MATHEMATICS PAPER IA

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. 10X2 =20

1. If $f(y)=\frac{y}{\sqrt{1-y^2}}$ and $g(y)=\frac{y}{\sqrt{1+y^2}}$ then show that (fog)(y) = y.

2. Prove that the real valued function $f(x)=\frac{x}{e^x-1}-\frac{x}{2}+1$ is an even function on R – {0}.

3.Write direction ratios of the vector $a=\bar{i}+\bar{j}-2\bar{k}$ and hence calculate its direction cosines.

4.If $\overline{OA}=\bar{i}+\bar{j}+\bar{k},\ \overline{AB}=3\bar{i}-2\bar{j}+\bar{k},\ \overline{BC}=\bar{i}+2\bar{j}-2\bar{k}$ and $\overline{CD}=2\bar{i}+\bar{j}+3\bar{k}$ then find the vector of $\overline{OD}$.

5. If $|\bar{a}|=13, |\bar{b}|=5$ and $\bar{a}\cdot\bar{b}=60$, then find $|\bar{a}\times\bar{b}|$.

6. Evaluate $\sin^2 82\frac{1°}{2}-\sin^2 22\frac{1°}{2}$.

7. Prove that $\cos 48^0 \cos 12^0=\frac{3+\sqrt{5}}{8}$

8.Prove that $\cos h^{-1}x=\log_e \; x-\sqrt{x^2-1}$

9. Show that the value of the determinant of skew-symmetric matrix of order three is always zero.

10. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$.

SECTION B

SHORT ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING 5 X 4 = 20

11.If $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along the positive directions of the coordinate axes, then show that the four points $4\bar{i}+5\bar{j}+\bar{k}, -\bar{j}-\bar{k}$, $3\bar{i}+9\bar{j}+4\bar{k}$ and $-4\bar{i}+4\bar{j}+4\bar{k}$ are coplanar.

12. The points O,A,B,X and Y are such that $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OX}=3\vec{a}$ and $\overrightarrow{OY}=3\vec{b}$ .find $\overrightarrow{BX}$ and $\overrightarrow{AY}$ in terms of $\vec{a}$ and $\vec{b}$ further if P divides AY in the ratio 1:3 then express $\overrightarrow{BP}$ in terms of $\vec{a}$ and $\vec{b}$.

13. The angle in semi circle is a right angle

14. Prove that $\sin^2\alpha + \cos^2(\alpha + \beta) + 2\sin\alpha\sin\beta\cos(\alpha + \beta)$ is independent of α.

15. Solve the following equations

$$6\tan^2 x - 2\cos^2 x = \cos 2x$$

16. If $Tan^{-1}x + Tan^{-1}y + Tan^{-1}z = \pi$ then prove that $x + y + z = xyz$

17. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

SECTION C

LONG ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING 5 X 7 = 35

18. If $f: A \to B$ is a bijection, then $f^{-1} of = I_A, fof^{-1} = I_B$

19. by induction prove that $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$upto n terms $= \frac{n}{24}[2n^2 + 9n + 13]$

20. Find the shortest distance between the lines $\bar{r} = 6\bar{i} + 2\bar{j} + 2\bar{k} + \lambda(\bar{i} - 2\bar{j} + 2\bar{k})$ and $\bar{r} = -4\bar{i} - \bar{k} + \mu(3\bar{i} - 2\bar{j} - 2\bar{k})$.

21. In a triangle ABC prove that

(i) $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$

22. If $\frac{a^2+b^2}{a^2-b^2} = \frac{\sin C}{\sin\ A-B}$ then prove that triangle ABC is either isosceles or right angled

23. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

24. By Gauss Jordan method Solve x + y + z = 9

2x + 5y + 7z = 52

2x + y – z = 0

SOLUTIONS

1. If $f(y)=\dfrac{y}{\sqrt{1-y^2}}$ and $g(y)=\dfrac{y}{\sqrt{1+y^2}}$ then show that (fog)(y) = y.

Sol. Given that

$$f(y)=\frac{y}{\sqrt{1-y^2}} \text{ and } g(y)=\frac{y}{\sqrt{1+y^2}}$$

$$\therefore \text{fog}(y)=f[g(y)]=f\left[\frac{y}{\sqrt{1-y^2}}\right]$$

$$=\frac{y}{\sqrt{1+y^2}}\Bigg/\sqrt{1-\left(\frac{y}{\sqrt{1+y^2}}\right)^2}$$

$$=\frac{y}{\sqrt{1+y^2}}\times\frac{\sqrt{1+y^2}}{1+y^2-y^2}=y$$

$$\therefore \text{fog}(y)=y$$

2.Prove that the real valued function $f(x)=\dfrac{x}{e^x-1}-\dfrac{x}{2}+1$ is an even function on R – {0}.

Sol. $f(x)=\dfrac{x}{e^x-1}-\dfrac{x}{2}+1 \quad \ldots(1)$

Let $x \in R-\{0\}$

Consider

$$f(x)=\frac{-x}{e^{-x}-1}+\frac{x}{2}+1$$

$$=\frac{-x}{\frac{1}{e^x}-1}+\frac{x}{2}+1$$

$$=\frac{-xe^x}{1-e^x}+\frac{x}{2}+1=\frac{-xe^x}{-(e^x-1)}+\frac{x}{2}+1$$

$$=\frac{xe^x}{e^x-1}+\frac{x}{2}+1 \qquad \ldots(2)$$

Consider f(x) – f(–x)

$$= \frac{x}{e^x - 1} - \frac{x}{2} + 1 - \frac{xe^x}{e^x - 1} - \frac{x}{2} - 1$$

$$= \frac{x - xe^x}{e^x - 1} - \frac{2x}{2}$$

$$= \frac{x(e^x - 1)}{(e^x - 1)} - x$$

$$= x - x = 0$$

$$f(x) - f(-x) = 0$$

$$\Rightarrow f(-x) = f(x)$$

$\therefore$ f is an even function.

3.Write direction ratios of the vector $a = \bar{i} + \bar{j} - 2\bar{k}$ and hence calculate its direction cosines.

Sol. Note that direction ratios a, b, c of a vector $r = x\bar{i} + y\bar{j} + z\bar{k}$ are just the respective components c, y and z of the vector. So, for the given vector, we have a = 1, b=1, c= –2. Further, if l, m and n the direction cosines of the given vector, then

$$l = \frac{a}{|r|} = \frac{1}{\sqrt{6}}, m = \frac{b}{|r|} = \frac{1}{\sqrt{6}}, n = \frac{c}{|r|} = -\frac{2}{\sqrt{6}} \text{ as } |r| = \sqrt{6}$$

Thus, the direction cosines are

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right).$$

4.If $\overline{OA} = \bar{i} + \bar{j} + \bar{k}$, $\overline{AB} = 3\bar{i} - 2\bar{j} + \bar{k}$, $\overline{BC} = \bar{i} + 2\bar{j} - 2\bar{k}$ and $\overline{CD} = 2\bar{i} + \bar{j} + 3\bar{k}$ then find the vector of $\overline{OD}$.

Sol. $\overline{OD} = \overline{OA} + \overline{AB} + \overline{BC} + \overline{CD}$

$$= \bar{i} + \bar{j} + \bar{k} + 3\bar{i} - 2\bar{j} + \bar{k} + \bar{i} + 2\bar{j} - 2\bar{k} + 2\bar{i} + \bar{j} + 3\bar{k}$$

$$\overline{OD} = 7\bar{i} + 2\bar{j} + 3\bar{k}$$

5. If $|\bar{a}| = 13, |\bar{b}| = 5$ and $\bar{a} \cdot \bar{b} = 60$, then find $|\bar{a} \times \bar{b}|$.

Sol. Given $|\bar{a}| = 13, |\bar{b}| = 5$ and $\bar{a} \cdot \bar{b} = 60$

We know that

$$|\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}\cdot\bar{b})^2$$
$$= 169\cdot 25 - 3600$$
$$= 25(169-144) = 625$$
$$|\bar{a}\times\bar{b}|^2 = 625$$
$$\therefore |\bar{a}\times\bar{b}| = 25$$

