MATHEMATICS PAPER IB

COORDINATE GEOMETRY(2D &3D) AND CALCULUS.

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. 10X2 =20

1.If the portion of a straight line intercepted between the axes of co-ordinates is bisected at (2p, 2q), write the equation of the straight line.

2. Transform equation (2 + 5k) x – 3 (1 + 2k) y + (2 – k) = 0 into form $L_1 + \lambda L_2 = 0$ and find the point of concurrency of the family of straight lines represented by the equation.

3. P is a variable point which moves such that 3PA = 2PB. If A = (–2, 2, 3) and B =(13,–3,13)prove that P satisfies the equation $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$.

4. Show that the plane through (1, 1, 1), (1, –1, 1) and (–7, –3, –5) is parallel to y-axis.

5. Check the continuity of f given by $f(x) = \begin{cases} \dfrac{x^2-9}{x^2-2x-3} & \text{if } 0 < x < 5 \text{ and } x \neq 3 \\ 1.5 & \text{if } x = 3 \end{cases}$ At the point 3.

6.Show that f, given by $f(x) = \dfrac{x - |x|}{x}$ $(x \neq 0)$ is continuous on $R - \{0\}$.

7. If $y = ae^{nx} + be^{-nx}$ then prove that $y'' = n^2 y$

8. find the derivative of $\sin\left[\tan^{-1} e^{-x}\right]$

9. Find approximate value of $\sqrt{82}$

10. It is given that Rolle's theorem holds for the function f(x) = $x^3 + bx^2 + ax$ on [1, 3] with $c = 2t + \frac{1}{\sqrt{3}}$. Find the values of a and b.

SECTION B

SHORT ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING **5 X 4 = 20**

11. Find the equation of locus of P, if the line segment joining (2, 3) and (–1, 5) subtends a right angle at P.

12. When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of

$$3x^2 + 10xy + 3y^2 = 9$$

13. Find the points on the line 3x – 4y – 1 = 0 which are at a distance of 5 units form the point (3, 2).

14. $\underset{x \to 0}{Lt}\left[\frac{1 + x^{\frac{1}{8}} - 1 - x^{\frac{1}{8}}}{x}\right]$

15. A container is in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of 2m^3/minute, how fast is the height of water changing when the level is 4 m?

16. Find the angle between the curve $2y = e^{\frac{-x}{2}}$ and y-axis.

17.. If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

SECTION C

LONG ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING 5 X 7 =35

18. If (h, k) is the image of (x_1, y_1) w.r.t the line ax + by + c = 0 ($a \neq 0, b \neq 0$), then prove that $\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(x_1+by_1+c)}{a^2+b^2}$.

19. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of intersecting lines, then show that the square of the distance of their point of intersection from the origin is $\frac{c(a+b)-f^2-g^2}{ab-h^2}$. Also show that the square of this distance is $\frac{f^2+g^2}{h^2+b^2}$ if the given lines are perpendicular.

20. Show that the straight lines $y^2-4y+3=0$ and $x^2+4xy+4y^2+5x+10y+4=0$ from a parallelogram and find the lengths of its sides.

21. If a ray makes angle α, β, γ and δ with the four diagonals of a cube find $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$

22. If $y = x\sqrt{a^2+x^2} + a^2 \log\ x + \sqrt{x^2+a^2}$, show that $\frac{dy}{dx} = 2\sqrt{a^2+x^2}$

23. If the tangent at any point on the curve $\mathbf{x^{2/3}+y^{2/3}=a^{2/3}}$ intersects the coordinate axes in A, B show that the length AB is constant.

24. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be 20 feet, find the maximum area.

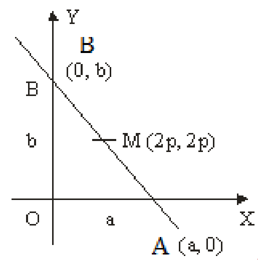
SOLUTIONS

1. If the portion of a straight line intercepted between the axes of co-ordinates is bisected at (2p, 2q), write the equation of the straight line.

