MATHEMATICS PAPER IA

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. 10X2 =20

1. If A = {–2, –1, 0, 1, 2} and f : A → B is a surjection defined by $f(x) = x^2 + x + 1$, then find B.

2. Find the domain of the function $f(x) = \log(x^2 - 4x + 3)$

3. Find a vector in the direction of vector $a = \bar{i} - 2\bar{j}$ that has magnitude 7 units.

4. Let $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}$, $\bar{b} = \bar{i} + \bar{j} + \bar{k}$ and $\bar{c} = \bar{j} + 2\bar{k}$, find the unit vector in the opposite direction of $\bar{a} + \bar{b} + \bar{c}$.

5. Find the angle between the planes $\bar{r}\cdot(2\bar{i} - \bar{j} + 2\bar{k}) = 3$ and $\bar{r}\cdot(3\bar{i} + 6\bar{j} + \bar{k}) = 4$.

6. prove that $\cot\left(\frac{\pi}{20}\right)\cot\left(\frac{3\pi}{20}\right)\cot\left(\frac{5\pi}{20}\right)\cot\left(\frac{7\pi}{20}\right)\cot\left(\frac{9\pi}{20}\right) = 1$

7. Draw the graph of $y = \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

8. Prove that $\cos hx - \sin hx\ ^{n} = \cos h\, nx - \sin hnx$

9. If I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and E = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bE)^3 = a^3I + 3a^2bE$.

Solve the following system of homogeneous equations.

10. If A = $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then find AA^T. Do A and A^T commute with respect to multiplication of matrices?

SECTION B

SHORT ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING 5 X 4 = 20

11. If $\bar{a}+\bar{b}+\bar{c}=\alpha\bar{d}$, $\bar{b}+\bar{c}+\bar{d}=\beta\bar{a}$ and $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors then show that $\bar{a}+\bar{b}+\bar{c}+\bar{d}=0$.

12.If $\bar{a}+\bar{b}+\bar{c}=0, |\bar{a}|=3, |\bar{b}|=5$ and $|\bar{c}|=7$ then find the angle between $\bar{a}$ and $\bar{b}$.

13. Prove that $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$

14. If α, β are solutions of the equation $a\cos\theta+b\sin\theta=c$ where a, b, c $\in$ R and if $a^2+b^2>0$ $\cos\alpha\neq\cos\beta$ *and* $\sin\alpha \neq \sin\beta$ then show that

$\cos\alpha\cos\beta=\frac{c^2-b^2}{a^2+b^2}$ (iv)_ $\sin\alpha\sin\beta=\frac{c^2-a^2}{a^2+b^2}$

15. Prove that $Tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+Tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}$

16. If a : b : c = 7 : 8 : 9 then find $\cos A=\cos B=\cos C$

17. If AB = I or BA = I, then prove that A is invertible and B = A^{-1}.

SECTION C

LONG ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING 5 X 7 = 35

18.If $f:A\to B$, $g:B\to C$ are two bijections then $(gof)^{-1}=f^{-1}og^{-1}$.

19. Prove that 2.3+ 3.4 + 4.5+ …. upto n terms $\frac{n(n^2+6n+11)}{3}$

20.If $\bar{a},\bar{b}$ and $\bar{c}$ represent the vertices A, B and C respectively of ΔABC, then prove that $|(\bar{a}\times\bar{b})+(\bar{b}+\bar{c})+(\bar{c}\times\bar{a})|$ is twice the area of ΔABC.

21. If A,B,C are angles of a triangle then prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

22. Show that $a^2\cot A+b^2\cot B+c^2\cos C=\frac{abc}{R}$.

23. Prove that $\det\begin{bmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{bmatrix}$

$=(a-b)(b-c)(c-a)(ab+bc+ca)$

24. By Matrix inversion method solve : $2x - y + 8z = 13$

$3x + 4y + 5z = 18$

$5x - 2y + 7z = 20$

Solutions

1. If A = {–2, –1, 0, 1, 2} and f : A → B is a surjection defined by $f(x) = x^2 + x + 1$, then find B.

Sol. Given that

$f(x) = x^2 + x + 1$

$f(-2) = (-2)^2 - 2 + 1 = 4 - 2 + 1 = 3$

$f(-1) = (-1)^2 - 1 + 1 = 1 - 1 + 1 = 1$

$f(0) = (0)^2 - 0 + 1 = 1$

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = 2^2 + 2 + 1 = 7$

Thus range of f, f(A) = {1, 3, 7}

Since f is onto, f(A) = B

∴ B = {3, 1, 7}

2. Find the domain of the function $f(x)=\log(x^2-4x+3)$

$x^2-4x+3>0$

$(x-1)(x-3)>0$

$x<1$ or $x>3$

Since the coefficient of x^2 is +ve

Domain of f is R – [1, 3]

