MATHEMATICS PAPER IA

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. **10X2 =20**

1. Find the domain and range of the $f(x) = \frac{x}{2-3x}$

2. Find the inverse of the function
   If $a, b \in R$, $f: R \to R$ defined by $f(x) = ax + b$ $(a \neq 0)$

3. If the position vectors of the points A, B, C are $-2\bar{i} + \bar{j} - \bar{k}$, $-4\bar{i} + 2\bar{j} + 2\bar{k}$ and $6\bar{i} - 3\bar{j} - 13\bar{k}$ respectively and $\overline{AB} = \lambda\overline{AC}$ then find the value of λ.

4. ABCDE is a pentagon. If the sum of the vectors AB, AE, BC, DE, ED and AC is λAC, then find the value of λ.

5. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be unit vectors such that $\bar{b}$ is not parallel to $\bar{c}$ and $\bar{a} \times (\bar{b} \times \bar{c}) = \frac{1}{2}\bar{b}$. Find the angles made y $\bar{a}$ with each of $\bar{b}$ and $\bar{c}$.

6. What is the value of tan 20° + tan 40° + $\sqrt{3}$ tan 20° tan 40°?

7. Prove that $\cos 20° \cos 40° - \sin 5° \sin 25° = \frac{\sqrt{3}+1}{4}$.

8. If $\sin hx = \frac{3}{4}$ find $\cos h2x$ *ad* $\sin h2x$

9. For any square matrix A, show that AA′ is symmetric.

10. Is $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ symmetric or skew symmetric 7.

SECTION B

SHORT ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING 5 X 4 = 20

11. If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors such that $\bar{a}$ is perpendicular to the plane of $\bar{b}, \bar{c}$ and the angle between $\bar{b}$ and $\bar{c}$ is $\pi/3$, then find $|\bar{a}+\bar{b}+\bar{c}|$.

12. If $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$, then prove that $a \tan\beta = b \tan\alpha$.

13. Prove that tan 70° – tan 20° = 2 tan 50°.

14. Solve the following and write the general solutions

$$\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$$

15. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then prove that `$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

16. In aΔABC, show that the sides a, b, c are in A.P. if and only if r_1, r_2, r_3 are in H.P.

17. Apply the test of rank to examine whether the following equations are consistent.
2x – y + 3z = 8, –x + 2y + z = 4, 3x + y = 4z = 0 and, if consistent, find the complete solution.

SECTION C

LONG ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING 5 X 7 = 35

18. If $f: A \to B$, $g: B \to A$ are two functions such that $gof = I_A$ and $fog = I_B$ then $f: A \to B$ is a bijection and $f^{-1} = g$.

19. $1^2 + (1^2+2^2) + (1^2+2^2+3^3) + \ldots\ldots\ldots$ upto n terms $= \frac{n(n+1)^2(n+2)}{12}$

20. If $[\bar{b}\ \ \bar{c}\ \ \bar{d}] + [\bar{c}\ \ \bar{a}\ \ \bar{d}] + [\bar{a}\ \ \bar{b}\ \ \bar{d}] = [\bar{a}\ \ \bar{b}\ \ \bar{c}]$ then show that $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar.

21. If $\cos A+\cos B+\cos C=\frac{3}{2}$ then show that the triangle is equilateral

22. Show that $a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}=s+\frac{\Delta}{R}$

23. Show that

$$\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$$

24. If $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$, then show that $A^{-1}=A^3$.

SOLUTIONS

1. Find the domain and range of the $f(x)=\frac{x}{2-3x}$

Sol. $f(x)=\frac{x}{2-3x}$

$2-3x\neq 0$

$2\neq 3x$

$x\neq\frac{2}{3}$

Domain of f is $R-\left\{\frac{2}{3}\right\}$

$\frac{x}{2-3x}=y$

$\Rightarrow x=y(2-3x)$

$\Rightarrow x=2y-3yx$

$\Rightarrow x+3yx=2y$

$\Rightarrow x(1+3y)=2y$

$\Rightarrow x=\frac{2y}{1+3y}$

$\Rightarrow 1+3y\neq 0$

$\Rightarrow 3y\neq -1$

$y\neq -\frac{1}{3}$

$\therefore$ Range of f is $R-\left\{-\frac{1}{3}\right\}$.

