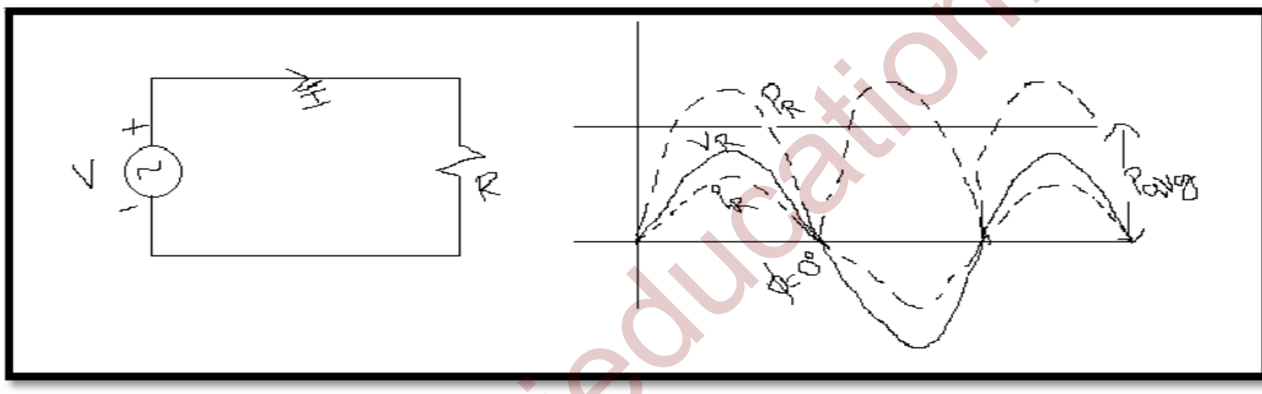
Steady State AC Circuit Analysis

In this unit, we will study about mathematical representation of sinusoidal quantities and steady state analysis of AC circuits excited by sinusoidal voltage. We have seen that by applying Kirchhoff's laws to a circuit and we get differential equations. The solution of these equations will be in two parts. One will be transient solution which will be short period and other is steady state solution which determined by algebraic equations.

- Here fundamental wave form we consider here is sinusoidal.

Phasor relations between voltage and currents in passive elements:

Resistor:



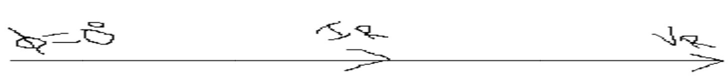
Let $\bar{V} = V_m \sin \omega t$ be the voltage and

$\bar{I} = I_m \sin \omega t$ be the current

$\bar{I} = \frac{\bar{V}}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t$

Therefore, $\bar{V} = \bar{I}$ R

Phasor diagram:



Power factor = cos Ø = cos 0 = 1 [UPF]

Instantaneous power, P (t) = V (t) * I (t)

P (t) = $V_m \sin \omega t * I_m \sin \omega t$

= $V_m\ I_m\ sin^2\ \omega t$

= $V_m\ I_m [\frac{1-\cos 2\omega t}{2}]$

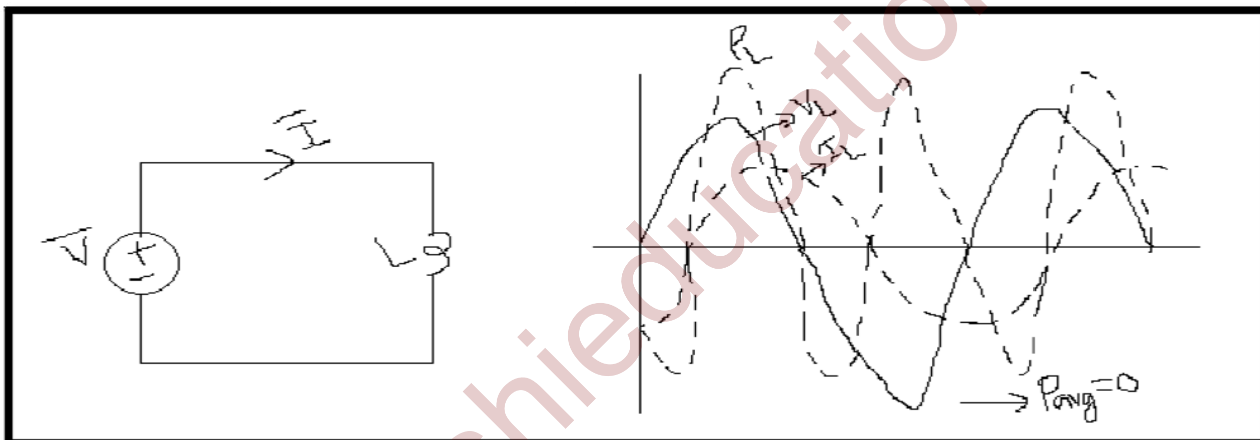
$= \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t \quad (\text{if } \cos 2\omega t = 0)$

$= \frac{V_m I_m}{\sqrt{2}\sqrt{2}}$

$P_{avg} = V_{RMS} I_{RMS}$ watts

- If a sinusoidal voltage is given to a resistor, the same sinusoidal current flows through it.
- Current and voltage are in phase in pure resistance.
- Resistor does not create a phase shift between voltage and current. (∅=0 and p.f=1)
- Pure resistive circuits in AC are called as "UPF Circuits"
- Resistor always dissipates electrical energy in the form of heat. So average power exists.