3.Find a vector in the direction of vector $a = \bar{i} - 2\bar{j}$ that has magnitude 7 units.

Sol. The unit vector in the direction of the given vector 'a' is

$$\hat{a} = \frac{1}{|a|}a = \frac{1}{\sqrt{5}}(\bar{i} - 2\bar{j}) = \frac{1}{\sqrt{5}}\bar{i} - \frac{2}{\sqrt{5}}\bar{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of a is

$$7a = 7\left(\frac{1}{\sqrt{5}}\bar{i} - \frac{2}{\sqrt{5}}\bar{j}\right) = \frac{7}{\sqrt{5}}\bar{i} - \frac{14}{\sqrt{5}}\bar{j}$$

4.Let $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}$, $\bar{b} = \bar{i} + \bar{j} + \bar{k}$ and $\bar{c} = \bar{j} + 2\bar{k}$, find the unit vector in the opposite direction of $\bar{a} + \bar{b} + \bar{c}$.

Sol. $\bar{a} + \bar{b} + \bar{c} = 2\bar{i} + 4\bar{j} - 5\bar{k} + \bar{i} + \bar{j} + \bar{k} + \bar{j} + 2\bar{k}$

$$= 3\bar{i} + 6\bar{j} - 2\bar{k}$$

$\therefore$ Unit vector in the direction of

$$\bar{a} + \bar{b} + \bar{c} = \pm\frac{\bar{a} + \bar{b} + \bar{c}}{|\bar{a} + \bar{b} + \bar{c}|}$$

$$= \pm\frac{3\bar{i} + 6\bar{j} - 2\bar{k}}{\sqrt{49}} = \pm\frac{3\bar{i} + 6\bar{j} - 2\bar{k}}{7}$$

5.Find the angle between the planes $\bar{r}\cdot(2\bar{i} - \bar{j} + 2\bar{k}) = 3$ and $\bar{r}\cdot(3\bar{i} + 6\bar{j} + \bar{k}) = 4$.

Sol. Given $\bar{r}\cdot(2\bar{i} - \bar{j} + 2\bar{k}) = 3$

$$\bar{r}\cdot(3\bar{i} + 6\bar{j} + \bar{k}) = 4$$

Given equation $\bar{r}\cdot\bar{n}_1 = p,\ \bar{r}\cdot\bar{n}_2 = q$

Let θ be the angle between the planes.

Then $\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

$$= \frac{(2\bar{i} - \bar{j} + 2\bar{k})\cdot(3\bar{i} + 6\bar{j} + \bar{k})}{\sqrt{2\bar{i} - \bar{j} + 2\bar{k}}\sqrt{3\bar{i} + 6\bar{j} + \bar{k}}}$$

$$= \frac{6 - 6 + 2}{\sqrt{9}\sqrt{46}} = \frac{2}{3\sqrt{46}}$$

$$\cos\theta = \frac{2}{3\sqrt{46}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{3\sqrt{46}}\right)$$

6. prove that $\cot\left(\frac{\pi}{20}\right)\cot\left(\frac{3\pi}{20}\right)\cot\left(\frac{5\pi}{20}\right)\cot\left(\frac{7\pi}{20}\right)\cot\left(\frac{9\pi}{20}\right) = 1$