6. Evaluate $\sin^2 82\frac{1^\circ}{2} - \sin^2 22\frac{1^\circ}{2}$.

Sol. Put $A = \sin^2 82\frac{1^\circ}{2}$ and $B = \sin^2 22\frac{1^\circ}{2}$, then

$$\sin^2 82\frac{1^\circ}{2} - \sin^2 22\frac{1^\circ}{2}$$
$$= \sin^2 A - \sin^2 B$$
$$= \sin(A+B)\sin(A-B)$$
$$= \sin 105^\circ \sin 60^\circ$$
$$= \sin(90^\circ + 15^\circ)\sin 60^\circ$$
$$= \cos 15^\circ \sin 60^\circ$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

7. Prove that $\cos 48^0 \cos 12^0 = \frac{3+\sqrt{5}}{8}$

Solution:

$$\cos 48^0 \cos 12^0 = \frac{1}{2}\; 2\cos 48^0 + \cos 12^0 \;= \frac{1}{2}\; \cos 60^0 + \cos 36^0$$
$$\frac{1}{2}\left\{\frac{1}{2} + \frac{\sqrt{5}+1}{4}\right\} = \frac{2+\sqrt{5}+1}{8} = \frac{\sqrt{5}+3}{8}$$

8.Prove that $\cos h^{-1}x = \log_e \; x - \sqrt{x^2-1}$

Solution:

Let $\cos h^{-1}x = y \Rightarrow x = \cos hy$

$$x = \frac{e^y + e^{-y}}{2} \Rightarrow 2x = e^y + \frac{1}{e^y}$$
$$2xe^y = \; e^y \;^2 + 1 \Rightarrow \; e^y \;^2 - 2xe^y + 1$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \Rightarrow e^y = x \pm \sqrt{x^2 - 1}$$

$$e^y = x + \sqrt{x^2 - 1} \Rightarrow y = \log_e \ x + \sqrt{x^2 - 1}$$

$$\boxed{\cos h^{-1} x = \log_e \ x + \sqrt{x^2 - 1}}$$

9. Show that the value of the determinant of skew-symmetric matrix of order three is always zero.

Sol. Let us consider a skew-symmetric matrix of order 3 × 3, say

$$A = \begin{bmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} \because |B| = |B^T|$$

$$= -|A| \Rightarrow 2|A| = 0$$

Hence |A| = 0.

10. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$.

Sol. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -5 - 6 = -11 \neq 0$

Hence A is invertible.

The cofactor matrix of A = $\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$$\therefore \text{Adj}A = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj}A}{\det A} = -\frac{1}{11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

11.If $\overline{i}, \overline{j}, \overline{k}$ are unit vectors along the positive directions of the coordinate axes, then show that the four points $4\overline{i}+5\overline{j}+\overline{k}, -\overline{j}-\overline{k}$, $3\overline{i}+9\overline{j}+4\overline{k}$ and $-4\overline{i}+4\overline{j}+4\overline{k}$ are coplanar.

Sol. Let O be a origin, then

$$\overline{OA} = 4\overline{i}+5\overline{j}+\overline{k},\ \overline{OB} - \overline{j} - \overline{k}$$

$$\overline{OC} = 3\overline{i}+9\overline{j}+4\overline{k}, \text{ and } \overline{OD} = -4\overline{i}+4\overline{j}+4\overline{k}$$

$$\overline{AB} = \overline{OB} - \overline{OA} = -4\overline{i} - 6\overline{j} - 2\overline{k}$$

$$\overline{AC} = \overline{OC} - \overline{OA} = -\overline{i} + 4\overline{j} + 3\overline{k}$$

$$\overline{AD} = \overline{OD} - \overline{OA} = -8\overline{i} - \overline{j} + 3\overline{k}$$

$$\left[\overline{AB}\quad \overline{AC}\quad \overline{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4[12+3] + 6[-3+24] - 2[1+32]$$