2. Find the inverse of the function

If $a,b\in R$, $f:R\to R$ defined by $f(x)=ax+b\ (a\neq 0)$

Sol: Let $f^{-1}(x)=y\Rightarrow x=f(y)$

$$x=ay+b\Rightarrow y=\frac{x-b}{a}$$

$$\therefore f^{-1}(x)=\frac{x-b}{a}$$

3. If the position vectors of the points A, B, C are $-2\bar{i}+\bar{j}-\bar{k},\ -4\bar{i}+2\bar{j}+2\bar{k}$ and $6\bar{i}-3\bar{j}-13\bar{k}$ respectively and $\overline{AB}=\lambda\overline{AC}$ then find the value of λ.

Sol. Let O be the origin and $\overline{OA}=-2\bar{i}+\bar{j}-\bar{k}$, $\overline{OB}=-4\bar{i}+2\bar{j}+2\bar{k}$, $\overline{OC}=6\bar{i}-3\bar{j}-13\bar{k}$

Given $\overline{AB}=\lambda\overline{AC}$

$$\overline{OB}-\overline{OA}=\lambda\left[\overline{OC}-\overline{OA}\right]$$

$$-4\bar{i}+2\bar{j}+2\bar{k}+2\bar{i}-\bar{j}+\bar{k}=$$

$$\lambda\left[6\bar{i}-3\bar{j}-13\bar{k}+2\bar{i}-\bar{j}+\bar{k}\right]$$

$$-2\bar{i}+\bar{j}+3\bar{k}=\lambda\left[8\bar{i}-\bar{j}-12\bar{k}\right]$$

Comparing $\bar{i}$ coefficient on both sides

$$-2=\lambda 8\Rightarrow\lambda=-\frac{2}{8}\Rightarrow\lambda=\frac{-1}{4}$$

4. ABCDE is a pentagon. If the sum of the vectors AB, AE, BC, DE, ED and AC is λAC, then find the value of λ.

Sol. Given that,

$$\overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED}+\overline{AC}=\lambda\overline{AC}$$

$$\Rightarrow(\overline{AB}+\overline{BC})+(\overline{AE}+\overline{ED})+(\overline{DC}+\overline{AC})=\lambda\overline{AC}$$

$$\Rightarrow\overline{AC}+\overline{AD}+\overline{DC}+\overline{AC}=\lambda\overline{AC}$$

$$\Rightarrow\overline{AC}+\overline{AC}+\overline{AC}=\lambda\overline{AC}$$

$$\Rightarrow 3\overline{AC}=\lambda\overline{AC}$$

$\therefore \lambda = 3$

5. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be unit vectors such that $\bar{b}$ is not parallel to $\bar{c}$ and $\bar{a} \times (\bar{b} \times \bar{c}) = \frac{1}{2}\bar{b}$. Find the angles made y $\bar{a}$ with each of $\bar{b}$ and $\bar{c}$.

Sol. $\frac{1}{2}\bar{b} = \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

Such $\bar{b}$ and $\bar{c}$ are non-coplanar vectors, equating corresponding coefficients on both sides, $\bar{a} \cdot \bar{c} = \frac{1}{2}$ and $\bar{a} \cdot \bar{b} = 0$.

$\therefore \bar{a}$ makes angle $\pi/3$ with $\bar{c}$ and is perpendicular to $\bar{b}$.

6. What is the value of tan 20° + tan 40° + $\sqrt{3}$ tan 20° tan 40°?

Sol. tan 20° + tan 40° + $\sqrt{3}$ tan 20° tan 40°

Consider 20° + 40° = 60°

Tan(20° + 40°) = tan 60°

$$\frac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°} = \sqrt{3}$$

$$\tan 20° + \tan 40° = \sqrt{3}(1 - \tan 20° \tan 40°)$$

$$\tan 20° + \tan 40° = \sqrt{3} - \sqrt{3}\tan 20° \tan 40°$$

$$\tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40° = \sqrt{3}$$

7. Prove that $\cos 20° \cos 40° - \sin 5° \sin 25° = \frac{\sqrt{3}+1}{4}$.

Sol. $\cos 20° \cos 40° - \sin 5° \sin 25°$

$$= \frac{1}{2}[2\cos 40° \cos 20° - 2\sin 25° \sin 5°]$$

$$= \frac{1}{2}[\cos(40° + 20°) + \cos(40° - 20°)$$
$$+ \cos(25° + 5°) - \cos(25° - 5°)]$$