Sol. $\cot\left(\frac{\pi}{20}\right) = \cot 9° = \frac{1}{\tan 9°}$

$$\cot\left(\frac{3\pi}{20}\right) = \cot 27° = \frac{1}{\tan 27°}$$

$$\cot\left(\frac{5\pi}{20}\right) = \cot 45° = 1$$

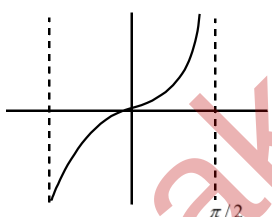
$$\cot\left(\frac{7\pi}{20}\right) = \cot 63° = \cot(90° - 27°) = \tan 27°$$

$$\cot\left(\frac{9\pi}{20}\right) = \cot 81° = \cot(90° - 9°) = \tan 9°$$

$$\therefore \cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cot\frac{5\pi}{20}\cot\frac{7\pi}{20}\cot\frac{9\pi}{20}$$

$$= \frac{1}{\tan 9°}\frac{1}{\tan 27°}\cdot 1\cdot \tan 27°\cdot \tan 9° = 1$$

7. Draw the graph of $y = \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



8. Prove that $\cos hx - \sin hx\ ^{n} = \cos h\, nx - \sin hnx$

Solution:

$$\cos hx - \sin hx\ ^{n} = \left\{\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right\}^n$$

$$= \left\{\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right\} = \ e^{-x}\ \ = e^{-nx}$$

$$\text{RHS= coshnx-sinhnx} = \frac{e^{nx} + e^{-nx}}{2} - \left(\frac{e^{nx} - e^{-nx}}{2}\right)$$

$$= \frac{e^{nx} + e^{-nx} - e^{nx} + e^{-nx}}{2} = e^{-nx}$$

9.If I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and E = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bE)^3 = a^3I + 3a^2bE$.

Sol. $aI + bE = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$$(aI + bE)^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$(aI + bE)^3 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}$$

$$= a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= a^3I + 3a^2bE$$

Solve the following system of homogeneous equations.

10.. If A = $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then find AA^T. Do A and A^T commute with respect to multiplication of matrices?

Sol. $A^T = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

11.Since $AA^T \neq A^TA$, A and A^T do not commute with respect to multiplication of matricesIf $\bar{a} + \bar{b} + \bar{c} = \alpha\bar{d}$, $\bar{b} + \bar{c} + \bar{d} = \beta\bar{a}$ and $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors then show that $\bar{a} + \bar{b} + \bar{c} + \bar{d} = 0$.

Sol. $\bar{a} + \bar{b} + \bar{c} - \alpha\bar{d} = \bar{0}$ …(1)

$\beta\bar{a} - \bar{b} - \bar{c} - \alpha\bar{d} = \bar{0}$ …(2)

(2) × (–1) ⇒

$-\beta\bar{a} + \bar{b} + \bar{c} + \alpha\bar{d} = \bar{0}$ …(3)

(1) = (3)

$\bar{a}+\bar{b}+\bar{c}-\alpha\bar{d}=-\beta\bar{a}+\bar{b}+\bar{c}+\bar{d}$

$-\beta=1\Rightarrow\beta=-1$

$-\alpha=1\Rightarrow\alpha=-1$

By substituting $\alpha = -1$ in (1) we get

$\bar{a}+\bar{b}+\bar{c}+\bar{d}=0$.

12. If $\bar{a}+\bar{b}+\bar{c}=0, |\bar{a}|=3, |\bar{b}|=5$ and $|\bar{c}|=7$ then find the angle between $\bar{a}$ and $\bar{b}$.

Sol. Given $|\bar{a}|=3, |\bar{b}|=5, |\bar{c}|=7$ and

$\bar{a}+\bar{b}+\bar{c}=0$

$\bar{a}+\bar{b}=-\bar{c}$

Squaring on both sides

$\bar{a}^2+\bar{b}^2+2\bar{a}\cdot\bar{b}=\bar{c}^2$

$\Rightarrow|\bar{a}|^2+|\bar{b}|^2|2[|\bar{a}||\bar{b}|\cos(\bar{a},\bar{b})]=|\bar{c}|^2$

$\Rightarrow 9+25+2[3.5\cos(\bar{a},\bar{b})]=49$

$\Rightarrow 2[15\cos(\bar{a},\bar{b})]=49-34$

$\Rightarrow \cos(\bar{a},\bar{b})=\frac{15}{30}$

$\Rightarrow \cos(\bar{a},\bar{b})=\frac{1}{2}=\cos\frac{\pi}{3}$

$\Rightarrow (\bar{a},\bar{b})=\frac{\pi}{3}$

$\therefore$ Angle between $\bar{a}$ and $\bar{b}$ is 60°.