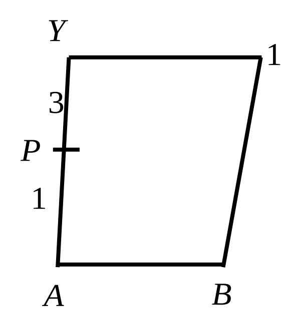
$$= -4\times 15 + 6\times 21 - 2\times 33$$

$$= -60 + 126 - 66$$

$$= -126 + 126 = 0$$

12. The points O,A,B,X and Y are such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OX} = 3\vec{a}$ and $\overrightarrow{OY} = 3\vec{b}$ .find $\overrightarrow{BX}$ and $\overrightarrow{AY}$ in terms of $\vec{a}$ and $\vec{b}$ further if P divides AY in the ratio 1:3 then express $\overrightarrow{BP}$ in terms of $\vec{a}$ and $\vec{b}$.

Sol: $\overrightarrow{BX} = \overrightarrow{OX} = \overrightarrow{OB} = 3\vec{a} - \vec{b}$



$$\overrightarrow{AY} = \overrightarrow{OY} = \overrightarrow{Oa} = 3\vec{b} - \vec{a}$$

$$\overrightarrow{OP} = \frac{1\times\overrightarrow{OY} + 3\overrightarrow{OA}}{4}$$

$$\overrightarrow{OP} = \frac{3\vec{b} + 3\vec{a}}{4}$$

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \frac{3\vec{b}+3\vec{a}}{4} - \vec{b} = \frac{3\vec{b}+3\vec{a}}{4} - 4\vec{b}$$

$$= \frac{1}{4}(3\vec{a} - \vec{b})$$

13. The angle in semi circle is a right angle

Proof: Let APB be a semi circle with centre at O.

OA = OB = OP also $\overrightarrow{OB} = -\overrightarrow{OA}$

$\overrightarrow{AP}.\overrightarrow{BP} = (\overrightarrow{OP} - \overrightarrow{OA}).(\overrightarrow{OP} - \overrightarrow{OA})$

$= (\overrightarrow{OP} - \overrightarrow{OA}).(\overrightarrow{OP} + \overrightarrow{OA}) \qquad \because \overrightarrow{OB} = -\overrightarrow{OA}$

$= (\overrightarrow{OP})^2 - (\overrightarrow{OA})^2 \qquad \because (\vec{a} + \vec{b}0.(\vec{a} - \vec{b}) = (\vec{a})^2 - (\vec{b})^2$

$= \left|\overrightarrow{OP}\right|^2 - \left|\overrightarrow{OA}\right|^2 = OP^2 - OP^2 = 0 \qquad \{\because OA = OP\}$

$\overrightarrow{AP}.\overrightarrow{BP} = 0 \qquad \therefore \overrightarrow{AP} \perp r\,\overrightarrow{PB}$ Hence $\angle APB = 90^0$

Hence angle in semi –cricle is 90^0

14. Prove that $\sin^2\alpha + \cos^2(\alpha + \beta) + 2\sin\alpha\sin\beta\cos(\alpha + \beta)$ is independent of α.

Sol. Given expression,

$\sin^2\alpha + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= \sin^2\alpha + 1 - \sin^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 + [\sin^2\alpha - \sin^2(\alpha+\beta)] + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 + \sin(\alpha+\alpha+\beta)\sin(\alpha-\alpha-\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 + \sin(2\alpha+\beta)\sin(-\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= 1 - \sin(2\alpha+\beta)\sin\beta + [2\sin\alpha\cos(\alpha+\beta)]\sin\beta$

$= 1 - \sin(2\alpha+\beta)\sin\alpha + [\sin(\alpha+\alpha+\beta) + \sin(\alpha-\alpha-\beta)]\sin\beta$

$= 1 - \sin(2\alpha+\beta)\sin\alpha + [\sin(2\alpha+\beta) - \sin\beta]\sin\beta$

$= 1 - \sin(2\alpha+\beta)\sin\alpha + \sin(2\alpha+\beta)\sin\beta - \sin^2\beta$

$= 1 - \sin^2\beta = \cos^2\beta$

Thus the given expression is independent of α.

