\frac{2}{\sqrt{3}}$ *and* $\beta = \frac{\sqrt{3}}{4}$

Now,

$$\alpha + \beta = -\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{4} = \frac{-8+3}{4\sqrt{3}} = -\frac{5}{4\sqrt{3}} \text{ and, } \alpha\beta = \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{4} = -\frac{1}{2}$$

Also,

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{5}{4\sqrt{3}} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{-2\sqrt{3}}{4\sqrt{3}} = -\frac{1}{2}$$

Hence, sum of the roots $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ and, Product of the roots $\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

14. Find the zeros of the quadratic polynomial $f(x) = abx^2 + (b^2 - ac)x - bc$, and verify the relationship between the zeros and its coefficients.

Solution:

$$f(x) = abx^2 + (b^2 - ac)x - bc,$$

$$\Rightarrow f(x) = abx^2 + b^2x - acx - bc,$$

$$\Rightarrow f(x) = bx(ax+b) - c(ax+b)$$

$$\Rightarrow f(x) = (ax+b)(bx-c)$$

The zeros of f(x) are given by f(x) = 0.

Now, f(x)=0

$$\Rightarrow (ax+b)(bx-c) = 0$$

$$\Rightarrow ax+b = 0 \text{ or, } bx - c = 0$$

$$\Rightarrow x = -\frac{b}{c} \text{ or, } x = \frac{c}{b}$$

Thus, the zeros of f(x) are: $\alpha = -\frac{b}{a}$ and, $\beta = \frac{c}{b}$

Now,

$$\alpha + \beta = -\frac{b}{a} + \frac{c}{b} = \frac{ac - b^2}{ab} \text{ and, } \alpha\beta = -\frac{b}{a} \times \frac{c}{b} = -\frac{c}{a}$$

Also,

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\left(\frac{b^2 - ac}{ab}\right) = \frac{ac - b^2}{ab} \text{ and, } \frac{\text{Constant term}}{\text{Coefficient of } x^2} = -\frac{bc}{ab} = -\frac{c}{a}$$

Hence,

Sum of the zeros $= -\frac{Coefficient\ of\ x}{Coefficient\ of\ x^2}$ and, $Product\ of\ the\ zeros\ \frac{Constant\ term}{Coefficient\ of\ x^2}$

15. Find the zeros of the polynomial $x^2 + \frac{1}{6}x - 2,$ and verify the relation between the coefficients and zeros of the polynomial.

Solution: Let $f(x) = x^2 + \frac{1}{6}x - 2.$ Then,

$$f(x) = \frac{1}{6}(6x^2 + x - 12) = \frac{1}{6}(6x^2 + 9x - 8x - 12)$$

$$\Rightarrow f(x) = \frac{1}{6}\{(6x^2 + 9x) - (8x + 12)\} = \frac{1}{6}\{3x(2x+3) - 4(2x+3)\} = \frac{1}{6}(2x+3)(3x-4)$$

The zeros of f(x) are given by f(x) = 0.

Now, $f(x) = 0 \Rightarrow \frac{1}{6}(2x+3)(3x-4) = 0 \Rightarrow 2x + 3 = 0\ or,\ 3x - 4 = 0 \Rightarrow x = \frac{-3}{2}\ or,\ x = \frac{4}{3}$

Hence, $\alpha = \frac{-3}{2}\ and\ \beta = \frac{4}{3}$ are the zeros of the given polynomial.

Now,

$$\alpha + \beta = \left(-\frac{3}{2}\right) + \frac{4}{3} = -\frac{1}{6}\ and,\ \alpha\beta = \left(\frac{-3}{2}\right)\left(\frac{4}{3}\right) = -2$$

The given polynomial is $f(x) = x^2 + \frac{1}{6}x - 2.$

$$-\frac{Coefficient\ of\ x}{Coefficient\ of\ x^2} = -\left(\frac{-1/6}{1}\right) = \frac{-1}{6}\ and,\ \frac{Constant\ term}{Coefficient\ of\ x^2} = \frac{-2}{1} = -2$$

Clearly,

$$\alpha + \beta = -\frac{Coefficient\ of\ x}{Coefficient\ of\ x^2}\ and,\ \alpha\beta = \frac{Constant\ term}{Coefficient\ of\ x^2}$$

Hence, the relation between the coefficients and zeros is verified.

