MATHEMATICS PAPER IA

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. 10X2 =20

1. If A = {1, 2, 3, 4} and f : A → R is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range of f.

2. If f(x + y) = f(xy) ∀x, y ∈ R then prove that f is a constant function.

3.In the two dimensional plane, prove by using vector methods, the equation of the line whose intercepts on the axes are $\bar{a}$ and $\bar{b}$ is $\frac{x}{a} + \frac{y}{b} = 1$.

4.If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B and C respectively of ΔABC then find the vector equation of median through the vertex A

5. Find the area of the parallelogram whose diagonals are $3\bar{i} + \bar{j} - 2\bar{k}$ and $\bar{i} - 3\bar{j} + 4\bar{k}$.

6.Prove that $\sin^2 \frac{\pi}{10} + \sin^2 \frac{4\pi}{10} + \sin^2 \frac{6\pi}{10} + \sin^2 \frac{9\pi}{10} = 2$

7.Prove that $\frac{1}{\cos 290°} + \frac{1}{\sqrt{3} \sin 250°} = \frac{4}{\sqrt{3}}$.

8. If $\sin hx = 3 \quad x = \log \; 3 - \sqrt{10}$

9. solve the system of equations :2x + 3y – z = 0
x – y – 2z = 0
3x + y + 3z = 0.

10. If A = $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ then show that $A^2 = -I$ where $i^2 = -1$.

SECTION B
SHORT ANSWER TYPE QUESTIONS
ANSWER ANY FIVE OF THE FOLLOWING 5 X 4 = 20

11. Show that the points $A(2\bar{i} - \bar{j} + \bar{k})$, $B(\bar{i} - 3\bar{j} - 5\bar{k})$, $C(3\bar{i} - 4\bar{j} - 4\bar{k})$ are the vertices of a right angle triangle.

12. Let $\bar{a}, \bar{b}, \bar{c}$ be mutually orthogonal vectors of equal magnitudes. Prove that the vector $\bar{a} + \bar{b} + \bar{c}$ is equally inclined to each of $\bar{a}, \bar{b}, \bar{c}$, the angle of inclination being $\cos^{-1}\frac{1}{\sqrt{3}}$.

13. If $\frac{\sin\alpha}{a} = \frac{\cos\alpha}{b}$, then prove that $a\sec 2\alpha + b\cos 2\alpha = b$.

14. If $|\tan x| = \tan x + \frac{1}{\cos x}$ and $x \in \; 0, 2\pi$ find the values of x

15. It $\cos^{-1} p + \cos^{-1} q + \cos^{-1} r = \pi$ prove that $p^2 + q^2 + r^2 + 2pqr = 1$

16. If $C = 60^0$ then show that (i) $\frac{a}{b+c} + \frac{b}{c+a} = 1$

(ii) $\frac{b}{c^2 - a^2} + \frac{a}{c^2 - b^2} = 0$

17. If A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then show that $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$ for all positive integers n.

SECTION C

LONG ANSWER TYPE QUESTIONS

ANSWER ANY FIVE OF THE FOLLOWING **5 X 7 = 35**

18. If $f: A \to B$, $g: B \to C$ are two one one onto functions then $gof: A \to C$ is also one one onto.

19. BY induction prove that $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

20. A line makes angles θ_1, θ_2, θ_3 and θ_4 with the diagonals of a cube. Show that $\cos^2\theta_1+\cos^2\theta_2+\cos^2\theta_3+\cos^2\theta_4=\frac{4}{3}$.

21. If A, B, C are the angles of a triangle then prove that cos2A+cos2B+cos2C = -4cosAcosBcosC-1

22. Theorem : If A is a non-singular matrix then A is invertible and $A^{-1}=\frac{\text{Adj}A}{\det A}$.

23. Solve the system of equations x + y + z = 9
2x + 5y + 7z = 52
2x + y – z = 0 by Gauss Jordan method

24. Gauss Jordan method Let an object be placed at some height h cm and let P and Q be two points of observation which are at a distance 10 cm apart on a line inclined at angle 15° to the horizontal. If the angles of elevation of the object from P and Q are 30° and 60° respectively then find h.