Sol. Let a, b be the intercepts of the line and AB be the line segment between the axes.

Then points A =(a, 0) and B = (0, b)

Equation of the line in the intercept form is $\frac{x}{a}+\frac{y}{b}=1$ --- (1)



Mid -point of AB is M=$\left(\frac{a}{2},\frac{b}{2}\right)=(2p,2q)$ given

$\Rightarrow \frac{a}{2}=2p, \frac{b}{2}=2q \Rightarrow a=4p, b=4q$

Substituting in (1), $\frac{x}{4p}+\frac{y}{4q}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}=4$

2. Transform equation (2 + 5k) x – 3 (1 + 2k) y + (2 – k) = 0 into form $L_1+\lambda L_2=0$ and find the point of concurrency of the family of straight lines represented by the equation.

Sol. Given equation is

(2 + 5k) x – 3(1 + 2k) y + (2 –k) = 0 $\Rightarrow$ (2x - 3y + 2) + k (5x – 6y – 1) = 0

which is of the form $L_1+\lambda L_2=0$ where $L_1=2x-3y+2=0$ and $L_2=5x-6y-1=0$

therefore given equation represents a family of straight lines.

Solving above two lines,

x		y		1	
-3	2	2	-3		
-6	-1	5	-6		

$\frac{x}{3+12}=\frac{y}{10+2}=\frac{1}{-12+15} \Rightarrow x=\frac{15}{3}=5, y=\frac{12}{3}=4$

The point of concurrency is P(5, 4).

3. P is a variable point which moves such that 3PA = 2PB. If A = (–2, 2, 3) and B = (13, –3, 13) prove that P satisfies the equation $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$.

Sol. Given points are :

A(–2, 2, 3) and B = (13, –3, 13)

Let P(x, y, z) be any point on the locus.

Given condition is : 3PA = 2PB

$\Rightarrow 9\,PA^2 = 4\,PB^2$

$9[(x + 2)^2 + (y - 2)^2 + (z - 3)^2]$

$= 4[(x - 13)^2 + (y + 3)^2 + (z - 13)^2]$

$\Rightarrow 9(x^2 + 4x + 4 + y^2 - 4y + 4 + z^2 - 6z + 9)$

$= 4(x^2 - 26x + 169 + y^2 + 6y + 9 + z^2 - 26z + 169)$

$9x^2 + 9y^2 + 9z^2 + 36x - 36y - 54z + 153 = 4x^2 + 4y^2 + 4z^2 - 104x + 24y - 104z + 1388$

$5x^2 + 5y^2 + 5z^2 + 140x - 60y + 50z - 1235 = 0$

Dividing with 5 locus of P is :

$x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$.

4. Show that the plane through (1, 1, 1), (1, –1, 1) and (–7, –3, –5) is parallel to y-axis.

Sol. Equation of the plane through (1, 1, 1), (1, –1, 1) and (–7, –3, –5) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & -2 & 0 \\ -8 & -4 & -6 \end{vmatrix} = 0 \Rightarrow 3x - 4z + 1 = 0$$

D.rs of normal to the plane aer 3,0,-4

d.rs of y axis are 0,1,0

$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 3.0 + 0.1 - 4.0 = 0$

Normal to the plane is perpendicular to the y-axis.

henceplnae is parallel to Y-axis.

5. Check the continuity of f given by $f(x) = \begin{cases} \frac{x^2-9}{x^2-2x-3} & \text{if } 0 < x < 5 \text{ and } x \neq 3 \\ 1.5 & \text{if } x = 3 \end{cases}$ At the point 3.

Sol : Given f(3) =1.5.

$$\underset{x\to 3}{Lt} f(x) = \underset{x\to 3}{Lt} \frac{x^2-9}{x^2-2x-3}$$

$$= \underset{x\to 3}{Lt} \frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{3+3}{3+1} = \frac{6}{4} = 1.5 = f(3)$$

$\therefore f(x)$ is continuous at $x = 3$.