13. Prove that $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$

Solution:

$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$

$\left(\sin\frac{\pi}{8}\right)^4+\left\{\sin^2\frac{\pi}{2}-\frac{\pi}{8}\right\}^4+\left\{\sin^2\left(\frac{\pi}{2}+\frac{\pi}{8}\right)\right\}^2+\left\{\sin\left(\pi-\frac{\pi}{8}\right)\right\}^4$

$\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}$

$2\left\{\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right\}=2\left\{\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)^2-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right\}$

$=2-4\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$

$$= 2 - \left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2 = 2 - \sin^2\frac{\pi}{4} = 2 - \frac{1}{2} = \frac{3}{2}$$

14. If α, β are solutions of the equation $a\cos\theta + b\sin\theta = c$ where a, b, c $\in$ R and if $a^2 + b^2 > 0$ $\cos\alpha \neq \cos\beta$ *and* $\sin\alpha \neq \sin\beta$ then show that

$\cos\alpha\cos\beta = \dfrac{c^2 - b^2}{a^2 + b^2}$ (iv)_ $\sin\alpha\sin\beta = \dfrac{c^2 - a^2}{a^2 + b^2}$

Solution:

$$a\cos\theta + b\sin\theta = c \Rightarrow (b\sin\theta)^2 = (c - a\cos\theta)^2$$

$$b^2(1 - \cos^2\theta) = c^2 + a^2\cos^2\theta - 2ac\cos\theta$$

$$(a^2 + b^2)\cos^2\theta - 2ac\cos\theta = (c^2 - b^2) = 0$$

Since α, β are solutions $\cos\alpha, \cos\beta$ are roots of above equation

$\therefore$ sum of roots $= \cos\alpha + \cos\beta = \dfrac{2ac}{a^2 + b^2}$

Product of roots $= \cos\alpha\cos\beta = \dfrac{c^2 - b^2}{a^2 + b^2}$

15. Prove that $Tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + Tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$

Solution:

Let $\cos^{-1}\frac{a}{b} = \alpha \Rightarrow \cos\alpha = \frac{a}{b}$

L.H.S $\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

$$\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}} + \frac{1 - \tan\frac{\alpha}{2}}{1 + \tan\frac{\alpha}{2}} = \frac{\left(1 + \tan\frac{\alpha}{2}\right)^2 + \left(1 - \tan\frac{\alpha}{2}\right)^2}{1 - \tan^2\frac{\alpha}{2}}$$

$$\frac{2\left\{1 + \tan^2\frac{\alpha}{2}\right\}}{1 - \tan^2\frac{\alpha}{2}} = 2\sec\alpha = \frac{2b}{a}$$

16. If a : b : c = 7 : 8 : 9 then find $\cos A = \cos B = \cos C$

Soltion : -

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{64k^2+81k^2-49k^2}{2\ 8k\ \ 9k} = \frac{\cancel{96k^2}^{12}}{2\times \cancel{8k}\times 9\cancel{k}}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{49k^2+81k^2-64k^2}{2\times 7k\times 9k} = \frac{66^{33}}{\cancel{2}\times 63} = \frac{11}{21}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49k^2+64k^2-81k^2}{2\times 7k\times 8k} = \frac{32k^2}{20k\times 8k} = \frac{2}{7}$$

$$\therefore \cos A = \cos B = \text{co}\,C = \frac{2}{3} = \frac{11}{21} = \frac{2}{7} = \frac{2}{3}\times 21 = \frac{11}{21}\times 21 = \frac{2}{7}\times 21$$

$$=14:11:6$$

17. If AB = I or BA = I, then prove that A is invertible and B = A^{-1}.

Sol. Given AB = I $\Rightarrow$ |AB| = | I |

= |A| |B| = 1

= |A| $\neq$ 0

$\therefore$ A is a non-singular matrix.

and BA = I $\Rightarrow$ |BA| = | I |

$\Rightarrow$ |B| |A| = 1 $\Rightarrow$ |A| $\neq$ 0

$\therefore$ A is a non-singular matrix.

