MATHEMATICS PAPER IIB

COORDINATE GEOMETRY AND CALCULUS.

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. 10X2 =20

1. Find the condition that the tangents Drawn from (0,0) to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be perpendicular to each other.

2. Find the pole of $ax + by + c = 0$ $(c \neq 0)$ With respect to $x^2 + y^2 = r^2$.

3. Find the equation of the radical axis of the circles $x^2 + y^2 - 3x - 4y + 5 = 0$, $3(x^2 + y^2) - 7x + 8y + 11 = 0$

4. Find the coordinates of the point on the parabola $y^2 = 8x$ whose focal distance is 10.

5. Find the equation of normal at $\theta = \frac{\pi}{3}$ to the hyperbola $3x^2 - 4y^2 = 12$.

6. Evaluate $\int (\tan x + \log \sec x) e^x dx$ on $\left(\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right) n \in Z$

7. Evaluate $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$ for $x \neq \pm a$

8. $\int_0^{\pi/2} \cos^7 x \cdot \sin^2 x dx$

9. $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$

10. Obtain the differential equation which corresponds to the circles which touch the Y-axis at the origin.

SECTION B

SHORT ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING **5 X 4 = 20**

11. If the parametric values of two points A and B lying on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are 30° and 60° respectively, then find the equation of the chord joining A and B.
12. Find the equation of the circle which cuts the circles $x^2 + y^2 - 4x - 6y + 11 = 0$, $x^2 + y^2 - 10x - 4y + 21 = 0$ orthogonally and has the diameter along the straight line $2x + 3y = 7$.

13. If S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.

14. Find the centre eccentricity, foci directrices and length of the latus rectum of the hyperbola $4x^2 - 9y^2 - 8x - 32 = 0$,

15.Evaluate $\int \frac{\sec x}{(\sec x + \tan x)^2} dx$ on $I \subset R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$.

16.Evaluate $\int \frac{1}{\cos(x-a)\cos(x-b)} dx$ on $I \subset R \setminus \left(\left\{a + \frac{(2n+1)\pi}{2} : n \in Z\right\} \cup \left\{b + (2n+1)\frac{\pi}{2} : n \in Z\right\}\right)$

17.solve the equation $(x^2 + y^2)dy = 2xy\,dx$

SECTION C

LONG ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING 5 X 7 = 35

18. The equation to the pair of tangents to the circle S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

19. Find the equations of circles which touch $2x - 3y + 1 = 0$ at (1, 1) and having radius $\sqrt{13}$.

20.

If a normal chord at a point t on the parabola $y^2 = 4ax$, subtends a right angle at vertex, then prove that $t = \pm\sqrt{2}$.

21. If $I_{m,n} = \int \sin^m x \cos^n x\, dx$, then show that $I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$ for a positive integer n and an integer $m \geq 2$.

22. $\int \frac{dx}{(1-x)\sqrt{3-2x-x^2}}$ on (–1, 3).

23. $\int_{-1}^{3/2} |x \sin \pi x|\, dx$

24. $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

SOLUTIONS

1. Find the condition that the tangents Drawn from (0,0) to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be perpendicular to each other.

Sol. Let θ *be* the angle between the pair of

Tangents then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$

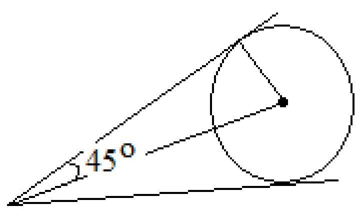
given $\theta = \frac{\pi}{2}$, radius r= $\sqrt{g^2 + f^2 - c}$

$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 + c = c$

$$\tan 45^0 = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}}$$

$\Rightarrow 1 = \frac{\sqrt{g^2+f^2-c}}{\sqrt{0+0+0+0+c}}$

$\Rightarrow g^2 + f^2 - c = c$



$\Rightarrow g^2 + f^2 = 2c$

This is the required condition

2. Find the pole of $ax + by + c = 0$ $(c \neq 0)$ With respect to $x^2 + y^2 = r^2$.

Sol. Let (x_1, y_1) be pole. Then the polar equation is $S_1=0$.

$\Rightarrow xx_1 + yy_1 - r^2 = 0$ ______(i)

But polar is $ax + by + c = 0$ ______(ii)

(i) and (ii) both are same lines

$$\Rightarrow \frac{x_1}{a} = \frac{y_1}{b} = \frac{-r^2}{c} \Rightarrow x_1 = -\frac{-a}{c}r^2, y_1 = \frac{-br^2}{c}$$

$$\therefore pole\left(\frac{-ar^2}{c}, \frac{-br^2}{c}\right)$$

3. Find the equation of the radical axis of the circles $x^2 + y^2 - 3x - 4y + 5 = 0$, $3(x^2 + y^2) - 7x + 8y + 11 = 0$

Sol. let $S \equiv x^2 + y^2 - 3x - 4y + 5 = 0$

$S' = x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3} = 0$

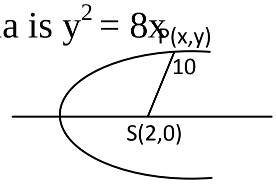
Radical axis is $S - S' = 0$

$$(x^2 + y^2 - 3x - 4y + 5) - \left(x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3}\right) = 0$$

$$-\frac{2}{3}x-\frac{20}{3}y+\frac{4}{3}=0 \Rightarrow x+10y-2=0$$

4. Find the coordinates of the point on the parabola $y^2 = 8x$ whose focal distance is 10.

Sol. Equation of the parabola is $y^2 = 8x$



$4a = 8 \Rightarrow a = 2$

$\Rightarrow S = (2, 0)$

let P(x, y) be a point on the parabola

Given SP = 10

$\Rightarrow |x+a| = 10 \Rightarrow x+2 = \pm 10$

$\Rightarrow x = 8$ or -12

5. Find the equation of normal at $\theta = \frac{\pi}{3}$ to the hyperbola $3x^2 - 4y^2 = 12$.

Sol: The given equation of hyperbola is

$3x^2 - 4y^2 = 12$

$$\Rightarrow \frac{x^2}{4}-\frac{y^2}{3}=1$$

The equation of normal at P(a sec θ, b tanθ) to the hyperbola S = 0 is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