Solutions

1. If A = {1, 2, 3, 4} and f : A → R is a function defined by $f(x)=\frac{x^2-x+1}{x+1}$ then find the range of f.

Sol. Given that

$$f(x)=\frac{x^2-x+1}{x+1}$$

$$f(1)=\frac{1^2-1+1}{1+1}=\frac{1}{2}$$

$$f(2)=\frac{2^2-2+1}{2+1}=\frac{3}{3}=1$$

$$f(3)=\frac{3^2-3+1}{3+1}=\frac{7}{4}$$

$$f(4)=\frac{4^2-4+1}{4+1}=\frac{13}{5}$$

$\therefore$ Range of f is $\left\{\frac{1}{2},1,\frac{7}{4},\frac{13}{5}\right\}$

2. If f(x + y) = f(xy) ∀x, y ∈ R then prove that f is a constant function.

Sol. f(x + y) = f(xy)

Let f(0) = k

then f(x) = f(x + 0) = f(x· 0) = f(0) = k

⇒ f(x + y) = k

∴ f is a constant function.

3.In the two dimensional plane, prove by using vector methods, the equation of the line whose intercepts on the axes are $\bar{a}$ and $\bar{b}$ is $\frac{x}{a}+\frac{y}{b}=1$.

Sol. Let A = $(\bar{a},0)$ and $B=(0,\bar{b})$

$\therefore \bar{A}=\bar{a}i,\ \bar{B}=\bar{b}j$

the equation of the line is $\bar{r}=(1-t)\bar{a}i+t(\bar{b}j)$

If $\bar{r}=x\bar{i}+y\bar{j}$, then $x=(1-t)\bar{a}$ and $y=t\bar{b}$

$\therefore \frac{x}{a}+\frac{y}{b}=1-t+t=1$

4.If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the vertices A, B and C respectively of ΔABC then find the vector equation of median through the vertex A

Sol: $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OC}=\vec{c}$ be the given vertices

Let D be the mid point of $BC=\frac{\vec{b}+\vec{c}}{2}$

The vector equation of the line passing through the points $\vec{a}$, $\vec{b}$ is $\vec{r}=(1-t)\vec{a}+t\vec{b}$

5. Find the area of the parallelogram whose diagonals are $3\bar{i}+\bar{j}-2\bar{k}$ and $\bar{i}-3\bar{j}+4\bar{k}$.

Sol. Given $\overline{AC}=3\bar{i}+\bar{j}-2\bar{k},\overline{BD}=\bar{i}-3\bar{j}+4\bar{k}$

Area of parallelogram = $\frac{1}{2}|\overline{AC}\times\overline{BD}|$

$=\frac{1}{2}\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}$

$=\left|\frac{1}{2}\left[\bar{i}(4-6)-\bar{j}(12+2)+\bar{k}(-9-1)\right]\right|$

$=\left|\frac{1}{2}[-2\bar{i}-14\bar{j}-10\bar{k}]\right|$

$=|-\bar{i}-7\bar{j}-5\bar{k}|$

$=\sqrt{1+49+25}=\sqrt{75}$

$\therefore$ Area of parallelogram = $5\sqrt{3}$ sq.units.

6.Prove that $\sin^2\frac{\pi}{10}+\sin^2\frac{4\pi}{10}+\sin^2\frac{6\pi}{10}+\sin^2\frac{9\pi}{10}=2$

Solution:

$\sin^2\frac{\pi}{10}+\sin^2\frac{4\pi}{10}+\sin^2\frac{6\pi}{10}+\sin^2\frac{9\pi}{10}$

$\sin^2\frac{\pi}{10}+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{10}\right)+\sin^2\left(\frac{\pi}{2}+\frac{\pi}{10}\right)+\sin^2\left(\pi-\frac{\pi}{10}\right)$

$\sin^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}+\sin^2\frac{\pi}{10}=2\left\{\sin^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}\right\}=2$

$$=\sin^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}=1$$

7.Prove that $\frac{1}{\cos 290°}+\frac{1}{\sqrt{3}\sin 250°}=\frac{4}{\sqrt{3}}$.