6. Show that f, given by $f(x) = \frac{x-|x|}{x}(x \neq 0)$ is continuous on $R - \{0\}$.

Sol : $\cdot f(x) = \begin{cases} \frac{x-x}{x}, x > 0 \\ \frac{x+x}{x}, x<0 \end{cases} \Rightarrow f(x) = \begin{cases} 0, x > 0 \\ \frac{2x}{x}, x < 0 \end{cases}$

$$\Rightarrow f(x) = \begin{cases} 0, x > 0 \\ 2, x < 0 \end{cases}$$

Left limit at x=0 is $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} 2 = 2$

Right limit at x=0 is $\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} 0 = 0$

$\underset{x\to 0-}{Lt} f(x) \neq \underset{x\to 0^+}{Lt} f(x) \therefore \underset{x\to 0}{Lt} f(x)$ *does* not exist.

Hence the function is not continuous at x=0.

When x<0, f(x) = 2, a constant. And it is continuous for all x<0.

When x>0 , f(x) = 0, which is continuos for all x>0.

Hence the function is continuous on R-{0}.

7. If $y = ae^{nx} + be^{-nx}$ then prove that $y'' = n^2 y$

Sol : $y = ae^{nx} + be^{-nx}$

$y_1 = na\,e^{nx} - nb\,e^{-nx}$

$y_2 = n^2.ae^{nx} + n^2\ be^{-nx}$

$y'' = n^2\ \ ae^{nx} + b.e^{-nx}\ \ = n^2 y$

8. find the derivative of $\sin\left[\tan^{-1}\ e^{-x}\ \right]$

Diff. w.r.t.x,

$\frac{dy}{dx} = \frac{d}{dx}\sin\left[\tan^{-1}\ e^{-x}\ \right]$

$= \cos\left[\tan^{-1}\ e^{-x}\ \right].\left[\tan^{-1}\ e^{-x}\ \right]^1$

$= \cos\ \tan^{-1}\ e^{-x}\quad \frac{1}{1+\ e^{-x}\ ^2}\ e^{-x}\ ^1 = \frac{-e^{-x}}{1+e^{-2x}}.\cos\left[\tan^{-1}\ e^{-x}\ \right]$

9. Find approximate value of $\sqrt{\mathbf{82}}$

Sol: let f(x) = $\sqrt{x}$,x = 81, Δ x = 1

Now

$f(x\ +\delta x)\ = f\ \ x\ \ +f^1\ \ x\ \ \ \delta x = \sqrt{x} + \frac{1}{2\sqrt{x}}.\Delta x$, put x = 81, Δ x = 1

$= \sqrt{81} + \frac{1}{2\sqrt{81}}.1 = 9 + \frac{1}{2.9} = 9 + \frac{1}{18} = 9 + 0.056 = 9.056$

10. It is given that Rolle's theorem holds for the function f(x) = $x^3 + bx^2 + ax$ on [1, 3] with $c = 2t + \frac{1}{\sqrt{3}}$. Find the values of a and b.

Sol. Given f(x) = $x^3 + bx^2 + ax$

$$f'(x) = 3x^2 + 2bx + a$$

$$\therefore f'(x) = 0 \Leftrightarrow 3c^2 + 2bc + a = 0$$

$$\Leftrightarrow c = \frac{-2b \pm \sqrt{4b^2 - 12a}}{6}$$

$$c = \frac{-b \pm \sqrt{b^2 - 3a}}{3}$$

$$2 + \frac{1}{\sqrt{3}} = \frac{-b \pm \sqrt{b^2 - 3a}}{3}$$

$$\frac{-b}{3} = 2 \text{ and } \frac{\sqrt{b^2 - 3a}}{3} = \frac{1}{\sqrt{3}}$$

$$\Leftrightarrow b = 6 \text{ and } b^2 - 3a = 3$$

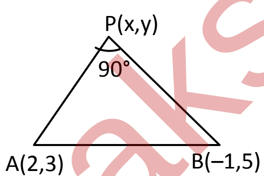
$$\Rightarrow 36 - 3 = 3a \Rightarrow 33 = 3a \Rightarrow a = 11$$

Hence a = 11, b = –6.