15. Solve the following equations

(i) $6\tan^2 x - 2\cos^2 x = \cos 2x$

Solution :

$6\tan^2 x = 4\cos^2 x - 1$

$6\sin^2 x = \cos^2 x \;\; 4\cos^2 x - 1 \Rightarrow 6 \;\; 1 - \cos^2 x \;\; = 4\cos^4 x - \cos^2 x$

$4\cos^4 x + 5\cos^2 x - 6 = 0 \Rightarrow 4\cos^4 x + 8\cos^2 x - 3\cos^2 x - 6 = 0$

$4\cos^2 x \;\; \cos^2 x + 2 \;\; - 3 \;\; \cos^2 x + 2 \;\; = 0$

$\cos^2 x = \frac{3}{4} \Rightarrow \cos^2 x = \cos^2 \frac{\pi}{6}$

$x = n\pi \pm \pi/6$

16. If $Tan^{-1}x + Tan^{-1}y + Tan^{-1}z = \pi$ then prove that $x + y + z = xyz$

Solution:

Let $\tan^{-1} x = \alpha$ $\tan^{-1} y = \beta$ $\tan^{-1} z = \gamma$

$x = Tan\alpha$ $y = Tan\beta$ $z = Tan\gamma$

Given $\alpha + \beta + \gamma = \pi \Rightarrow \alpha + \beta = \pi - x$

$$Tan\ \alpha + \beta\ = Tan(\pi - \gamma) \Rightarrow \frac{Tan\alpha + Tan\beta}{1 - Tan\alpha Tan\beta} = Tan\gamma$$

$$Tan\alpha + Tan\beta = -Tan\gamma + Tan\alpha Tan\beta Tan\gamma \Rightarrow Tan\alpha + Tan\beta + Tan\gamma\ Tan\alpha Tan\beta Tan\gamma$$

$$= x + y + z\ = xyz$$

17. Find the inverse of A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Det A = 1(4 – 3) – 2(6 – 3) + 1(3 – 2)

= 1 – 6 + 1 = –4

The cofactors of elements of A are

$A_{11} = +(4-3) = 1$

$A_{12} = -(6-3) = -3$

$A_{13} = +(3-2) = 1$

$A_{21} = -(4-1) - 3$

$A_{22} = +(2-1) = 1$

$A_{23} = -(1-2) = 1$

$A_{31} = +(6-2) = 4$

$A_{32} = -(3-3) = 0$

$A_{33} = +(2-6) = -4$

$$\therefore \text{Adj A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj A}}{\det A} = -\frac{1}{4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

18. If $f: A \to B$ is a bijection, then $f^{-1}of = I_A, fof^{-1} = I_B$

Proof: Since $f: A \to B$ is a bijection $f^{-1}: B \to A$ is also a bijection and

$f^{-1}(y) = x \Leftrightarrow f(x) = y \forall y \in B$

$f: A \to B$, $f^{-1}: B \to A \Rightarrow f^{-1}of: A \to A$

Clearly $I_A: A \to A$ such that $I_A(x) = x, \forall x \in A$.

Let $x \in A$

$x \in A, f: A \to B \Rightarrow f(x) \in B$

Let y = f(x)

$y = f(x) \Rightarrow f^{-1}(y) = x$

$(f^{-1}of)(x) = f^{-1}[f(x) = f^{-1}(y) = x = I_A(x)$

$\therefore (f^{-1}of)(x) = I_A(x) \forall x \in A$ $\quad \therefore f^1 of = I_A$

$f^1: B \to A$, $f: A \to B \to fof^1: B \to B$

Clearly $I_B: B \to B$ such that $I_B(y) = y \forall y \in B$

Let $y \in B$

$y \in B, f^{-1}: B \to A = f^1(y) \in A$

Let $f^1(y) = x$

$f^1(y) = x \Rightarrow f(x) = y$

$(fof^1)(y) = f[f^1(y)] = f(x) = y = I_B(y)$

$\therefore (fof^{-1})(y) = I_B(y) \forall y \in B$ $\quad \therefore fof^{-1} = I_B$

19. by induction prove that $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots\ldots$upto n terms

$= \frac{n}{24}[2n^2 + 9n + 13]$

Sol: $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots\ldots\ldots\ldots\ldots + \frac{1^3+2^3+3^3+\ldots..+n^3}{1+3+5+\ldots.+(2n-1)} = \frac{n}{24}\left[2n^2 + 9n + 13\right]$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S $= \frac{1^3}{1} = 1$