$$= \frac{1}{2}[\cos 60° + \cos 20° + \cos 30° - \cos 20°]$$

$$= \frac{1}{2}[\cos 60° + \cos 30°]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}+1}{4}$$

8. If $\sin hx = \frac{3}{4}$ find $\cos h2x \;\; ad \;\; \sin h2x$

Solution:

$$\sin hx = \frac{3}{4}$$

$$\cos h^2 x = 1 + \sin h^2 x$$

$$\cos h^2 x = 1 + \frac{9}{16} \Rightarrow \cos h^2 x = \frac{25}{16}$$

$$\cos hx = \frac{5}{4}$$

$$\sin^2 x = 2\sin hx \cos hx = 2 \times \frac{3}{4} \times \frac{5}{4} = \frac{15}{8}$$

$$\cos^2 x = \cos \text{h}^2 x + \sin h^2 x = \frac{25}{16} + \frac{9}{16} = \frac{34}{16} = \frac{17}{8}$$

9. For any square matrix A, show that AA′ is symmetric.

Sol. A is a square matrix

$$(AA')' = (A')'A' = A \cdot A'$$

$$\because (AA')' = AA'$$

$\Rightarrow AA'$ is a symmetric matrix.

10. Is $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ symmetric or skew symmetric 7.

Sol. Let A = $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix} = -A$$

$\therefore$ A is a skew symmetric matrix.

11. If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors such that $\bar{a}$ is perpendicular to the plane of $\bar{b}, \bar{c}$ and the angle between $\bar{b}$ and $\bar{c}$ is π/3, then find $|\bar{a} + \bar{b} + \bar{c}|$.

Sol. $\bar{a}$ perpendicular to plane contain $\bar{b}$ and $\bar{c}$.

$$\Rightarrow \bar{a} \cdot \bar{b} = 0, \; \bar{a} \cdot \bar{c} = 0$$

Consider

$|\bar{a}+\bar{b}+\bar{c}|^2 = (\bar{a}+\bar{b}+\bar{c})^2$

$= \bar{a}^2 + \bar{b}^2 + \bar{c}^2 + 2\bar{a}\bar{b} + 2\bar{b}\bar{c} + 2\bar{c}\,\bar{a}$

$= |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 + 0$

$+2|\bar{b}||\bar{c}|\cos(\bar{b},\bar{c}) + 0$

$= 1+1+2+2(1)(1)\cos\frac{\pi}{3}$

$= 3+2\times\frac{1}{2} = 3+1 = 4$

$\therefore |\bar{a}+\bar{b}+\bar{c}| = 2$

12. If $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$, then prove that $a\tan\beta = b\tan\alpha$.

Sol. Given that $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$

By using componendo and dividendo, we get

$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)} = \frac{a+b+a-b}{a+b-a+b} = \frac{2a}{2b} = \frac{a}{b}$

$\Rightarrow \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)} = \frac{a}{b}$

$\Rightarrow \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\alpha\cos\beta+\cos\alpha\sin\beta} = \frac{a}{b}$

$\Rightarrow \frac{2\sin\alpha\cos\beta}{2\cos\alpha\sin\beta} = \frac{a}{b}$

$\Rightarrow \tan\alpha\cot\beta = \frac{a}{b}$

$\Rightarrow b\tan\alpha = a\tan\beta$

hence, $a\tan\beta = b\tan\alpha$

13. Prove that tan 70° – tan 20° = 2 tan 50°.

Sol. tan 50° = tan(70° – 20°)

$$= \frac{\tan 70° - \tan 20°}{1 + \tan 70° \tan 20°}$$

$\Rightarrow \tan 70° - \tan 20°$

$= \tan 50°(1 + \tan 70° \cdot \tan 20°)$

$= \tan 50°(1 + \tan 70° \cdot \tan(90° - 70°)]$

$= \tan 50°[1 + \tan 70° \cdot \cot 70°]$

$= \tan 50°[1 + 1]$

$= 2\tan 50°$

$\therefore \tan 70° - \tan 20° = 2\tan 50°$

14.Solve the following and write the general solutions

$\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$

Solution :