11. Find the equation of locus of P, if the line segment joining (2, 3) and (–1, 5) subtends a right angle at P.

Sol. Given points A(2, 3), B(–1, 5).

Let P(x, y) be any point in the locus.



Given condition is : $\angle APB = 90°$

$\Rightarrow$ (slope of $\overline{AP}$) (slope of $\overline{BP}$) = –1

$$\Rightarrow \frac{y-3}{x-2} \cdot \frac{y-5}{x+1} = -1$$

$$(y-3)(y-5) + (x-2)(x+1) = 0$$

$$x^2 + y^2 - x - 8y + 13 = 0$$

$\therefore$ Locus of P is $x^2 + y^2 - x - 8y + 13 = 0$

12.When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of

$3x^2+10xy+3y^2=9$

Sol. Given equation is $3x^2+10xy+3y-9=0$ ....(1)

Angle of rotation of axes is $\theta=\frac{\pi}{4}$ .Let (X,Y) be the new co-ordinates of $x.y$

$$x=X\cos\theta-Y\sin\theta= X\cos\frac{\pi}{4}-y\sin\frac{\pi}{4}=\frac{X-Y}{\sqrt{2}}$$

$$y=X\sin\theta+Y\cos\theta=X\sin\frac{\pi}{4}+Y\cos\frac{\pi}{4}=\frac{X+Y}{\sqrt{2}}$$

Transformed equation of (1) is $3\left(\frac{X-Y}{\sqrt{2}}\right)^2+10\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+3\left(\frac{X+Y}{\sqrt{2}}\right)^2-9=0$

$$\Rightarrow 3\frac{X^2-2XY+Y^2}{2}+10\frac{X^2-Y^2}{2}+3\frac{X^2+2XY+Y^2}{2}-9=0$$

$$\Rightarrow 3X^2-6XY+3Y^2+10X^2-10Y^2+3X^2+6XY+3Y^2-18=0$$

$$\Rightarrow 16X^2-4Y^2-18=0\Rightarrow 8X^2-2Y^2=9$$

13. Find the points on the line 3x – 4y – 1 = 0 which are at a distance of 5 units form the point (3, 2).

Sol. Equation of the line is 3x – 4y – 1 = 0 $\Rightarrow$ slope of the line is tanθ = ¾

$\Rightarrow$ sinθ = 3/5 and cosθ = 4/5

Given Point is (3,2) = (x_1,y_1) and r = 5.

Co-ordinates of any point on the given line at a distance r are

$(x_1\pm r\cos\theta, y_1\pm r\sin\theta)$

Co-ordinates of the points are $\left(3+5.\frac{4}{5},2+5.\frac{3}{5}\right)=(7,5)$ And

$\left(3-5.\frac{4}{5},2-5.\frac{3}{5}\right)=(-1,-1)$

14. $\underset{x\to0}{Lt}\left[\frac{1+x^{\frac{1}{8}}-1-x^{\frac{1}{8}}}{x}\right]$

Sol : $\underset{x\to 0}{Lt}\frac{(1+x)^{\frac{1}{8}}-(1-x)^{\frac{1}{8}}}{x}$

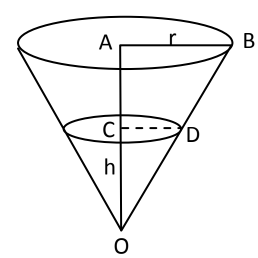
$= \underset{x\to 0}{Lt}\frac{(1+x)^{\frac{1}{8}}-1+1-(1-x)^{\frac{1}{8}}}{x}$

$= \underset{1+x\to 1}{Lt}\frac{(1+x)^{\frac{1}{8}}-1}{(1+x)-1} + \underset{1-x\to 1}{Lt}\frac{(1-x)^{\frac{1}{8}}-1^{8}}{(1-x)-1}$

$= \frac{1}{8}1^{-7/8} + \frac{1}{8}1^{-7/8} = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$

15. A container is in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of $2m^3$/minute, how fast is the height of water changing when the level is 4 m?