SOL

$\cos 290° = \cos(270°+20°)=\sin 20°$

$\sin 250° = \sin(270°-20°)=\cos 20°$

$$\text{L.H.S.}=\frac{1}{\sin 20°}-\frac{1}{\sqrt{3}\cos 20°}=\frac{\sqrt{3}\cos 20°-\sin 20°}{\sqrt{3}\sin 20°\cos 20°}$$

$$=\frac{2\left(\frac{\sqrt{3}}{2}\cos 20°-\frac{1}{2}\sin 20°\right)}{\frac{\sqrt{3}}{2}(2\sin 20°\cos 20°)}=\frac{4}{\sqrt{3}}\left[\frac{\sin 60°\cos 20°-\cos 60°\sin 20°}{\sin 40°}\right]$$

$$=\frac{4}{\sqrt{3}}\frac{\sin(60°-20°)}{\sin 40°}=\frac{4}{\sqrt{3}}=\text{R.H.S.}$$

8. If $\sin hx=3 \quad x=\log\ 3-\sqrt{10}$

Solution:

$\sin hx=3 \Rightarrow x=\sin h^{-1}3$

$\sin h^{-1}y=\cos\ \ y+\sqrt{y^2+1}$

$\sin h^{-1}3=\log\ 3+\sqrt{9+1}$

$=\log\ 3+\sqrt{10}$

9. solve the system of equations :2x + 3y – z = 0
x – y – 2z = 0
3x + y + 3z = 0.

Sol. The coefficient matrix is $\begin{bmatrix}2&3&-1\\1&-1&-2\\3&1&3\end{bmatrix}$

$$\det \text{ of }\begin{bmatrix}2&3&-1\\1&-1&-2\\3&1&3\end{bmatrix}=\begin{bmatrix}2&3&-1\\1&-1&-2\\3&1&3\end{bmatrix}$$

$=2(-3+2)-3(3+6)-1(1+3)$

$=-2-27-4=-33\neq 0,\ \rho(A)=3$

Hence the system has the trivial solution

$x = y = z = 0$ only.

10. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ then show that $A^2 = -I$ where $i^2 = -1$.

Sol. $A^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$

11. Show that the points $A(2\bar{i} - \bar{j} + \bar{k})$, $B(\bar{i} - 3\bar{j} - 5\bar{k})$, $C(3\bar{i} - 4\bar{j} - 4\bar{k})$ are the vertices of a right angle triangle.

Sol. We have

$AB = (1-2)\bar{i} + (-3+1)\bar{j} + (-5-1)\bar{k}$
$= -\bar{i} - 2\bar{j} - 6\bar{k}$

$|AB| = \sqrt{1+4+36} = \sqrt{41}$

$BC = (3-1)\bar{i} + (-4+3)\bar{j} + (-4+5)\bar{k}$
$= 2\bar{i} - \bar{j} + \bar{k}$ and

$|BC| = \sqrt{4+1+1} = \sqrt{6}$

$CA = (2-3)\bar{i} + (-1+4)\bar{j} + (1+4)\bar{k}$
$= -\bar{i} + 3\bar{j} + 5\bar{k}$

$|CA| = \sqrt{1+9+25} = \sqrt{35}$

We have $|AB|^2 = |BC|^2 + |CA|^2$

Therefore, the triangle is a rt. Triangle.

12. Let $\bar{a}, \bar{b}, \bar{c}$ be mutually orthogonal vectors of equal magnitudes. Prove that the vector $\bar{a} + \bar{b} + \bar{c}$ is equally inclined to each of $\bar{a}, \bar{b}, \bar{c}$, the angle of inclination being $\cos^{-1}\frac{1}{\sqrt{3}}$.

Sol. Let $|\bar{a}| = |\bar{b}| = |\bar{c}| = \lambda$

Now, $|\bar{a}+\bar{b}+\bar{c}|^2=\bar{a}^2+\bar{b}^2+\bar{c}^2+2\Sigma\bar{a}\cdot\bar{b}$

$=3\lambda^2 (\because \bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{c}=\bar{c}\cdot\bar{a}=0)$

Let θ be the angle between $\bar{a}$ and $\bar{a}+\bar{b}+\bar{c}$

Then $\cos\theta=\dfrac{\bar{a}\cdot(\bar{a}+\bar{b}+\bar{c})}{|\bar{a}||\bar{a}+\bar{b}+\bar{c}|}=\dfrac{\bar{a}\cdot\bar{a}}{\lambda(\lambda\sqrt{3})}=\dfrac{1}{\sqrt{3}}$

Similarly, it can be proved that $\bar{a}+\bar{b}+\bar{c}$ inclines at an angle of $\cos^{-1}\dfrac{1}{\sqrt{3}}$ with b and c.