16. If $\alpha\ and\ \beta$ are the zeros of the quadratic polynomial $f(x) = x^2 - px + q,$ then find the values of (i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution: It is given that $\alpha\ and\ \beta$ are the zero of the polynomial $f(x) = x^2 + -px + q,$

$\therefore\quad \alpha+\beta=-\left(\frac{-p}{1}\right)=p \text{ and}, \alpha\beta=\frac{q}{1}=q$

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q$ $[\because \alpha+\beta=p \text{ and } \alpha\beta=q]$

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{p}{q}$

17. Find a quadratic polynomial, the sum and product of whose zeroes are $\sqrt{2}$ and $-\frac{3}{2}$ respectively. Also, find its zeroes.

Solution: Let α, β be the zeros of required polynomial. It is given that $\alpha+\beta=\sqrt{2}$ and $\alpha\beta=-\frac{3}{2}$.

The quadratic polynomial is $f(x)=x^2-(\alpha+\beta)x+\alpha\beta$ or, $f(x)=x^2-\sqrt{2}x-\frac{3}{2}$

Now, $f(x)=x^2-\sqrt{2}x-\frac{3}{2}$

$\Rightarrow f(x)=\frac{1}{2}(2x^2-2\sqrt{2}x-3)$

$\Rightarrow f(x)=\frac{1}{2}(2x^2-3\sqrt{2}x+\sqrt{2}x-3)$

$\Rightarrow f(x)=\frac{1}{2}\{\sqrt{2}x(\sqrt{2}x-3)+(\sqrt{2}x-3)\}$

$\Rightarrow f(x)=\frac{1}{2}(\sqrt{2}x-3)(\sqrt{2}x+1)$

The zeroes of f(x) are given by f(x) = 0.

Now, f(x)=0

$\Rightarrow \frac{1}{2}(\sqrt{2}x-3)(\sqrt{2}x+1)=0 \Rightarrow \sqrt{2}x-3=0 \text{ or}, \sqrt{2}x+1=0 \Rightarrow x=\frac{3}{\sqrt{2}} \text{ or}, x=-\frac{1}{\sqrt{2}}$

Hence, the zeroes of f(x) are $\frac{3}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$.

18. If α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$, then evaluate:

(i) $\alpha^2+\beta^2$ (ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (iii) $\alpha^3+\beta^3$

(iv) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ (v) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Solution: It is given that α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$,

$\alpha+\beta=-\frac{b}{c}$ and $\alpha\beta=\frac{c}{a}$

(i) We know that

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$\alpha^2+\beta^2=\left(\frac{-b}{a}\right)^2-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$$

(ii) $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\frac{c}{a}}=\frac{b^2-2ac}{ac}$$

(iii) We know that $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$\therefore\ \alpha^3+\beta^3=\left(\frac{-b}{a}\right)^3-3\frac{c}{a}\left(\frac{-b}{a}\right)=\frac{-b^3}{a^3}+\frac{3bc}{a^2}=\frac{-b^3+3abc}{a^3}=\frac{3abc-b^3}{a^3}$$

(iv) $$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{(\alpha\beta)^3}=\frac{\frac{3abc-b^3}{a^3}}{\left(\frac{c}{a}\right)^3}=\frac{3abc-b^3}{c^3}$$

(v) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{\frac{c}{a}}=\frac{3abc-b^3}{a^2c}=\frac{b(3ac-b^2)}{a^2c}$$

19. If α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$, then evaluate:

i) $\alpha^4+\beta^4$ **(ii)** $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}$

Solution It is given that α and β are the zeros of the quadratic polynomial $f(x)=ax^2+bx+c$,

$$\therefore\quad \alpha+\beta=-\frac{b}{a}\ \text{and},\ \alpha\beta=\frac{c}{a}$$