Inductor:



Let I = $I_m \sin \omega t$

And V = L $\frac{dI}{dt}$

V = L $\frac{d}{dt}(I_m \sin \omega t)$

V = $\omega L I_m \cos \omega t$

V = $\omega L I_m \sin(90 + \omega t)$

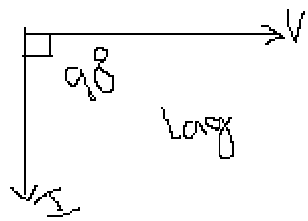
V = $\omega L I_m \sin(\omega t + 90)$

V = $\omega L I_m \sin \omega t$ [j]

Therefore, $\bar{V}$ = j $\omega L \bar{I}$ = j$X_L \bar{I}$

Where X_L = inductive reactance = ωL

Phasor diagram:



Power factor = cos ∅ = cos 90 = 0

- When sinusoidal voltage is given to an inductor, sinusoidal current flows through it.
- Current lags the voltage in pure inductance by 90 degrees.
- Inductor involves in creating a phase shift between voltage and current in it.
- A pure inductor never absorbs power, so average power is zero but it continuously stores and releases. Electrical energy in its electromagnetic form and back to the AC supply system.

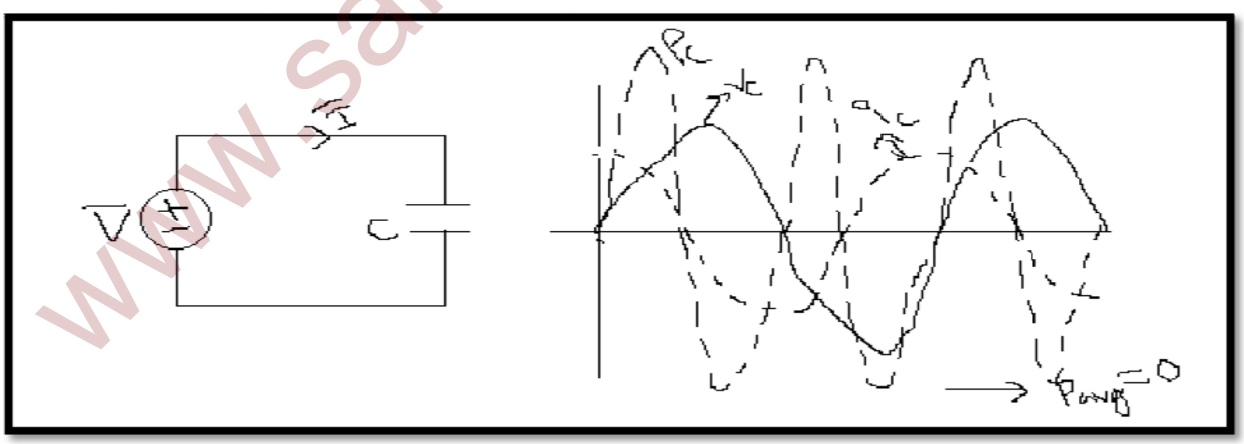
P (t) = V (t) * I (t)

$= \frac{V_m I_m}{2} \sin 2\omega t$

Energy stored, w = $\int_0^{\frac{\pi}{2}\omega} \frac{V_m I_m}{2} \sin 2\omega t = \frac{V_m I_m}{2\omega} = \frac{1}{2} L I_m{}^2$

- So an inductor is a passive lagging element in AC.

Capacitor:



Let V = $V_m \sin \omega t$

And I = C$\frac{dV}{dt}$

I = C $\frac{d}{dt}(V_m \sin \omega t)$

I = ω C$V_m \cos \omega t$

I = ω C$V_m \sin(90 + \omega t)$

I = ω C$V_m \sin(\omega t + 90)$

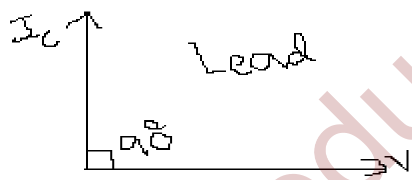
I = ω C$V_m \sin \omega t$ [j]