13. If $\dfrac{\sin\alpha}{a}=\dfrac{\cos\alpha}{b}$, then prove that $a\sec 2\alpha+b\cos 2\alpha=b$.

Sol. Given that $\dfrac{\sin\alpha}{a}=\dfrac{\cos\alpha}{b}$

$\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{a}{b}$

$\therefore \tan\alpha=\dfrac{a}{b}$

L.H.S. = $a\sec 2\alpha+b\cos 2\alpha$

$=a\left[\dfrac{2\tan\alpha}{1+\tan^2\alpha}\right]+b\left[\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}\right]$

$=a\left[\dfrac{2\times\frac{a}{b}}{1+\left(\frac{a}{b}\right)^2}\right]+b\left[\dfrac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2}\right]$

$=\left[\dfrac{\frac{2a^2}{b}}{\frac{b^2+a^2}{b^2}}\right]+b\left[\dfrac{\frac{b^2-a^2}{b^2}}{\frac{b^2+a^2}{b^2}}\right]$

$=\dfrac{2a^2b}{a^2+b^2}+\dfrac{b(b^2-a^2)}{a^2+b^2}$

$=\dfrac{2a^2b+b^3-ba^2}{a^2+b^2}$

$=\dfrac{b^3+a^2b}{a^2+b^2}=\dfrac{b(b^2+a^2)}{a^2+b^2}=b=\text{R.H.S.}$

14. If $|\tan x|=\tan x+\dfrac{1}{\cos x}$ and $x\in 0, 2\pi$ find the values of x

Solution:

$\cos 1$ suppose $\tan x > 0$

$\therefore \tan x = \tan x + \frac{1}{\cos x} \Rightarrow \frac{1}{\cos x} = 0$ not possible

Case (ii) Suppose $\tan x < 0$

$\therefore -\tan x = \tan x + \frac{1}{\cos x} \Rightarrow -2\tan x = \frac{1}{\cos x}$

$$-2\sin x = 1 \Rightarrow \sin x = -\frac{1}{2}$$

x Lies in (iii) or (iv) quadrant

But $\tan x < 0$

$\therefore x = -\pi/6 \quad (or)\, 11\pi/6$

But $x \in \; 0, 2\pi$

$\therefore x = 11\pi/6$

15. It $\cos^{-1} p + \cos^{-1} q + \cos^{-1} r = \pi$ prove that $p^2 + q^2 + r^2 + 2pqr = 1$

Solution:

Let $\cos^{-1} p = \alpha \qquad \cos^{-1} q = \beta \qquad \cos^{-1} r = \delta$

$p = \cos\alpha \qquad q = \cos\beta \qquad r = \cos\delta$

Given $\alpha + \beta + \gamma = \pi \qquad \cos\; \alpha + \beta \; = \cos\; \pi - \gamma$

$\cos\alpha\cos\beta - \sin\alpha\sin\gamma = \cos\gamma$

$pq = -r + \sqrt{1-p^2}\;\sqrt{1-q^2} \Rightarrow pq + r = \sqrt{1-p^2}\;\sqrt{1-q^2}$

Squaring on both such

$p^2q^2 + r^2 + 2pqr = 1 - p^2 - q^2 + p^2q^2 \Rightarrow p^2 + q^2 + r^2 + 2pqr = 1$

16. If $C = 60^0$ then show that (i) $\frac{a}{b+c} + \frac{b}{c+a} = 1$ (ii) $\frac{b}{c^2 - a^2} + \frac{a}{c^2 - b^2} = 0$

Solution : -

Given $C = 60^0$

$C^2 = a^2 + b^2 - 2ab\cos C$

$C^2 = a^2 + b^2 - \not{2}ab\left(\frac{1}{\not{2}}\right)$

$C^2 + ab = a^2 + b^2$

$\frac{a}{b+c} + \frac{b}{c+a} =$

$\frac{ac + a^2 + bc + b^2}{b+c \quad c+a} = \frac{a^2 + b^2 + ac + bc}{bc + ab + c^2 + ac}$

But $a^2 + b^2 = c^2 + ab$

$$\frac{c^2+ab+ac+bc}{bc+ab+c^2+ac}=1$$

17. If A = $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ then show that $A^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$ for all positive integers n.