AB = I or BA = I, A is invertible.

$\therefore$ A^{-1} exists.

AB = I $\Rightarrow$ A^{-1} AB = A^{-1} I

$\Rightarrow$ IB = A^{-1} $\Rightarrow$ B = A^{-1}

$\therefore$ B = A^{-1}

18.If $f:A\to B$, $g:B\to C$ are two bijections then $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: $f:A\to B, g:B\to C$ are bijections $\Rightarrow gof:A\to C$ is bijection $\Rightarrow (gof)^{-1}:C\to A$ is a bijection.

$f:A\to B$ is a bijection $\Rightarrow f^{-1}:B\to A$ is a bijection

$g:B\to C$ is a bijection $\Rightarrow g^{-1}:C\to B$ is a bijection

$g^{-1}:C\to B$, $g^{-1}:B\to A$ are bijections $\Rightarrow f^{-1}og^{-1}:C\to A$ is a bijection

Let $z\in C$

$z\in C, g:B\to C$ is onto $\Rightarrow \exists y\in B \ni g(y)=z \Rightarrow g^{-1}(z)=y$

$y\in B, f:A\to B$ is onto $\Rightarrow \exists x\in A \ni f(x)=y \Rightarrow f^{-1}(y)=x$

$(gof)(x) = g[f(x)] = g(y) = z \Rightarrow (gof)^{-1}(z) = x$

$\therefore (gof)^{-1}(z) = x = f^{-1}(y) = f^{-1}[g^{-1}(z)] = (f^{-1}og^{-1})(z)$ $\therefore (gof)^{-1} = f^{-1}og^{-1}$

19. Prove that 2.3+ 3.4 + 4.5+ …. upto n terms $\dfrac{n(n^2+6n+11)}{3}$

Sol: 2, 3, 4………………… n terms $t_n = 2+(n-1)1 = n+1$

3, 4, 5……….. n terms $t_n = 3 + (n-1)1 = n+2$

$$2.3 + 3.4 + 4.5 + ... + (n+1)(n+2) = \frac{n(n^2+6n+11)}{3}$$

Let S_n be the given statement

For n = 1 L.H.S = 2.3 = 6

$$R.H.S = \frac{1(1+6+11)}{3} = 6$$

L. H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$\therefore 2.3 + 3.4 + 4.5 + ... + (k+1)(k+2) = \frac{k(k^2+6k+11)}{3} + (k+2)(k+3)$$

$$= \frac{k(k^2+6k+11) + 3(k^2+5k+6)}{3}$$

$$= \frac{k^3+9k^2+26k+18}{3}$$

$$= \frac{(k+1)\ \ k^2+8k+18}{3}$$

$k = -1$

$k=-1$	1	9	26	18
	0	−1	−8	−18
	1	8	18	0

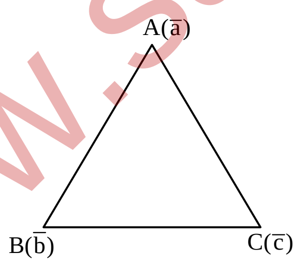
$$= \frac{(k+1)\ \ (k+1)^2 + 6(k+1) + 11}{3}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

20.If $\bar{a}, \bar{b}$ and $\bar{c}$ represent the vertices A, B and C respectively of ΔABC, then prove that $|(\bar{a}\times\bar{b}) + (\bar{b}+\bar{c}) + (\bar{c}\times\bar{a})|$ is twice the area of ΔABC.

Sol.



Let O be the origin,

$$\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$$

Area of ΔABC is $\Delta = \frac{1}{2}\left|(\overline{AB}\times\overline{AC})\right|$

$$= \frac{1}{2}\left|(\overline{OB}-\overline{OA})\times(\overline{OC}-\overline{OA})\right|$$

$$= \frac{1}{2}\left|(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})\right|$$

$$= \frac{1}{2}\left|\bar{b}\times\bar{c}-\bar{b}\times\bar{a}-\bar{a}\times\bar{c}+\bar{a}\times\bar{a}\right|$$

$$= \frac{1}{2}\left|\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+\bar{0}\right|$$

$$= \frac{1}{2}\left|\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}\right|$$

$2\Delta = \left|\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}\right|$. Hence proved.