(i) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$\Rightarrow\quad \alpha^4+\beta^4=\{(\alpha+\beta)^2-2\alpha\beta\}^2-2(\alpha\beta)^2$$

$$\Rightarrow\quad \alpha^4+\beta^4=\left\{\left(-\frac{b}{a}\right)^2-2\frac{c}{a}\right\}^2-2\left(\frac{c}{a}\right)^2 \qquad \left[\because \alpha+\beta=-\frac{b}{a},\alpha\beta=\frac{c}{a}\right]$$

$$\Rightarrow\quad \alpha^4+\beta^4=\left(\frac{b^2-2ac}{a^2}\right)^2-\frac{2c^2}{a^2}=\frac{(b^2-2ac)^2-2a^2c^2}{a^4}$$

(ii)

$$\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(b^2-2ac)^2-2a^2c^2}{a^4\times\left(\frac{c}{a}\right)^2}$$

$$=\frac{(b^2-2ac)^2-2a^2c^2}{a^2c^2}$$

20. Find a quadratic polynomial if the zeroes of it are 2 and $\frac{-1}{3}$ respectively.

Sol : Let the quadratic polynomial be $ax^2+bx+c, a\neq 0$ and its zeroes be α *and* β Here

$\alpha=2, \beta=\frac{-1}{3}$

Sum of the zeroes = $(\alpha+\beta)$

$$= 2+\left(\frac{-1}{3}\right)=\frac{5}{3}$$

Product of the zeroes = $(\alpha\beta)$

$$=2\left(\frac{-1}{3}\right)=\frac{-2}{3}$$

Therefore the quadratic polynomial

ax^2+bx+c *is* $k\,[x^2-(\alpha+\beta)\,x+\alpha\beta]$,

Where K is a constant = $K\left[x^2-\frac{5}{3}x-\frac{2}{3}\right]$

We can put different values of k.

When k=3, the quadratic polynomial will be $3x^2 - 5x - 2$.

21. Find a quadratic polynomial with zeroes -2 and $\frac{1}{3}$.

Sol: Let the quadratic polynomial be

$ax^2+bx+c, a\neq 0$ and its zeroes be α *and* β

Here $\alpha=-2; \beta=\frac{1}{3}$

Sum of the zeroes $=(\alpha+\beta)=(-2)+\frac{1}{3}$

$$= \frac{-6+1}{3} = \frac{-5}{3}$$

Product of the zeroes = $\alpha\beta = (-2)\left(\frac{1}{3}\right)$

$$= \frac{-2}{3}$$

$\therefore$ The quadratic polynomial $ax^2 + bx + c$ is

$K\left[x^2 - (\alpha+\beta)\, x + \alpha\beta\right]$ Where k is a constant

$= k\,[x^2 - \left(\frac{-5}{3}\right) x + \left(\frac{-2}{3}\right)]$

We can put different values of k

When k =3 the quadratic polynomial will be $3x^2 + 5x - 2$

22. What is the quadratic polynomial whose sum of zeroes is $\frac{-3}{2}$ and the product of zeroes is -1.

Sol : Sum of zeroes = $(\alpha+\beta) = \frac{-3}{2}$

Product of zeroes = $\alpha\beta = -1$

$\therefore$ The quadratic polynomial $ax^2 + bx + c$ is

$K\left[x^2 - (\alpha+\beta)\, x + \alpha\beta\right]$ where k is a constant.

$= k\,[x^2 - \left(\frac{-3}{2}\right) x + (-1)]$

We can put different values of k

When k =2 the quadratic polynomial will be $2x^2 + 3x - 2$.