$\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$

$2\cos 2\theta \,.\, \sin\theta - 2\left(2\cos^2\theta - 1\right) = 0$

$2\cos 2\theta \sin\theta - 2\cos 2\theta = 0 \Rightarrow 2\cos 2\theta\left(\sin\theta - 1\right) = 0$

$\cos 2\theta = 0 \quad : \quad \sin\theta = 1$

$2\theta = (2n+1)\pi/2 \qquad \theta = n\pi + (-1)^n \pi/2$

$\theta = (2n+1)\pi/4 \quad : \quad \theta = n\pi + (-1)^n \pi/2 \quad n \in z$

15. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then prove that \` $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

Solution:

Let $\sin^{-1}x = \alpha \quad \sin^{-1}y = \beta \quad \sin^{-1}z = \gamma$

$\sin\alpha = x \quad \sin\beta = y \quad \sin\gamma = z$

Given $\alpha + \beta + \gamma = \pi \Rightarrow \alpha + \beta = \pi - \gamma$

$\sin(\alpha + \beta) = \sin\gamma \; and \; \cos\gamma = -\cos(\alpha + \beta)$

$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$\sin\alpha\sqrt{1-\sin^2\alpha} + \sin\beta\sqrt{1-\sin^2\beta} + \sin\gamma\sqrt{1-\sin^2\gamma}$

$\frac{1}{2}\left[\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right] \quad = \quad \frac{1}{2}\left[2\sin(\alpha+\beta)\cos(\alpha-\beta) + \sin^2\gamma\right]$

$\frac{1}{2}\left[2\sin\gamma\cos(\alpha-\beta) + 2\sin\gamma\cos\gamma\right]$

$\sin\gamma\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] = 2\sin\alpha\sin\beta\sin\gamma = 2xyz$

16. In aΔABC, show that the sides a, b, c are in A.P. if and only if r_1, r_2, r_3 are in H.P.

Sol. r_1, r_2, r_3 are in H.P. $\Leftrightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P.

$\Leftrightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in A.P.

$\Leftrightarrow s-a, s-b, s-c$ are in A.P.

$\Leftrightarrow$ –a, –a, –c are in A.P.

$\Leftrightarrow$a, b, c are in A.P.

17. Apply the test of rank to examine whether the following equations are consistent.
2x – y + 3z = 8, –x + 2y + z = 4, 3x + y = 4z = 0 and, if consistent, find the complete solution.

Sol. The augmented matrix is $\begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

(on interchange R_1 and R_2)

$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 + 3R_1$

$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix}$

$R_3 \to 3R_3 - 7R_2$

$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix}$

Now det $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38 \end{bmatrix} = (-1)(3)(-38) = 144$

Hence rank (A) = rank [AD] = 3

$\therefore$ The system has a unique solution.

18. If $f:A\to B$, $g:B\to A$ are two functions such that $gof=I_A$ and $fog=I_B$ then $f:A\to B$ is a bijection and $f^{-1}=g$.

Proof: Let $x_1,x_2\in A,\ f(x_1)=f(x_2)$

$x_1,x_2\in A,\ f:A\to B\Rightarrow f(x_1),f(x_2)\in B$

$f(x_1),f(x_2)\in B,\ f(x_1)=f(x_2)$, $g=B\to A\Rightarrow g[f(x_1)]=g[f(x_2)]$

$\Rightarrow(gof)(x_1)=(gof)(x_2)\Rightarrow I_A(x_2)\Rightarrow x_1=x_2$

$\therefore x_1,x_2\in A,\ f(x_1)=f(x_2)\Rightarrow x_1=x_2 \therefore f:A\to B$ is one one

Let $y\in B$.