Sol. h = 8 m = OC



r = 6 m = AB

$\frac{dv}{dt} = 2\,m^3/\text{minute}$

ΔOAB and OCD are similar angle then

$\frac{CD}{AB} = \frac{OC}{OA}$

$\frac{r}{6} = \frac{h}{8} \Rightarrow r = h\frac{3}{4}$

Volume of cone $v = \frac{1}{3}\pi r^2 h$

$$v = \frac{1}{3}\pi h^2 \frac{9}{16} h$$

$$v = \frac{3}{16}\pi h^3$$

$$\frac{dv}{dt} = \frac{3}{16}\pi 3h^2 \frac{dh}{dt} \quad \because h = 16$$

$$2 = \frac{3}{16}\pi 3(16)\frac{dh}{dt} \Rightarrow \frac{2}{9\pi} = \frac{dh}{dt}$$

16. Find the angle between the curve $\mathbf{2y = e^{\frac{-x}{2}}}$ and y-axis.

Sol: Equation of y-axis is x = 0

The point of intersection of the curve $2y = e^{\frac{-x}{2}}$ and $x = 0$ is $P\left(0, \frac{1}{2}\right)$

The angle ψ made by the tangent to the curve $2y = e^{\frac{-x}{2}}$ at P with x – axis is given by $\tan\psi = \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = \left.\frac{-1}{4}e^{\frac{-x}{2}}\right|_{\left(0,\frac{1}{2}\right)} = \frac{-1}{4}$

Further, if ϕ is the angle between the $y-\text{axis}$ and $2y = e^{\frac{-x}{2}}$, then we have

$$\tan\phi = \left|\tan\left(\frac{\pi}{2} - \psi\right)\right| - |\cot\psi| = 4$$

$\therefore$ The angle between the curve and the y-axis is $\tan^{-1} 4$.

17.. If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Sol. $x^y = e^{x-y}$

$$\log x^y = \log e^{x-y}$$

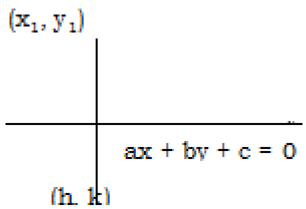
$$y \log x = x - y$$

$$y = \frac{x}{1+\log x}$$

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\cdot\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

18. If (h, k) is the image of (x_1, y_1) w.r.t the line ax + by + c = 0 (a ≠ 0, b ≠ 0), then $\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(x_1+by_1+c)}{a^2+b^2}$.



Proof:

Let $A(x_1, y_1)$, B(h, k)

Mid pointof isP = $\left(\frac{x_1+h}{2}, \frac{y_1+k}{2}\right)$

Since B is the image of A,thereforemid pointP lies on ax + by + c = 0.

$\Rightarrow a\left(\frac{x_1+h}{2}\right)+b\left(\frac{y_1+k}{2}\right)$+c =0

$\Rightarrow ax_1 + by_1 + ah + bk + 2c = 0$

$\Rightarrow ah + bk = -ax_1 + by_1 - 2c.$

Slope of $\overline{AB}$ is $\frac{k-y_1}{h-x_1}$

And Slope of given line is $-\frac{a}{b}$

$\overline{AB}$ is perpendicular to the given line

$\Rightarrow \left(\frac{k-y_1}{h-x_1}\right)\left(-\frac{a}{b}\right) = -1$

$\Rightarrow \frac{k-y_1}{b} = \frac{h-x_1}{a}$

By the law of multipliers in ratio and proportion

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{a\ h-x_1\ +b\ k-y_1}{a^2+b^2}$$

$$=\frac{ah+bk-ax_1-by_1}{a^2+b^2}$$

$$=\frac{-ax_1-by_1-2c-ax_1-by_1}{a^2+b^2}$$

$$=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

Hence $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$

19. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of intersecting lines, then show that the square of the distance of their point of intersection from the origin is $\frac{c(a+b)-f^2-g^2}{ab-h^2}$. Also show that the square of this distance is $\frac{f^2+g^2}{h^2+b^2}$ if the given lines are perpendicular.