23. If α, β, γ are the zeroes of the given cubic polynomials, find the values as given in the table.

S.No.	Cubic Polynomial	$\alpha+\beta+\gamma$	$\alpha\beta+\beta\gamma+\gamma\alpha$	$\alpha\beta\gamma$
1.	x^3+3x^2-x-2	$\frac{-b}{a}=\frac{-3}{1}=-3$	$\frac{c}{a}=\frac{-1}{1}=-1$	$\frac{-d}{a}=\frac{-(-2)}{1}=2$
2.	$4x^3+8x^2-6x-2$	$\frac{-b}{a}=\frac{-8}{4}=-2$	$\frac{c}{a}=\frac{-6}{4}=-\frac{-3}{2}$	$\frac{-d}{a}=\frac{-(-2)}{4}=\frac{1}{2}$
3.	x^3+4x^2-5x-2	$\frac{-b}{a}=\frac{-4}{1}=-4$	$\frac{c}{a}=\frac{-5}{1}=-5$	$\frac{-d}{a}=\frac{-(-2)}{1}=2$
4.	x^3+5x^2+4 $x^3+5x^2+0.x+4$	$\frac{-b}{a}=\frac{-5}{1}=-5$	$\frac{c}{a}=\frac{0}{1}=0$	$\frac{-d}{a}=\frac{-4}{1}=-4$

24. Verify that 3, -1, $-\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x)=3x^3-5x^2-11x-3,$ and then verify the relationship between the zeroes and the coefficients.

Sol : Comparing the given polynomial with

ax^3+bx^2+cx+d, *we get*

a=3, b= -5, c = -11, d = -3.

Further

$$p(3)=3\times3^3-(5\times3^2)-(11\times3)-3$$
$$=81-45-33-3=0,$$
$$p(-1)=-3-5+11-3=0,$$
$$p(-\frac{1}{3})=-\frac{1}{9}-\frac{5}{9}+\frac{11}{3}-3=-\frac{2}{3}+\frac{2}{3}=0$$

Therefore, 3, -1, and $-\frac{1}{3}$ are the zeroes of $3x^3-5x^2-11x-3$.

So, we take $\alpha=3,\ \beta=-1$ *and* $\gamma=-\frac{1}{3}$.

Now,

$$\alpha+\beta+\gamma=3+(-1)+\left(-\frac{1}{3}\right)=2-\frac{1}{3}=\frac{5}{3}$$

$$=\frac{-(-5)}{3}=\frac{-b}{a},$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=3x(-1)+(-1)\times\left(-\frac{1}{3}\right)+$$

$$\left(-\frac{1}{3}\right)\times 3=-3+\frac{1}{3}-1=\frac{-11}{3}=\frac{c}{a},$$

$$\alpha\beta\gamma=3\times(-1)\times\left(-\frac{1}{3}\right)=-1=\frac{-(-3)}{3}=\frac{-d}{a}$$

25. Find the quadratic polynomial in each case, with the given numbers as the sum and product of its zeroes respectively.

i) $\frac{1}{4},-1$ ii) $\sqrt{2},\frac{1}{3}$ iii) $0,\sqrt{5}$ iv) 1,1 v) $\frac{-1}{4},\frac{1}{4}$ vi) 4,1

Sol : i) Let the quadratic polynomial be $ax^2+bx+c,$ and its zeroes be α *and* β

We have, $\alpha+\beta=\frac{1}{4}=\frac{-(-1)}{4}=\frac{-b}{a}$

$$\alpha\beta=-1=\frac{-4}{4}=\frac{c}{a}$$

If we take a=4; b = -1; c = -4

∴ The quadratic polynomial which fits the given conditions is $4x^2-x-4$.

ii) Let the quadratic polynomial be $ax^2+bx+c,$ and its zeroes be α *and* β

We have, $\alpha+\beta=\sqrt{2}=\frac{-(-\sqrt{2})}{1}=\frac{-b}{a}$

$$=\frac{-(-3\sqrt{2})}{3}$$

$$\alpha\beta=\frac{1}{3}=\frac{c}{a}$$

If we take a =3, b= $-3\sqrt{2}$; c=1

∴ The quadratic polynomial which fits the given conditions is $3x^2-3\sqrt{2}\,x+1$

iii) Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α *and* β

We have, $\alpha + \beta = 0 = \frac{0}{1} = \frac{-b}{a}$

$$\alpha\beta = \sqrt{5} = \frac{\sqrt{5}}{1} = \frac{c}{a}$$

If we take a =1; b=0; c= $\sqrt{5}$

$\therefore$ The quadratic polynomial which fits the given conditions is

$x^2 + 0.\, x + \sqrt{5} = x^2 + \sqrt{5}$

iv) Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α *and* β

We have, $\alpha + \beta = 1 = \frac{-(-1)}{1} = \frac{-b}{a}$

$$\alpha\beta = 1 = \frac{1}{1} = \frac{c}{a}$$

If we take a =1; b= -1 ; c = 1

$\therefore$ The quadratic polynomial which fits the given conditions is $x^2 - x + 1$

v) Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α *and* β

We have, $\alpha + \beta = \frac{-1}{4} = \frac{-b}{a}$

$$\alpha\beta = \frac{1}{4} = \frac{c}{a}$$

If we take a =4; b= -1 ; c = 1

$\therefore$ The quadratic polynomial which fits the given conditions is $4x^2 + x + 1$

vi) Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α *and* β

We have, $\alpha + \beta = 4 = \frac{-(-4)}{1} = \frac{-b}{a}$

$$\alpha\beta = 1 = \frac{1}{1} = \frac{c}{a}$$

If we take a =1; b= -4 ; c = 1

$\therefore$ The quadratic polynomial which fits the given conditions is $x^2 - 4x + 1$

26. Divide $2x^2 + 3x + 1 \; by \; x + 2.$

Sol : $(2x-1)\,(x+2) + 3 = 2x^2 + 3x - 2 + 3$

$= 2x^2 + 3x + 1$

i.e., $2x^2 + 3x + 1 = (x+2)(2x-1) + 3$

Therefore, Dividend = Divisor x Quotient + Remainder

$$x+2\overline{)\,2x^2 + 3x + 1\,}\quad 2x - 1$$

$$2x^2 + 4x$$

$$- \quad -$$

$$-x + 1$$

$$-x - 2$$

$$+ \quad +$$

$$3$$

27. Divide $3x^2 - x^3 - 3x + 5 \; by \; x - 1 - x^2$, **and verify the division algorithm.**

Sol : Dividend = Divisor x Quotient + Remainder

$= (-x^2 + x - 1)(x-2) + 3$

$= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3$

$= -x^3 + 3x^2 - 3x + 5$

$$-x^2 + x - 1\overline{)\,-x^3 + 3x^2 - 3x + 5\,}\quad x - 2$$

$$-x^3 + x^2 - x$$

$$+ \quad - \quad +$$

$$2x^2 - 2x + 5$$

$$2x^2 - 2x + 2$$

$$- \quad + \quad -$$

$$3$$

28. Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ *and* $-\sqrt{2}$.

Sol : Since two of the zeroes are $\sqrt{2}$ *and* $-\sqrt{2}$, therefore we can divide by

$$(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2.$$

$$\begin{array}{r|l} & 2x^2 - 3x + 1 \\ x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \\ & 2x^4 \quad\quad - 4x^2 \\ & - \quad\quad\quad + \\ \hline & -3x^3 + x^2 + 6x - 2 \\ & -3x^3 \quad\quad + 6x \\ & + \quad\quad\quad - \\ \hline & x^2 - 2 \\ & x^2 - 2 \\ & - \quad + \\ \hline & 0 \\ \hline \end{array}$$

First term of quotient is $\frac{2x^4}{x^2} = 2x^2$

Second term of quotient is $\frac{-3x^3}{x^2} = -3x$

Third term of quotient is $\frac{x^2}{x^2}$

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 \_ 3x + 1)$

Now, by splitting -3x, we factorize $2x^2 - 3x + 1$ as (2x-1) (x-1). So, its zeroes are given by $x = \frac{1}{2}$ and x=1. Therefore, the zeroes of the given polynomial are $\sqrt{2}$, $-\sqrt{2}$ and 1 and $\frac{1}{2}$.