$y\in B, g:B\to A\Rightarrow g(y)\in A$

19. $1^2+(1^2+2^2)+(1^2+2^2+3^3)+..........$ upto n terms $=\dfrac{n(n+1)^2(n+2)}{12}$

Sol: $1^2+(1^2+2^2)+(1^2+2^2+3^2)+........+(1^2+2^2+3^2+.....n^2)=\dfrac{n(n+1)^2(n+2)}{12}$

$1^2+(1^2+2^2)+(1^2+2^2+3^2)+........=\dfrac{n(n+1)(2n+1)}{6}=\dfrac{n(n+1)^2(n+2)}{12}$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S = 1

R.H.S = $\dfrac{1(1+1)^2(1+2)}{12}$

Assume S_k is true

$1^2+(1^2+2^2)+(1^2+2^2+3^2)+...+\dfrac{k(k+1)(2k+1)}{6}=\dfrac{k(k+1)^2(k+2)}{12}$

Adding $(k+1)^{th}$ term i.e. $=\dfrac{(k+1)(k+2)(2k+3)}{6}$ on both sides

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+.....+=\frac{(k+1)(k+2)(2k+3)}{6}=\frac{k(k+1)^2(k+2)}{12}+\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{k(k+1)^2(k+2)+2(k+1)(k+2)(2k+3)}{12}$$

$$=\frac{(k+1)(k+2)\ \ k^2+k+4k+6}{12}$$

$$=\frac{(k+1)(k+2)\ \ k^2+5k+6}{12}$$

$$=\frac{(k+1)(k+2)(k+2)(k+3)}{12}$$

$$=\frac{(k+1)(k+2)^2(k+3)}{12}$$

20.If $\left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{d}\right]+\left[\bar{a}\ \ \bar{b}\ \ \bar{d}\right]=\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$ then show that $\bar{a},\bar{b},\bar{c},\bar{d}$ are coplanar.

Sol. Let O be the origin, then

$\overline{OA}=\bar{a},\overline{OB}=\bar{b},\overline{OC}=\bar{c},\overline{OD}=\bar{d}$ are position vectors.

Then $\overline{AB}=\bar{b}-\bar{a},\overline{AC}=\bar{c}-\bar{a}$ and $\overline{AD}=\bar{d}-\bar{a}$

The vectors $\overline{AB},\overline{AC},\overline{AD}$ are coplanar.

$\therefore\left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right]=0$

$\Rightarrow\left[\bar{b}-\bar{a}\ \ \ \bar{c}-\bar{a}\ \ \ \bar{d}-\bar{a}\right]=0$

$\Rightarrow(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})\cdot(\bar{d}-\bar{a})=0$

$\Rightarrow(\bar{b}\times\bar{c}-\bar{b}\times\bar{a}-\bar{a}\times\bar{c}+\bar{a}\times\bar{a})\cdot(\bar{d}-\bar{a})=0$

$\Rightarrow(\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a})\cdot(\bar{d}-\bar{a})=0$

$(\because\bar{a}\times\bar{a}=0)$

$\Rightarrow(\bar{b}\times\bar{c})\cdot\bar{d}+(\bar{a}\times\bar{b})\cdot\bar{d}+(\bar{c}\times\bar{a})\cdot\bar{d}-(\bar{b}\times\bar{c})\cdot\bar{a}-(\bar{a}\times\bar{b})\cdot\bar{a}-(\bar{c}\times\bar{a})\cdot\bar{a}=0$

$\Rightarrow(\bar{b}\times\bar{c})\cdot\bar{d}+(\bar{a}\times\bar{b})\cdot\bar{d}+(\bar{c}\times\bar{a})\cdot\bar{d}-(\bar{b}\times\bar{c})\cdot\bar{a}=0$

$\Rightarrow\left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right]+\left[\bar{a}\ \ \bar{b}\ \ \bar{d}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{d}\right]=\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