R.H.S = $= \frac{1}{24}[2+9+13] = 1$

L.H.S = R.H.S

Hence $S_{(1)}$ is true

Assume S_k is true

$$\therefore \frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....+\frac{1^3+2^3+3^3+.......+k^3}{1+3+5+.....2(k-1)}=\frac{k}{24}\left[2k^2+9k+13\right]$$

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....+\frac{k^2(k+1)^2}{4k^2}=\frac{k}{24}\left[2k^2+9k+13\right]$$

Adding $\frac{(k+2)^2}{4}$ on both sides

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{13+5}+.....+\frac{(k+1)^2}{4}+\frac{(k+2)^2}{4}=\frac{k}{24}\;2k^2+9k+13\;+\frac{(k+2)^2}{4}$$

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{13+5}+....+\frac{(k+2)^2}{4}=\frac{k\;\;2k^2+9k+13\;\;+6(k^2+4k+4)}{24}$$

$$=\frac{2k^2+9k+13+6k^2+24k+24}{24}$$

$$=\frac{2k^2+15k^2+37k+24}{24}$$

$k=-1$	2	15	37	2
	0	–2	–13	–24
	2	13	24	0

$$=\frac{(k+1)\;\;(2k^2+13k+24}{24}$$

$$=\frac{(k+1)\;\;2(k^2+2k+1)+9(k+1)+13}{24}$$

$$=\frac{(k+1)\;\;2(k+1)^2+9(k+1)+13}{24}$$

$\therefore S_{k+1}$ is true

$\therefore S_n$ is true $\forall n\in N$

20. Find the shortest distance between the lines $\bar{r}=6\bar{i}+2\bar{j}+2\bar{k}+\lambda(\bar{i}-2\bar{j}+2\bar{k})$ and $\bar{r}=-4\bar{i}-\bar{k}+\mu(3\bar{i}-2\bar{j}-2\bar{k})$.

Sol. Given lines are

$\bar{r}=6\bar{i}+2\bar{j}+2\bar{k}+\lambda(\bar{i}-2\bar{j}+2\bar{k})$

$\bar{r}=-4\bar{i}-\bar{k}+\mu(3\bar{i}-2\bar{j}-2\bar{k})$

Let $\bar{a}=6\bar{i}+2\bar{j}+2\bar{k},\bar{b}=\bar{i}-2\bar{j}+2\bar{k}$

$\bar{c}=-4\bar{i}-\bar{k},\bar{d}=3\bar{i}-2\bar{j}-2\bar{k}$

Shortest distance between the given lines is

$$\frac{\left|[\bar{a}-\bar{c}\;\;\bar{b}\;\;\bar{d}]\right|}{|\bar{b}\times\bar{d}|}$$

$\bar{a}-\bar{c}=10\bar{i}+2\bar{j}+3\bar{k}$

$$\left|[\bar{a}-\bar{c}\ \ \bar{b}\ \ \bar{d}]\right|=\begin{vmatrix}10 & 2 & 3\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$$

$=10(4+4)-2(-2-6)+3(-2+6)$

$=80+16+12=108$

$$[\bar{b}\times\bar{d}]=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$$

$=\bar{i}(4+4)-\bar{j}(-2-6)+\bar{k}(-2+6)$

$=8\bar{i}+8\bar{j}+4\bar{k}$

$|\bar{b}\times\bar{d}|=\sqrt{64+64+16}=\sqrt{144}=12$

$\therefore$ Distance = $\frac{108}{12}=9$ units.