Sol. Let S(n) be the statement that

$$A^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$$

Given A = $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$\Rightarrow A^1=\begin{bmatrix}\cos 1\theta & -\sin 1\theta\\ \sin 1\theta & \cos 1\theta\end{bmatrix}\Rightarrow$ S(1) is true.

Assume that S(k) is true.

$$\therefore A^k=\begin{bmatrix}\cos k\theta & -\sin k\theta\\ \sin k\theta & \cos k\theta\end{bmatrix}$$

Now A^{k+1} = A^kA

$$=\begin{bmatrix}\cos k\theta & -\sin k\theta\\ \sin k\theta & \cos k\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos k\theta\cos\theta-\sin k\theta\sin\theta & -\cos k\theta\sin\theta-\sin k\theta\cos\theta\\ \sin k\theta\cos\theta+\cos k\theta\sin\theta & -\sin k\theta\sin\theta+\cos k\theta\cos\theta\end{bmatrix}=\begin{bmatrix}\cos(k\theta+\theta) & -\sin(k\theta+\theta)\\ \sin(k\theta+\theta) & \cos(k\theta+\theta)\end{bmatrix}=\begin{bmatrix}\cos(k+1)\theta & -\sin(k+1)\theta\\ \sin(k+1)\theta & \cos(k+1)\theta\end{bmatrix}$$

$\therefore$ S(k + 1) is true.

By principle of Mathematical induction S(n) is true for all n $\in$N.

$\therefore A^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$ for all positive integers n.

18. If $f:A\to B$, $g:B\to C$ are two one one onto functions then $gof:A\to C$ is also one one be onto.

Sol: i) Let $x_1,x_2\in A$and $f(x_1)=f(x_2)$.

$x_1,x_2\in A,\ f:A\to B\Rightarrow f(x_1),\ f(x_2)\in B$

$f(x_1),\ f(x_2)\in B,\to C,\ f(x_2)\Rightarrow g[f(x_1)]=g[f(x_2)]\Rightarrow(gof)(x_1)=(gof)(x_2)$

$x_1,x_2\in A,(gof)(x_1)=(gof):A\to C$ is one one$\Rightarrow x_1=x_2$

$\therefore x_1,x_2\in A,\ f(x_1)=f(x_2)\Rightarrow x_1=x_2$.

$\therefore f: A \to B$ is one one.

ii) Proof :let $z \in C$

$z \in C, g: B \to C$ is onto $\Rightarrow \exists y \in B \ni f(x) = y$

Now $(gof)(x) = g(y) = z$

$\therefore z \in C \Rightarrow \exists x \in A \ni (gof)(x) = z$.

$\therefore gof: A \to C$ is onto.

19. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Sol: Let $S_{(n)}$ be the given statement

For n = 1 L.H.S $\frac{1}{1.3} = \frac{1}{3}$

R.H.S = $\frac{1}{2+1} = \frac{1}{3}$

Assume S_k is true

$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

Adding $(k+1)^{th}$ term i.e. $\frac{1}{(2k+1)(2k+3)}$ on both sides

$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

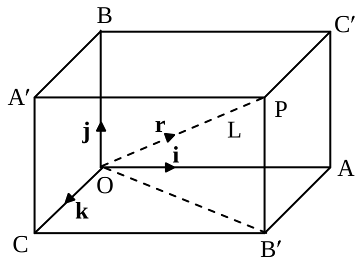
$= \frac{k+1}{2k+3}$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

20. A line makes angles θ_1, θ_2, θ_3 and θ_4 with the diagonals of a cube. Show that $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 = \frac{4}{3}$.

Sol.



Let OAB′C, BC′PA′ be a unit cube.

Let $\overline{OA}=\overline{i},\overline{OB}=\overline{j}$ and $\overline{OC}=\overline{k}$

$\overline{OP}, AA', BB', CC'$ be its diagonals.

Let $\overline{r}=x\overline{i}+y\overline{j}+z\overline{k}$ be a unit vector along a line L.