21. If $\cos A+\cos B+\cos C=\frac{3}{2}$ then show that the triangle is equilateral

Solution: -

$$\cos A+\cos B+\cos C=\frac{3}{2}\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$$

$$2\cos\left(90^0-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$$

$$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=3/2$$

$$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-2\sin^2\frac{C}{2}=\frac{1}{2}$$

$$4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-4\sin^2\frac{C}{2}=1\Rightarrow 1+4\sin^2\frac{C}{2}-4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)=0$$

$$\left(2\sin\frac{C}{2}\right)^2-2\left(2\sin\frac{C}{2}\cos\frac{A-B}{2}\right)+\cos^2\left(\frac{A-B}{2}\right)-\cos^2\left(\frac{A-B}{2}\right)+1=0$$

$$\left\{2\sin\frac{C}{2}-\cos\left(\frac{A-B}{2}\right)\right\}^2+\sin^2\left(\frac{A-B}{2}\right)=0$$

$$\therefore 2\sin\frac{C}{2} - \cos\left(\frac{A-B}{2}\right) = 0 \; and \; \sin\frac{A-B}{2} = 0$$

$$\therefore 2\sin\frac{C}{2} = \cos\left(\frac{A-B}{2}\right) \; and \; A - B = 0$$

$$\therefore 2\sin\frac{C}{2} = 1 \Rightarrow \frac{C}{2} = 30^0 \Rightarrow c = 60^0$$

$$A = B \therefore A = B = 60^0$$

Hence trangile is equilateral

22. Show that $a\cos^2\frac{A}{2} + b\cos^2\frac{B}{2} + c\cos^2\frac{C}{2} = s + \frac{\Delta}{R}$

Solution: -

$$a\cos^2\frac{A}{2} + b\cos^2\frac{B}{2} + c\cos^2\frac{C}{2}$$

$$\frac{a \; 1+\cos A}{2} + \frac{b \; 1+\cos B}{2} + \frac{c \; 1+\cos C}{2}$$

$$\frac{a + a\cos A + b + b\cos B + C + c\cos C}{2} = \frac{a+b+c \; +}{2}$$

$$\frac{a+b+c \; + \; a\cos A + b\cos B + c\cos C}{2}$$

$$\frac{2s + 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C}{2}$$

$$\frac{2S + R \; \sin 2A + \sin 2B + \sin 2C}{2}$$

$$\frac{2S + R \; 2\sin \; A+B \; \cos \; A-B \; + \sin 2C}{2}$$

$$\frac{2S + R \; 2\sin C\cos \; A-B \; + 2\sin C\cos C}{2}$$

$$\frac{2S + 2R\sin C \; \cos \; A-B \; + \cos C}{2}$$

$$\frac{2S + 2R\sin C \; \cos \; A-B \; - \cos \; A+B}{2}$$

$$\frac{2S + 4R\sin A\sin B\sin C}{2} = S + 2\text{R}\sin A\sin B\sin C$$

$$\frac{S + 2R^2\sin A\sin B\sin C}{R} = S + \frac{\Delta}{R}$$

23. Show that

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

Sol. L.H.S = $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2 - a^2 & b^2 - a^3 \\ 0 & c^2 - a^2 & c^3 - a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2 + ba + a^2 \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$= -(a-b)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b-c & b^2 - c^2 + a(b-c) \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$$

$$= -(a-b)(c-a)(b-c)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & 1 & b+c+a \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$$

$$= -(a-b)(b-c)(c-a)$$
$$[(c^2 + ca + a^2) - (b+c+a)(c+a)]$$

$$= -(a-b)(b-c)(c-a)$$
$$[c^2 + ca + a^2 - b(c+a) - (c+a)^2 =$$

$$= -(a-b)(b-c)(c-a)$$
$$[c^2 + ca + a^2 - bc - ab - c^2 - 2ca - a^2]$$

$$= -(a-b)(b-c)(c-a)[-ab - bc - ca]$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

24. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then show that $A^{-1} = A^3$.

Sol. $A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$

$$A^4 = A^2A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

$\therefore A^4 = I$

$\det A = 3(1) - 3(-2) + 4(-2) = 1$

$\because A \neq 0 \Rightarrow A^{-1}$ exists

$\because A^4 = I$

Multiply with A^{-1}

$A^4(A^{-1}) = I(A^{-1})$

$\Rightarrow A^3(AA^{-1}) = A^{-1} \Rightarrow A^3(I) = A^{-1}$

$\therefore A^{-1} = A^3$