Sol. Let the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represent the lines

$l_1x + m_1y + n_1 = 0 \qquad \ldots(1)$

$l_2x + m_2y + n_2 = 0 \qquad \ldots(2)$

$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) =$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$l_1l_2 = a,\ m_1m_2 = b,\ n_1n_2 = c$

$l_1m_2+l_2m_1=2h, l_1n_2+l_2n_1=2g,$
$m_1n_2+m_2n_1=2f$

Solving (1) and (2)

$$\frac{x}{m_1n_2-m_2n_1}=\frac{y}{l_2n_1-l_1n_2}=\frac{1}{l_1m_2-l_2m_1}$$

The point of intersection,

$$P\left[\frac{m_1n_2-m_2n_1}{l_1m_2-l_2m_1},\frac{l_2n_1-l_1n_2}{l_1m_2-l_2m_1}\right]$$

$$OP^2=\frac{(m_1n_2-m_2n_1)^2+(l_2n_1-l_1n_2)^2}{(l_1m_2-l_2m_1)^2}$$

$$=\frac{(m_1n_2+m_2n_1)^2-4m_1m_2n_1n_2+(l_1n_2+l_2n_1)^2-4l_1l_2n_1n_2}{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}$$

$$=\frac{4f^2-4abc+4g^2-4ac}{4h^2-4ab}$$

$$=\frac{c(a+b)-f^2-g^2}{ab-h^2}$$

If the given pair of lines are perpendicular, then a + b = 0

$\therefore$ a = –b

$$OP^2=\frac{0-f^2-g^2}{(-b)b-h^2}=\frac{f^2+g^2}{h^2+b^2}.$$

20. Show that the straight lines $y^2-4y+3=0$ and $x^2+4xy+4y^2+5x+10y+4=0$ from a parallelogram and find the lengths of its sides.

Sol. Equation of the first pair of lines is

$y^2-4y+3=0, \Rightarrow\ y-1\ \ y-3\ =0$

$\Rightarrow y-1=0 \text{ or } y-3=0$

$\Rightarrow$Equations of the lines are y – 1 = 0 ……..(1)and y – 3 = 0 ……..(2)

Equations of (1) and (2) are parallel.

Equation of the second pair of lines is $x^2+4xy+4y^2+5x+10y+4=0$

$\Rightarrow\ x+2y\ ^2+5\ x+2y\ +4=0$

$\Rightarrow\ x+2y\ ^2+4\ x+2y\ +\ x+2y\ +4=0$

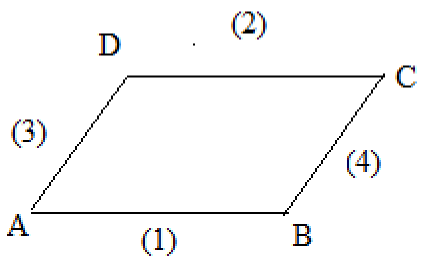
$\Rightarrow\ x+2y\ \ x+2y+4\ +1\ x+2y+4\ =0$

$\Rightarrow\ x+2y+1\ \ x+2y+4\ =0$

$\Rightarrow x+2y+1=0, x+2y+4=0$

Equations of the lines are $x+2y+1=0$ ……..(3)and $x+2y+4=0$ ……(4)

Equations of (3) and (4) are parallel .



Solving (1), (3) x + 2 + 1= 0, x = - 3

Co-ordinates of A are (-3, 1)

Solving (2), (3) x + 6 + 1 = 0, x = - 7

Co-ordinates of D are (-7,3)