29. Divide the polynomial p(x) by the polynomial g(x) and find the quotient and remainder in each of the following:

(i) p(x) = $x^3 - 3x^2 + 5x - 3, g(x) = x^2 - 2$

(ii) $p(x) = x^4 - 3x^2 + 4x + 5, g(x) = x^2 + 1 - x$

Sol : i) p(x) = $x^3 - 3x^2 + 5x - 3, g(x) = x^2 - 2$

$(x^3 - 3x^2 + 5x - 3) \div (x^2 - 2)$

$$\begin{array}{r} x-3 \\ x^2-2\overline{)x^3-3x^2+5x-3} \\ x^3 \quad -2x \\ - \quad + \\ \hline -3x^2+7x-3 \\ -3x^2 \quad +6 \\ + \quad - \\ \hline 7x-9 \end{array}$$

Quotient = x-3; Remainder = 7x -9

ii) $p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

$$g(x) = x^2 - x + 1$$

$(x^4 + 0x^3 - 3x^2 + 4x + 5) \div (x^2 - x + 1)$

$$\begin{array}{r} x^2+x-3 \\ x^2-x+1\overline{)x^4+0x^3-3x^2+4x+5} \\ x^4-x^3+x^2 \\ - \quad + \quad - \\ \hline x^3-4x^2+4x \\ x^3-x^2+x \\ - \quad + \quad - \\ \hline -3x^2+3x+5 \\ -3x^2+3x-3 \\ + \quad - \quad + \\ \hline 8 \end{array}$$

Quotient = x^2+x-3; Remainder = 8

30. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time and the product of its zeroes as 2, -7, -14 respectively.

Sol : Let α, β, γ are the zeroes of the cubic polynomial of ax^3+bx^2+cx+d

Given $\alpha + \beta + \gamma = 2$

$\alpha\beta + \beta\gamma + \gamma\alpha = -7$

$\alpha\beta\gamma = -14$

Then the cubic polynomial is

$$x^3 - x^2(\alpha + \beta + \gamma) + x\,(\alpha\beta + \beta\gamma + \gamma\beta) - \alpha\beta\gamma$$
$$= x^3 - x^2(2) + x(-7) - (-14)$$
$$= x^3 - 2x^2 - 7x + 14$$

31. Draw the graph of $x^2 - 3x - y$ and find the zeroes. Justify the answers $y = x^3 - 3x - 4$

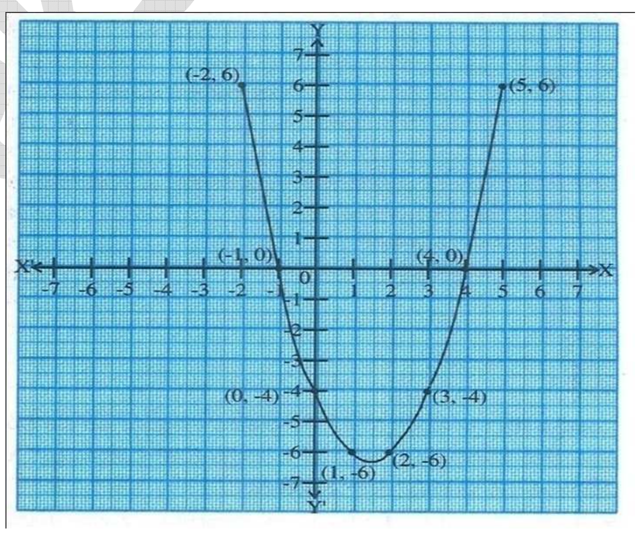
x	-2	-1	0	1	2	3	4	5
$y = x^3 - 3x - 4$	6	0	-4	-6	-6	-4	0	6
(x,y)	(-2,6)	(-1,0)	(0,4)	(1,-6)	(2,-6)	(3,-4)	(4,0)	(5,6)

- 1 and 4 are zeroes of the quadratic polynomial because (-1,0) and (4,0) are intersection points of X – anis

Justify : $x^2 - 3x - 4 = x^2 - 4x + 1x - 4$

$$= x\,(x-4) + 1\,(x-4) = (x+1)\,(x-4)$$

Zeroes of p(x) = (-1, 4)



32. Draw the graph of $x^3 - 4x$

$y = x^3 - 4x$

x	-2	-1	0	1	2
$y = x^3 - 4x$	0	3	0	-3	0
(x,y)	(-2,0)	(-1,3)	(0,0)	(1,-3)	(2,0)

-2, 0, 2 are the zeroes of cubic polynomial