$\bar{I}$ = j $\omega c \bar{V}$

Therefore, $\bar{V} = \frac{\bar{I}}{jX_c}$ = - j$X_c \bar{I}$ (where $X_c = \frac{1}{\omega c}$)

Where X_c = capacitive reactance

Phasor diagram:



Power factor = cos ∅ = cos 90 = 0

- If a sinusoidal voltage is given to a capacitor, sinusoidal current flows through it.
- Current leads the voltage by 90 degree in a pure capacitor.
- Capacitor creates a phase shift in voltage and current.
- A pure capacitor never absorbs the power, so average power is zero but it continuously stores and releases electrical energy in its electrostatic form from and back to the AC system.

$$P(t) = V(t) * I(t)$$

$$= \frac{V_m I_m}{2} \sin 2\omega t$$

Energy stored, w = $\int_0^{\frac{\pi}{2}\omega} \frac{V_m I_m}{2} \sin 2\omega t = \frac{V_m I_m}{2\omega} = \frac{1}{2} C V_m^{\ 2}$

- A capacitor is a passive leading element in AC.

Impedance:

The impedance of a branch or circuit element or complete network is defined as the ratio of voltage function to current function. With sinusoidal voltage and current the Z will have magnitude and angle.

$$\text{Impedance, } Z = \frac{Voltage\ function}{current\ function} = \frac{V_m}{I_m} = \frac{V_{rms}}{I_{rms}} \text{ ohms}$$

Admittance:

The reciprocal of impedance is called as admittance, and is denoted by Y. The unit of Y is mho or seimens.

$$Y = \frac{current\ function}{voltage\ function} = \frac{I_m}{V_m} = \frac{I_{rms}}{V_{rms}} \text{ mho}$$

Phase Angle:

If voltage and current both are sinusoidal functions of time, the phase difference is called as phase angle. The phase angle never exceeds± 90. The phase angle is the angle of phasor current with phasor voltage is taken as the account.

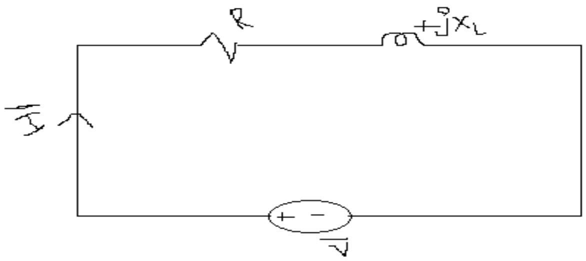
Impedance form table:

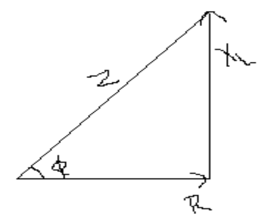
Element	v-i relation	V across element For i=$I_m \sin \omega t$	Phase angle	Impedance
R	V= ir	R$I_m \sin \omega t$	0	R
L	V = L$\frac{di}{dt}$	$\omega L I_m \sin(\omega t + 90)$	90 (lagging)	ωL
C	V=$\frac{1}{C}\int i\, dt$	$\frac{I_m}{\omega C}\sin(\omega t - 90)$	90 (leading)	$\frac{1}{\omega c}$

Admittance form table:

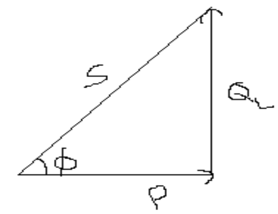
element	v-i relation	I across element For v=$V_m \sin \omega t$	Phase angle	Admittance
R	I = Gv	$\frac{V_m}{R}\sin \omega t$	0	$\frac{1}{R}$
L	I = $\frac{1}{L}\int v\, dt$	$\frac{V_m}{\omega L}\sin(\omega t - 90)$	90 (lagging)	$\frac{1}{\omega L}$
C	I = C$\frac{dv}{dt}$	$\omega c V_m \sin(\omega t + 90)$	90 (leading)	ωC

Series RL circuit:





impedance traingle



power traingle

Applying KVL,

-V + IR + jIX_L = 0

V = I [R + jX_L] = IZ   where Z = R + jX_L in ohms

Impedance triangle:

|Z| = $\sqrt{R^2 + {X_L}^2}$

Impedance angle from triangular circuit, $\emptyset = \tan^{-1}\frac{\omega L}{R}$

Z = R + jX_L → I^2 Z = I^2R + j$X_L I^2$

V*I = I^2R + j$X_L I^2$ → S = P + jQ_L

Where S is apparent power or total power, P is active or real power measured in watts and Q_L is Reactive power measured in VAR.