Which makes angles θ_1, θ_2, θ_3 and θ_4 with $\overline{AA'},\overline{BB'},\overline{CC'}$ and $\overline{OP}$.

$\Rightarrow |\overline{r}|=\sqrt{x^2+y^2+z^2}=1$

We have $\overline{OB'}=\overline{OA}-\overline{OC}=\overline{i}+\overline{k}$

$\overline{OP}=\overline{OB'}-\overline{B'P}=\overline{i}+\overline{k}+\overline{j}\,[\because \overline{B'O}=\overline{OB}=\overline{j}]$

$=\overline{i}+\overline{j}+\overline{k}$

$\overline{AA'}=\overline{OA'}-\overline{OA}=\overline{j}+\overline{k}-\overline{i}=-\overline{i}+\overline{j}+\overline{k}$

$\overline{BB'}=\overline{OB'}-\overline{OB}=\overline{i}+\overline{k}-\overline{j}=\overline{i}-\overline{j}+\overline{k}$

$\overline{CC'}=\overline{OC'}-\overline{OC}=\overline{i}+\overline{j}-\overline{k}$

Let $(\overline{r},\overline{OP})=\theta_1$

$$\cos\theta_1=\frac{\overline{r}\cdot\overline{OP}}{|\overline{r}||\overline{OP}|}=\frac{(x\overline{i}+y\overline{j}+z\overline{k})\cdot(\overline{i}+\overline{j}+\overline{k})}{1\cdot\sqrt{1+1+1}}$$

$$=\frac{x+y+z}{\sqrt{3}} \quad ...(1)$$

Similarly $(\overline{r},AA')=\theta_2$

$$\Rightarrow \cos\theta_2=\frac{\overline{r}\cdot\overline{AA'}}{|\overline{r}||\overline{AA'}|}=\frac{(x\overline{i}+y\overline{j}+z\overline{k})\cdot(-\overline{i}+\overline{j}+\overline{k})}{1\cdot\sqrt{1+1+1}}$$

$$=\frac{-x+y+z}{\sqrt{3}} \quad ...(2)$$

$(\overline{r},\overline{BB'})=\theta_3$

$$\Rightarrow \cos\theta_3=\frac{\overline{r}\cdot\overline{BB'}}{|\overline{r}||\overline{BB'}|}$$

$$= \frac{(x\bar{i} + y\bar{j} + z\bar{k}) \cdot (\bar{i} - \bar{j} + \bar{k})}{1 \cdot \sqrt{1+1+1}}$$

$$= \frac{x - y + z}{\sqrt{3}} \qquad \ldots(3)$$

$$(\bar{r}, \overline{CC'}) = \theta_4$$

$$\Rightarrow \cos\theta_3 = \frac{\bar{r} \cdot \overline{CC'}}{|\bar{r}||\overline{CC'}|}$$

$$= \frac{(x\bar{i} + y\bar{j} + z\bar{k}) \cdot (\bar{i} + \bar{j} - \bar{k})}{1 \cdot \sqrt{1+1+1}}$$

$$= \frac{x + y - z}{\sqrt{3}} \qquad \ldots(4)$$

$$\therefore \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4$$

$$= \left(\frac{x+y+z}{\sqrt{3}}\right)^2 + \left(\frac{-x+y+z}{\sqrt{3}}\right)^2 + \left(\frac{x-y+z}{\sqrt{3}}\right)^2 + \left(\frac{x+y-z}{\sqrt{3}}\right)^2$$

$$(x+y+z)^2 + (-x+y+z)^2 = \frac{(x+y+z)^2 + (x-y+z)^2 + (x+y-z)^2}{3}$$

$$= \frac{2(x+y)^2 + 2z^2 + 2(x-y)^2 + 2z^2}{3} = \frac{2\left[(x+y)^2 + (x-y)^2\right] + 4z^2}{3}$$

$$= \frac{2\left[2x^2 + 2y^2\right] + 4z^2}{3}$$

$$= \frac{4x^2 + 4y^2 + 4z^2}{3} = \frac{4}{3}\left[x^2 + y^2 + z^2\right] = \frac{4}{3}(1) = \frac{4}{3}$$