Solving (1), (4) x + 2 + 4 = 0, x = - 6

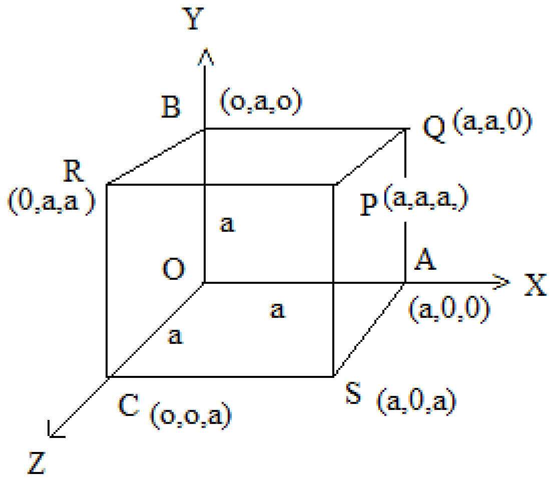
Co-ordinates of B are (-6, 1)

$$AB=\sqrt{(-3+6)^2+(1-1)^2}=\sqrt{9+0}=3$$

$$AD=\sqrt{(-3+7)^2+(1-3)^2}=\sqrt{16+4}=\sqrt{20}=2\sqrt{5}$$

Length of the sides of the parallelogram are $3, 2\sqrt{5}$

21. If a ray makes angle α, β, γ and δ with the four diagonals of a cube find $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$



Sol:

Let OABC;PQRS be the cube.

Let a be the side of the cube. Let one of the vertices of the cube be the origin O and the co-ordinate axes be along the three edges $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ passing through the origin.

The co-ordinate of the vertices of the cube with respect to the frame of reference OABC are as shown in figure are A (a,o,o), B(o,a,o), C(0,o,a) P(a,a,a) Q(a,a,o) R(o,a,a) and S(a,o,a)

The diagonals of the cube are $\overline{OP}, \overline{CQ}, \overline{AR} and \overline{BS}$. and their d.rs are respectively (a, a, a), (a, a, -a), (-a, a, a) and (a, -a, a).

Let the direction cosines of the given ray be l, m, n .

Then $l^2 + m^2 + n^2 = 1$

If this ray is making the angles α, β, γ and δ with the four diagonals of the cube, then

$$\cos\alpha = \frac{|a \times l + a \times m + a \times n|}{\sqrt{a^2 + a^2 + a^2}.1} = \frac{|l + m + n|}{\sqrt{3}}$$

Similarly, $\cos\beta = \dfrac{|l + m - n|}{\sqrt{3}}$

$$\cos\gamma = \frac{|-l + m + n|}{\sqrt{3}} \ and\ \cos\delta = \frac{|-l + m + n|}{\sqrt{3}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$$

$$\frac{1}{3}\{|l + m + n|^2 + |l + m - n|^2 + |-l + m + n|^2 + |l - m + n|^2\}$$

$$\frac{1}{3}[\ l + m + n\ ^2 +\ l + m - n\ ^2 +\ -l + m + n\ ^2 +\ l - m + n\ ^2]$$

$\frac{1}{3}[4\ l^2 + m^2 + n^2\ ] = \frac{4}{3}$ (since $l^2 + m^2 + n^2 = 1$)

22. If $y = x\sqrt{a^2 + x^2} + a^2 \log\ x + \sqrt{x^2 + a^2}$, show that $\frac{dy}{dx} = 2\sqrt{a^2 + x^2}$

sol : $y = x\sqrt{a^2 + x^2} + a^2 \log\ x + \sqrt{a^2 + x^2}$

diff. w.r.t x,

$$\frac{d}{dx}y = \frac{d}{dx}\ x\sqrt{a^2 + x^2} + a^2 \log\ x + \sqrt{a^2 + x^2}$$

$$\Rightarrow \frac{dy}{dx} = x.\frac{1}{2\sqrt{a^2 + x^2}}\sqrt{2}\,x + \sqrt{a^2 + x^2}.1$$

$$+\frac{a^2}{x + \sqrt{a^2 + x^2}}\left(1 - \frac{1}{2\sqrt{a^2 + x^2}}2x\right)$$