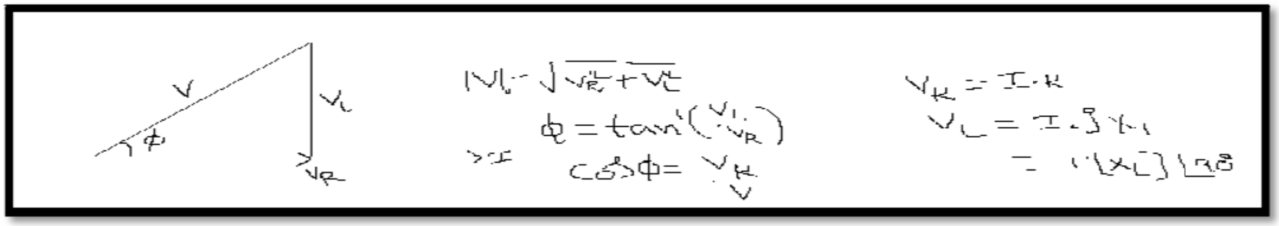
Power triangle:

|S| = $\sqrt{P^2 + {Q_L}^2}$ and $\emptyset = \tan^{-1}\frac{Q_L}{P}$

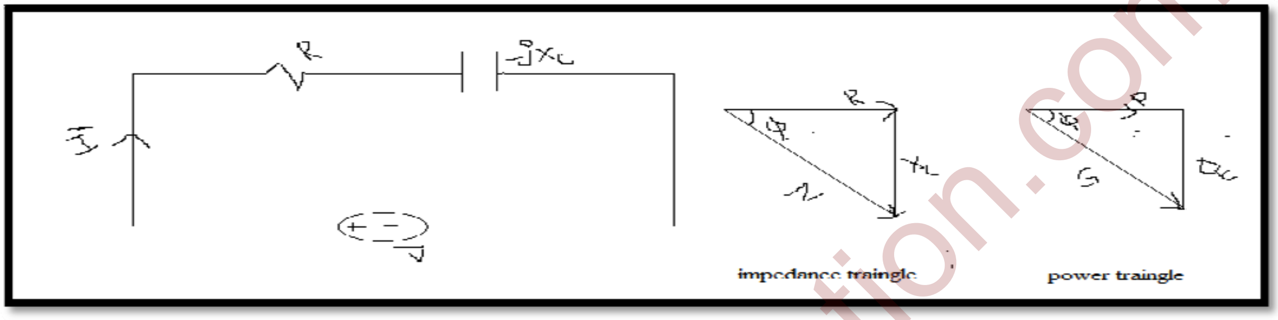
cos $\emptyset$ = $\frac{P}{S}$ → P = Scos $\emptyset$ = VIcos $\emptyset$ watts

sin $\emptyset$ = $\frac{Q_L}{S}$ → Q_L = Ssin $\emptyset$ = VIsin $\emptyset$ VAR

Phasor diagram:



Series RC circuit:



Applying KVL,

-V + IR - jIX_C = 0

V = I [R - jX_C] = IZ   where Z = R - jX_C in ohms

Impedance triangle:

$$|Z| = \sqrt{R^2 + {X_C}^2}$$

Impedance angle from triangular circuit, $\emptyset = \tan^{-1}\frac{X_C}{R} = \tan^{-1}\frac{1}{\omega CR}$

Z = R - jX_C → I^2 Z = I^2R - j$X_C I^2$

V*I = I^2R - j$X_C I^2$ → S = P - jQ_C

Where S is apparent power or total power, P is active or real power measured in watts and Q_L is Reactive power measured in VAR.

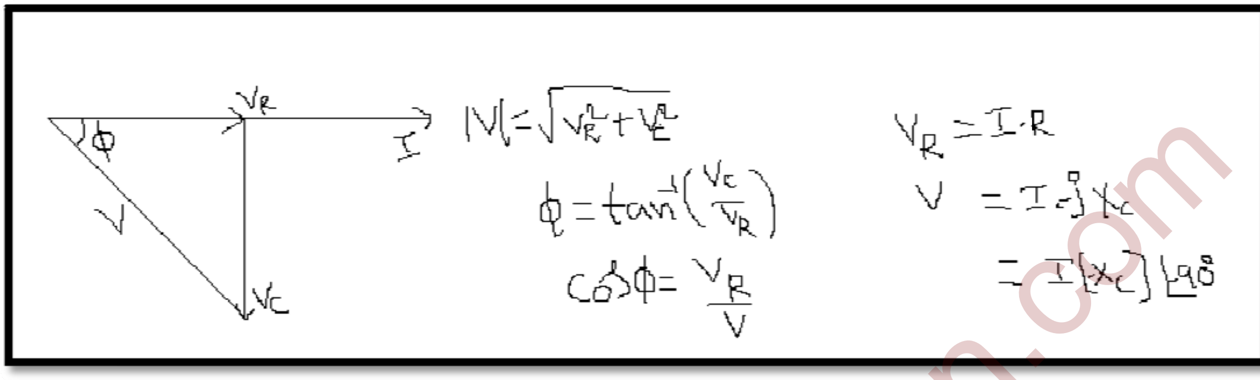
Power triangle:

$$|S| = \sqrt{P^2 + {Q_C}^2} \text{ and } \emptyset = \tan^{-1}\frac{Q_C}{P}$$

$$\cos\emptyset = \frac{P}{S} \rightarrow P = S\cos\emptyset = VI\cos\emptyset \text{ watts}$$

$$\sin\emptyset = \frac{Q_C}{S} \rightarrow Q_C = S\sin\emptyset = VI\sin\emptyset \text{ VAR}$$