21. If A, B, C are the angles of a triangle then prove that cos2A+cos2B+cos2C = -4cosAcosBcosC-1

cos2A+cos2B+cos2C =

$$= 2\cos\frac{2A+2B}{2}\cos\frac{2A-2B}{2}+\cos 2C$$
$$= 2\cos(A+B)\cos(A-B)+2\cos^2 C-1$$
$$= 2\cos(\pi-c)\cos(A-B)+2\cos^2 C-1$$
$$= -2\cos C\cos(A-B)+2\cos^2 C-1$$
$$= 2\cos C\left(-\cos(A-B)+\cos C\right)-1$$
$$= 2\cos C\left(-\cos(A-B)+\cos(\pi-(A+B))\right)-1$$
$$= 2\cos C\left(-\cos(A-B)-\cos(A+B)\right)-1$$
$$= 2\cos C\left(-2\cos A\cos B\right)-1$$
$$= -4\cos A\cos B\cos C-1$$

22. Theorem : If A is a non-singular matrix then A is invertible and $A^{-1}=\frac{\text{Adj}A}{\det A}$.

Sol. Proof : Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a non-singular matrix.

$\therefore \det A \neq 0.$

$$\text{Adj}A=\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A\cdot\text{Adj}A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$=\begin{bmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{bmatrix}$$

$$=\begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}=\det A\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= det A I

$$\therefore A\cdot\frac{\text{Adj}A}{\det A}=I$$

Similarly we can prove that $A\cdot\frac{\text{Adj}A}{\det A}=I$

$$\therefore A^{-1}=\frac{\text{Adj}A}{\det A}$$

23. Solve the system of equations x + y + z = 9
2x + 5y + 7z = 52
2x + y – z = 0 by Gauss Jordan method

Sol.

Gauss Jordan method :

Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_2$

$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -4 & -8 & -52 \end{bmatrix}$

$R_1 \to 3R_1 - R_2, R_3 \to 3R_3 + 4R_2$

$A \sim \begin{bmatrix} 3 & 0 & -2 & -7 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{bmatrix}$

$R_3 \to R_3\left(-\frac{1}{4}\right)$, we obtain

$A \sim \begin{bmatrix} 3 & 0 & -2 & -7 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

$R_1 \to R_1 + 2R_3, R_2 \to R_2 - 5R_3$, we get

$A \sim \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

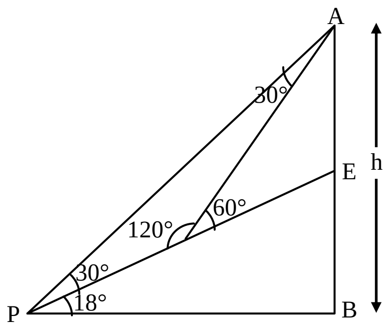
$R_1 \to R_1\left(\frac{1}{3}\right), R_2 \to R_2\left(\frac{1}{3}\right)$ we have

$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

$\therefore$ The given equations have a unique solution and solution is x = 1, y = 3, z = 5.

31.24. Gauss Jordan method Let an object be placed at some height h cm and let P and Q be two points of observation which are at a distance 10 cm apart on a line inclined at angle 15° to the horizontal. If the angles of elevation of the object from P and Q are 30° and 60° respectively then find h.

Sol.



A is the position of the object.

Given that AB = h cm

P and Q are points of observation.

Given that, PQ = 10 cm

We have,

$\angle BPE = 15°, \angle EPA = 30°, \angle EQA = 60°$

In ΔPQA,

P = 30°, Q = 120° and A = 30°

$\therefore$ By sine rule,

$$\frac{AP}{\sin 120°} = \frac{PQ}{\sin 30°}$$

$$\frac{AP}{\sin(180° - 60°)} = \frac{10}{1/2}$$

$$\frac{AP}{\sin 60°} = 20° \Rightarrow \frac{AP}{\sqrt{3}/2} = 20°$$

$$AP = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$

In ΔPBA, $\sin 45° = \frac{AB}{AP}$

$$\frac{1}{\sqrt{2}} = \frac{h}{10\sqrt{3}}$$

$$h = \frac{10\sqrt{3}}{\sqrt{2}} = \frac{5 \cdot 2 \cdot \sqrt{3}}{\sqrt{2}} = 5\sqrt{2}\sqrt{3} = 5\sqrt{6}\text{ cm}$$