21. If A,B,C are angles of a triangle then prove that
$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

SOL.

$A+B+C = 180^0$

$$LHS = \sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}$$

$$= \sin^2\frac{A}{2}+\sin\left(\frac{B}{2}+\frac{C}{2}\right).\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$= \sin^2\frac{A}{2}+\sin\left(90-\frac{A}{2}\right).\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$=1-\cos^2\frac{A}{2}+\cos\frac{A}{2}.\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$=1-\cos\frac{A}{2}\left(\cos\frac{A}{2}-\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right)$$

$$=1-\cos\frac{A}{2}\left(\cos\left(90-\left(\frac{B}{2}+\frac{C}{2}\right)\right)-\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right)$$

$$=1-\cos\frac{A}{2}\left(\sin\left(\frac{B}{2}+\frac{C}{2}\right)-\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right)$$

$$=1-\cos\frac{A}{2}\left(2\cos\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=RHS$$

22. Show that $a^2\cot A+b^2\cot B+c^2\cos C=\frac{abc}{R}$.

Sol. L.H.S. $a^2\cot A+b^2\cot B+c^2\cos C$

$$= 4R^2\sin^2 A\frac{\cos A}{\sin A} + 4R^2\sin^2 B\frac{\cos B}{\sin B} + 4R^2\sin^2 C\frac{\cos C}{\sin C} \text{(by sine rule)}$$

$$= 2R^2(2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C)$$

$$= 2R^2(\sin 2A + \sin 2B + \sin 2C)$$

$$= 2R^2(4\sin A\sin B\sin C)$$

$$= \frac{1}{R}(2R\sin A)(2R\sin B)(2R\sin C)$$

$$= \frac{abc}{R} = \text{R.H.S.}$$

23. Prove that $\det\begin{bmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{bmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$

$$= \frac{1}{abc}abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$\underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - R_3}{=}} \begin{vmatrix} 0 & a^2-c^2 & a^3-c^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$\underset{R_2 \to R_2 - R_1}{\overset{(a-c)(b-c)}{=}} \begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-c & b^2+bc+c^2 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-c & b^2-a^2+bc-ac \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & 1 & c+a+b \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} a+c & a^2+ac+c^2 \\ 1 & a+b+c \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)(ab+bc+ca)$$

24. By Matrix inversion method solve : $2x - y + 8z = 13$

$3x + 4y + 5z = 18$

$5x - 2y + 7z = 20$

Sol.

Matrix inversion method :

Let $A = \begin{bmatrix} 2 & -1 & 8 \\ 3 & 4 & 5 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 13 \\ 18 \\ 20 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = 28 + 10 = 38$$

$$B_1 = -\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -(21-25) = 4$$

$$C_1 = \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = -6 - 20 = -26$$

$$A_2 = -\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = -(-7+16) = -9$$

$$B_2 = \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = (14-40) = -26$$

$$C_2 = -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -(-4+5) = -1$$

$$A_3 = \begin{vmatrix} -1 & 8 \\ 4 & 5 \end{vmatrix} = -5 - 32 = -37$$

$$B_3 = -\begin{vmatrix} 2 & 8 \\ 3 & 5 \end{vmatrix} = -(10-24) = 14$$

$$C_3 = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8 + 3 = 11$$

$$\text{Adj} A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}$$

$$\text{Det} A = a_1A_1 + b_1B_1 + c_1C_1$$

$$= 2\cdot 38 + (-1)4 + 8(-26)$$

$$= 76 - 4 - 208 = -136$$

$$A^{-1} = \frac{\text{Adj}\,A}{\text{Det}\,A} = -\frac{1}{136}\begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}$$

$$X = A^{-1}D = -\frac{1}{280}\begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}\begin{bmatrix} 13 \\ 18 \\ 20 \end{bmatrix}$$

$$= -\frac{1}{136}\begin{bmatrix} 494 & -162 & -740 \\ 52 & -468 & +280 \\ -338 & -18 & +220 \end{bmatrix}$$

$$= -\frac{1}{136}\begin{bmatrix} -408 \\ -136 \\ -136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Solution is x = 3, y = 1, z = 1.