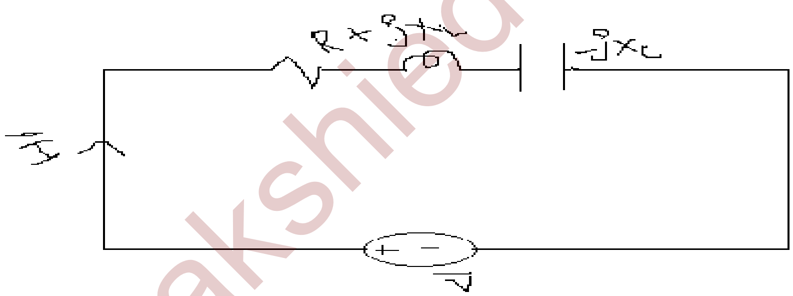
Phasor diagram:



Note:

- ✓ Reactive power controls the voltage while real power controls the frequency.
- ✓ -Q_C is generating VAR and $+Q_L$ is absorbing VAR.

Series RLC circuit*:*



Applying KVL,

$$-V + IR + jIX_L - jIX_C = 0$$

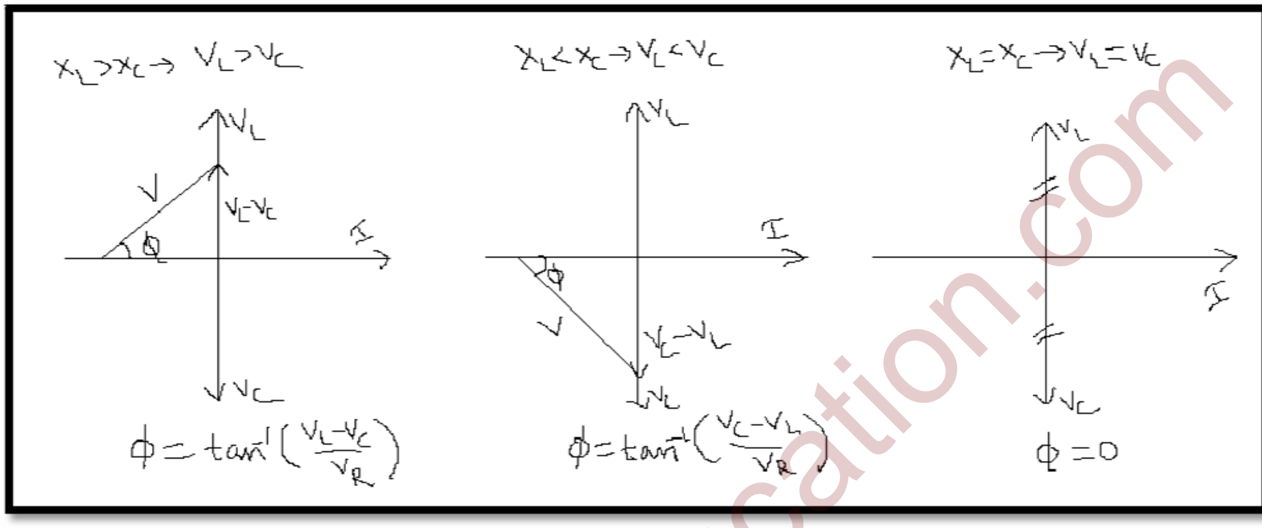
$$V = I\,[R + j\,(X_L - X_C)] \text{ where } Z = [R + j\,(X_L - X_C)] \text{ in ohms}$$

- If $X_L > X_C$ then Z = R + j X_{net} which behaves like a RL circuit and phase angle ∅ <90 lagging power factor.
- $X_L < X_C$ Then Z = R - j X_{net} which behaves like a RC circuit and phase angle ∅ >90 leading power factor.
- $X_L = X_C$ Then Z = R and ∅ = 90 unity power factor.