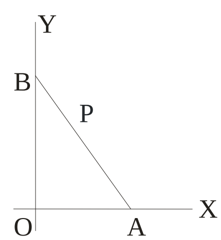
$$= \frac{x^2}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2} + \frac{a^2}{x + \sqrt{a^2 + x^2}}.\frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2}}$$

$$= \frac{x^2}{\sqrt{a^2 + x^2}} + \sqrt{a^2 + x^2} + \frac{a^2}{\sqrt{a^2 + x^2}}$$

$$= \frac{x^2 + a^2}{\sqrt{a^2 + x^2}} + \sqrt{a^2 + x^2}$$

$$= \sqrt{a^2 + x^2} + \sqrt{a^2 + x^2} = 2\sqrt{a^2 + x^2}$$

23. If the tangent at any point on the curve $\mathbf{x^{2/3} + y^{2/3} = a^{2/3}}$ intersects the coordinate axes in A, B show that the length AB is constant,



Sol: Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$

Let $x = a\ \cos^3\theta,\ y = a\sin^3\theta$

be the parametric equations of the curve .

Then any point P on the curve is $a\cos^3\theta, a\sin^3\theta$

$$\Rightarrow \frac{dy}{dx} = \frac{a.3\sin^2\theta.\ \cos\theta}{a.3\cos^2\theta.\ -\sin\theta} = \frac{-\sin\theta}{\cos\theta}$$

Equation of the tangent at P is

$$y - a\sin^3\theta = \frac{-\sin\theta}{\cos\theta}\ x - a\cos^3\theta$$

$$\Rightarrow \frac{y}{\sin\theta} - a\sin^2\theta = -\frac{x}{\cos\theta} + a\cos^2\theta$$

$$\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a\ \sin^2\theta + \cos^2\theta\ = a$$

$$\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$$

$\Rightarrow x - \text{int ercept } OA = a\cos\theta$

$y - \text{int ercept } OB = a\sin\theta$

Now $AB^2 = OA^2 + OB^2$

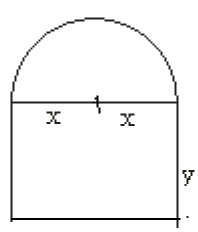
$= \ a\cos\theta\ ^2 + \ a\sin\theta\ ^2$

$= a^2\ \sin^2\theta + \cos^2\theta\ = a^2$

$\Rightarrow AB = a, \text{acons tan t}$

24. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be 20 feet, find the maximum area.

Sol: Let the length of the rectangle be 2x and breadth be y so that radius of the semi-circle is x.



Perimeter $= 2x + 2y + \pi.x = 20$

$2y = 20 - 2x - \pi x$

$$y = 10 - x - \frac{\pi}{2}.x$$

$$\text{Area} = 2xy + \frac{\pi}{2}.x^2$$

$$= 2x\left(10 - x - \frac{\pi x}{2}\right) + \frac{\pi}{2}x^2$$

$$= 20x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$$

Let f(x) $= 20x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$

$\Rightarrow f' = 20 - 4x - 2\pi x + \pi x$ and f'' =-4-2π+π= -4-π for max or min

for max or min $f'(x) = 0 \Rightarrow 20 - 4x - \pi x = 0$

$$\Rightarrow (\pi + 4)x = 20$$

$$\Rightarrow x = \frac{20}{\pi + 4}$$

$$f''(x) = -4 - \pi < 0$$

$\Rightarrow f(x)$ has a maximum when $x = \frac{20}{\pi + 4}$

$$y = 10 - x - \frac{\pi}{2}x = 10 - \frac{20}{\pi + 4} - \frac{\pi}{2}\frac{20}{\pi + 4}$$

$$= \frac{10\pi + 40 - 20 - 10\pi}{\pi + 4}$$

$$= \frac{20}{\pi + 4}$$

Maximum area $= 2xy + \frac{\pi}{2}.x^2$

$$= \frac{40}{\pi + 4}.\frac{20}{\pi + 4} + \frac{\pi}{2}\frac{400}{(\pi + 4)^2}$$

$$= \frac{800 + 200\pi}{(\pi + 4)^2} = \frac{200(\pi + 4)}{(\pi + 4)^2}$$

$= \frac{200}{\pi + 4}$ sq.feet.