21. In a triangle ABC prove that

(i) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$

Solution:

Given $A+B+C=\pi$

R.H..S

$$4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)=2\left\{\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\right\}\left\{2\cos\left(\frac{\pi-C}{4}\right)\right\}$$

$$=\left\{\cos\left(\frac{\pi-A+\pi-B}{}\right)+\cos\left(\frac{\pi-A-\pi+B}{4}\right)\right\}\left\{2\cos\left(\frac{\pi-C}{4}\right)\right\}$$

$\because 2\cos A\cos B=\cos\ \ A+B\ \ +\cos\ \ A+B$

$$=\left\{\cos\left\{\frac{\pi}{2}-\left(\frac{A+B}{4}\right)\right\}+\cos\left(\frac{A-B}{4}\right)\right\}2\cos\left(\frac{\pi-C}{4}\right)$$

$=2\cos\frac{\pi-C}{4}\sin\left(\frac{A+B}{4}\right)+2\cos\left(\frac{\pi-C}{4}\right)\cos\left(\frac{A-B}{4}\right)\ \because\ \cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)=\sin\left(\frac{A+B}{4}\right)$

$$=\sin\left(\frac{\pi-C+A+B}{4}\right)-\sin\left(\frac{\pi-C-A-B}{4}\right)+\cos\left(\frac{\pi-C+A-B}{4}\right)+\cos\left(\frac{\pi-C-A+B}{4}\right)$$

$$\left\{\begin{array}{l}\because 2\cos A\sin B=\sin\ \ A+B\ \ -\sin\ \ A-B\\ 2\cos A\cos B=\cos\ \ A+B\ \ +\cos\ \ A-B\end{array}\right\}$$

$$\therefore \sin\left(\frac{\pi - C + \pi - C}{4}\right) - \sin\left\{\frac{A+B+C-C-A-B}{4}\right\} + \cos\left(\frac{A+B+C-C+A-B}{4}\right)$$

$$+\cos\left\{\frac{A+B+C-C-A+B}{4}\right\}\left\{\begin{array}{l}\because \pi = A+B+C \\ ad\ A+B=\pi - C\end{array}\right\}$$

$$=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)+\cos\frac{A}{2}+\cos\frac{B}{2}$$

$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$$

22. If $\frac{a^2+b^2}{a^2-b^2} = \frac{\sin C}{\sin\ A-B}$ then prove that triangle ABC is either isosceles or right angled

Solution : -

Given $\frac{a^2+b^2}{a^2-b^2} = \frac{\sin C}{\sin\ A-B}$

$\Rightarrow\ a^2+b^2\ \sin\ A-B\ =\ a^2-b^2\ \sin C$

Using sine rule we have

$4R^2\ \{\sin^2 A+\sin^2 B\}\sin(A-B) = 4R^2\ \{\sin^2 A-\sin^2 B\}\sin C$

$\sin^2 A+\sin^2 B\ \sin\ A-B\ -\sin\ A-B\ \sin\ A+B\ \sin C=0$

But in triangle ABC $\sin\ A+B\ =\sin C$

$\therefore\ \sin^2 A+\sin^2 B\ \sin\ A-B\ -\sin\ A-B\ \sin C\ \ \sin C\ =0$

$\sin\ A-B\ \ \sin^2 A+\sin^2 B-\sin^2 C\ \ =0$

$\sin\ A-B\ =0\ or\ \sin^2 A+\sin^2 B=\sin^2 C$

$A=B\ or\ a^2+b^2=c^2$

$\therefore$ triangle either isosceles or right angled

23. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol. L.H.S. = $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

By applying $R_1 \Rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

By applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$$= (2)(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S.}$$

24. By Gauss Jordan method Solve $x + y + z = 9$

$2x + 5y + 7z = 52$

$2x + y - z = 0$

Sol.

Gauss Jordan method :

Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -4 & -8 & -52 \end{bmatrix}$$

$R_1 \to 3R_1 - R_2, R_3 \to 3R_3 + 4R_2$

$$A \sim \begin{bmatrix} 3 & 0 & -2 & -7 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{bmatrix}$$

$R_3 \to R_3\left(-\frac{1}{4}\right)$, we obtain

$$A \sim \begin{bmatrix} 3 & 0 & -2 & -7 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$R_1 \to R_1 + 2R_3, R_2 \to R_2 - 5R_3$, we get

$$A \sim \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$R_1 \to R_1\left(\frac{1}{3}\right), R_2 \to R_2\left(\frac{1}{3}\right)$ we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$\therefore$ The given equations have a unique solution and solution is x = 1, y = 3, z = 5.