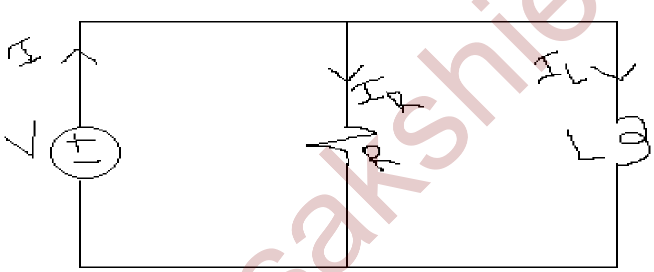
$Y = G \pm jB$

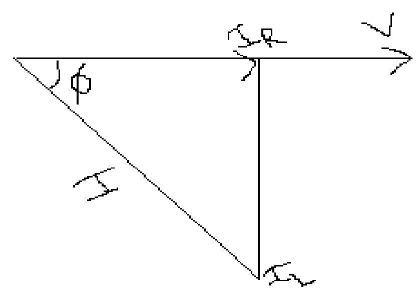
- ✓ Y = G – jB where $B_L = \frac{1}{X_L} = \frac{1}{\omega l} = \frac{1}{2\pi f l}$ → inductive susceptance (mho)
- ✓ Y = G + jB where $B_C = \frac{1}{X_C} = \omega c = 2\pi f c$ → capacitive susceptance (mho)

Phasor diagram:



Parallel RL circuit:





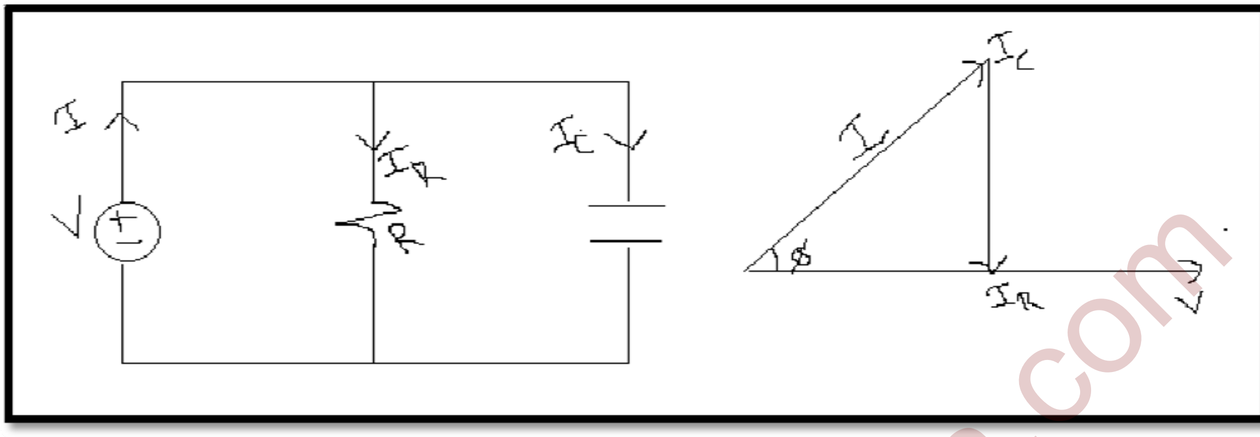
$I_R = \frac{V}{R}$ And $I_L = \frac{V}{jX_L} = \frac{V}{X_L} \lfloor{-90}$

I = $\sqrt{{I_R}^2 + {I_L}^2}$

$\emptyset = \tan^{-1}\frac{I_L}{I_R}$

$\cos\emptyset = \frac{I_R}{I}$ (Lagging)

Parallel RC circuit:



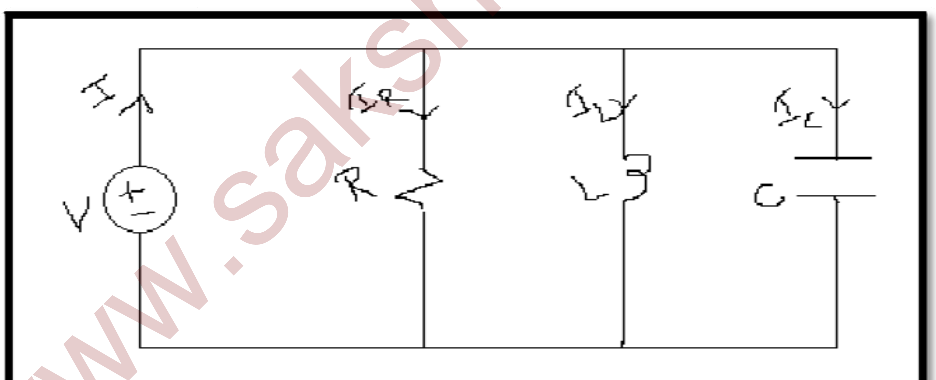
$I_R = \frac{V}{R}$ And $I_C = \frac{V}{-jX_C} = \frac{V}{X_C} \lfloor +90$

I = $\sqrt{{I_R}^2 + {I_C}^2}$

$\emptyset = \tan^{-1}\frac{I_C}{I_R}$

$\cos\emptyset = \frac{I_R}{I}$ (Leading)

Parallel RLC circuit:

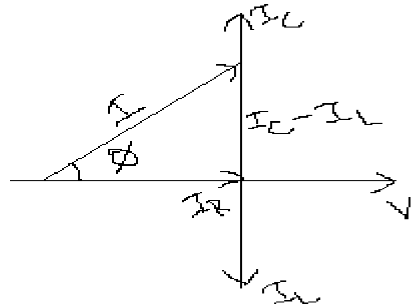


$I_R = \frac{V}{R}$

$I_L = \frac{V}{jX_L} = \frac{V}{X_L} \lfloor -90$

$I_C = \frac{V}{-jX_C} = \frac{V}{X_C} \lfloor +90$

Case1: if $X_L > X_C$ → $I_L < I_C$

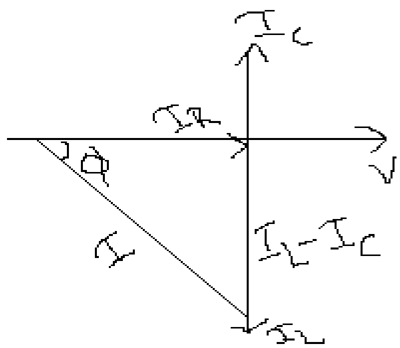


$$|I| = \sqrt{I_R^2 + (I_C - I_L)^2}$$

$$\phi = \tan^{-1}\left(\frac{I_C - I_L}{I_R}\right)$$

$$\cos\phi = \frac{I_R}{I} \text{ (Lead)}$$

Case2: if $X_L < X_C$ → $I_L > I_C$

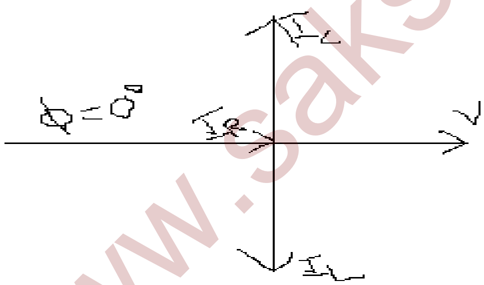


$$|I| = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$$\phi = \tan^{-1}\left(\frac{I_L - I_C}{I_R}\right)$$

$$\cos\phi = \frac{I_R}{I} \text{ (Lag)}$$

Case3: if $X_L = X_C$ → $I_L = I_C$



$$|I| = I_R$$

$$\phi = 0^\circ$$

$$\cos\phi = 1 \text{ (UPF)}$$

Concept of power and power factor:

In a DC system, power is given by the product of voltage and current and is measured in watts. In the AC system, since the voltage and current are the functions of time. So, the product of voltage and current at any instant is called the *instantaneous power*.

Power, P (t) = V (t) * I (t)

Bu we will take average power = $\frac{1}{T}\int_0^T v(t) * i(t)dt$ -------- (1)

Let us consider a general ac circuit, in which current may lead or lag the voltage by an angle $\emptyset$

$$v(t) = V_m \sin \omega t$$

$$i(t) = I_m \sin(\omega t \pm \emptyset)$$

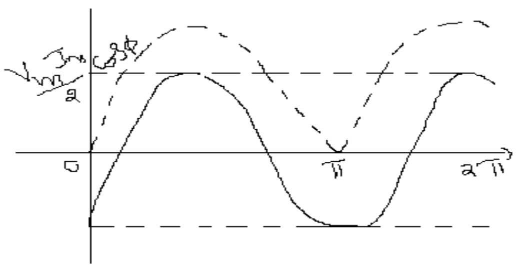
$$p(t) = v(t) * i(t)$$

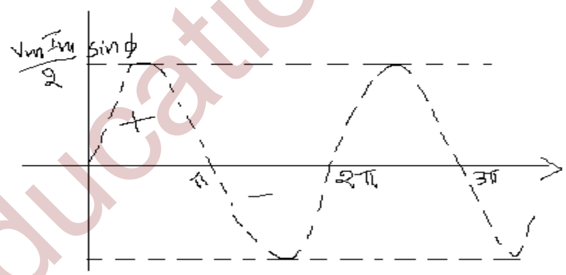
$$= V_m \sin \omega t \, I_m \sin(\omega t \pm \emptyset)$$

$$= \frac{1}{2}(2V_m \sin \omega t \, I_m \sin(\omega t \pm \emptyset)) \text{ ------------ } (2)$$

According to trigonometric formula (2) can be written as,

$$p(t) = \frac{V_m I_m}{2} \cos \emptyset - \frac{V_m I_m}{2} (\cos \emptyset \cos 2\omega t) \pm \frac{V_m I_m}{2} (\sin 2\omega t \, I_m \sin \emptyset) \text{ ---- } (3)$$





From equation (3) if we plot a graph, considering first two terms it is always positive and represents the real power.

The average value of first two terms is, P $= \frac{V_m I_m}{2} \cos \emptyset = \text{VI}\cos \emptyset$ --------- (4)

Equation (4) is known as real power or active power and it is expressed in watts.

From equation (3) if we plot a graph, if we consider next two it is sinusoidal variation and represents the oscillating part of energy between the source and reactive elements and this contributes for reactive power Q. This is measured in reactive volt amperes (VAR).

Reactive power $= \frac{V_m I_m}{2} \sin \emptyset = \text{VI}\sin \emptyset$ -------- (5)

Different types power and definitions:

Active power:

The active power or real power in an AC circuit is given by the product of voltage and current and cosine of the phase angle. It is always positive.

$$P = \text{VI}\cos \emptyset \text{ watts}$$

Reactive power:

The reactive power in an ac circuit is given by the product of voltage and current and ***sin*** of the phase angle.

$$\mathbf{Q = VIsin\,\emptyset\ VAR}$$

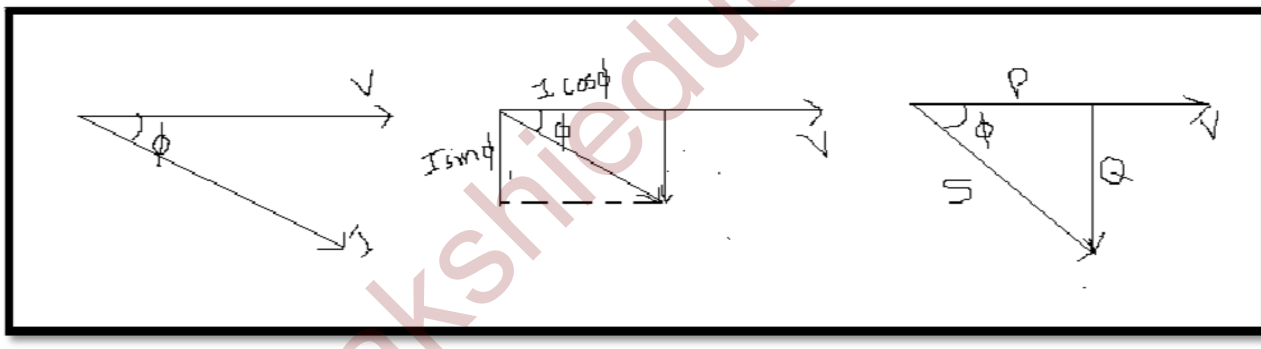
- If $\emptyset$ is leading, then reactive power is taken as positive and it is capacitive.
- If $\emptyset$ is lagging, then reactive power is taken as negative and it is inductive.

Apparent power:

The apparent power in AC circuit is the product of voltage and current. It is measured in *volt amperes*.

S = VI volt amps

The equations associated with the above components of power can be developed from phasor diagram by consulting the power triangle as shown in fig.



The component Isin$\emptyset$ is reactive or quadrature component of current. The product of voltage and this component gives the reactive power.

The component Icos$\emptyset$ is active or real or in phase component of current. The product of voltage and this component gives the active or real power.

By multiplying each side of current triangle with voltage, then the current triangle becomes the power triangle as shown in fig.

Apparent power, S = $\sqrt{P^2 + S^2}$

Other expressions for power:

Real power, P = VIcos$\emptyset$ = $\frac{VIR}{Z}$

$= \frac{V^2 R}{Z^2} = I^2$ R

Reactive power, Q = VIsin $\emptyset = I^2$ X

Apparent power, S = VI = I^2 Z

Power factor:

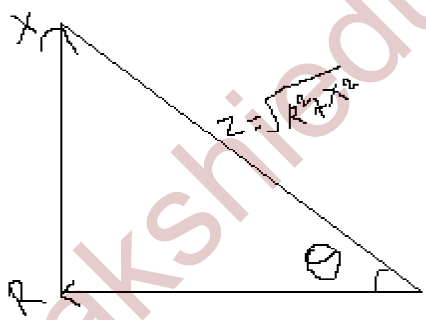
It is defined as the cosine of the phase angle$\emptyset$. It is also the ratio of real power to apparent power.

$$\text{Power factor} = \cos \emptyset = \frac{Real\ Power}{Apparent\ Power}$$

Depending on whether the current lags or leads the voltage the power factor is also taken as lagging or leading power factor.

The factor sin $\emptyset$ is called the reactive power.

The phase angle $\emptyset$ i.e. phase difference between V and I, is same as the impedance angle (θ). This can be depicted from the impedance triangle as shown in fig below:



$$Z = \sqrt{R^2 + X^2}$$

$$\cos \theta = \frac{R}{Z} \sin \theta = \frac{X}{Z}$$

$$\text{Power factor} = \cos \theta = \frac{R}{Z} = \frac{P}{S}$$