∴ Equation of normal at

$\theta = \frac{\pi}{3}$ when $a^2 = 4, b^2 = 3$

$$\frac{2x}{\sec\frac{\pi}{3}}+\frac{\sqrt{3}y}{\tan\frac{\pi}{3}}=4+3$$

$$\Rightarrow \frac{2x}{2}+\frac{\sqrt{3}y}{\sqrt{3}}=7$$

$\Rightarrow x+y=7.$

6. $\int(\tan x+\log\sec x)e^x dx$ on $\left(\left(2n-\frac{1}{2}\right)\pi,\left(2n+\frac{1}{2}\right)\pi\right) n\in Z$

Sol. let $f=\log|\sec x| \Rightarrow f'=\frac{1}{\sec x}\cdot\sec x\cdot\tan x\cdot$

$= \tan x$

7. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$ for $x \neq \pm a$

Proof :

$\int \frac{1}{a^2 - x^2} dx = \int \frac{1}{(a+x)(a-x)} dx$

$= \frac{1}{2a} \int \left(\frac{1}{a+x} + \frac{1}{a-x} \right) dx = \frac{1}{2a} \; \log|a+x| - \log|a-x| \; + c$

$= \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

8. $\int_0^{\pi/2} \cos^7 x \cdot \sin^2 x dx$

Sol. $I = \int_0^{\pi/2} \cos^7 x \cdot \sin^2 x dx$,

m=2,n=7

$\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx$ here m even,

n odd

$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$

$= \frac{7-1}{9} \times \frac{7-3}{7} \times \frac{7-5}{5} \times \frac{1}{2+1}$

$= \frac{6}{9} \cdot \frac{4}{7} \cdot \frac{2}{5} \cdot \frac{1}{3} = \frac{16}{315}$

9. $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$ …(1)

$I = \int_0^{\pi/2} \frac{\sin^5(\pi/2 - x) dx}{\sin^5(\pi/2 - x) + \cos^5(\pi/2 - x)}$

$\left(\because \int_0^a f(a-x) dx = \int_0^a f(x) \right)$

$$= \int_0^{\pi/2} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} \qquad \ldots(2)$$

Adding (1) and (2) ,

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\pi/2} 1 \cdot dx$$

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

10. Obtain the differential equation which corresponds to the circles which touch the Y-axis at the origin.

Sol. Equation of the given family of circles is

$x^2 + y^2 + 2gx = 0$, g arbitrary const …(i)

$x^2 + y^2 = -2gx$

Differentiating w.r.t. x

$2x + 2yy_1 = -2g$ …(ii)

Substituting in (i)

$x^2 + y^2 = x(2x + 2yy_1)$ by (ii)

$= 2x^2 + 2xyy_1$

$yy^2 - 2xyy_1 - 2x^2 = 0$

$y^2 - x^2 = 2xy\frac{dy}{dx}$.

11. If the parametric values of two points A and B lying on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are 30° and 60° respectively, then find the equation of the chord joining A and B.

Sol. Equation of the circle is

$x^2 + y^2 - 6x + 4y - 12 = 0$

Equation of the chord joining θ_2 and θ_1 is

$$(x+g)\cos\frac{\theta_1+\theta_2}{2} + (y+f)\sin\frac{\theta_1+\theta_2}{2} = r\cos\frac{\theta_1-\theta_2}{2}$$

$r = \sqrt{9+4+12} = 5$

$(x - 3)\cos 45° + (y + 2)\sin 45° = 5\cos 15°$

$$\Rightarrow \frac{(x-3)+(y+2)}{\sqrt{2}} = \frac{5(\sqrt{3}+1)}{2\sqrt{2}}$$

$\Rightarrow 2(x + y - 6) = 5\sqrt{3} + 5$

$\Rightarrow 2x+2y-7-5\sqrt{3}=0 \Rightarrow 2x+2y-(7+5\sqrt{3})=0$

12. Find the equation of the circle which cuts the circles $x^2 + y^2 - 4x - 6y + 11 = 0$, $x^2 + y^2 - 10x - 4y + 21 = 0$ orthogonally and has the diameter along the straight line $2x + 3y = 7$.

Sol. Let circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$

S=0 is Orthogonal to $x^2 + y^2 - 4x - 6y + 11 = 0$, $x^2 + y^2 - 10x - 4y + 21 = 0$

$\Rightarrow 2g(-2) + 2f(-3) = 11 + c$ —— (1)

$\Rightarrow 2g(-5) + 2f(-2) = 21 + c$ —— (2)

(1) –(2) $\Rightarrow -6g + 2f = 10$ —— (3)

centre (-g,-f) is on $2x + 3y = 7$,

$\therefore -2g - 3f = 7$ —— (4)

Solving (3) and (4)

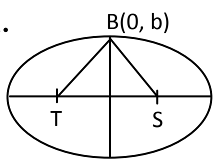
$f = -1$, $g = -2$,

sub. These values in (1), then $c = 3$

Equation of circle $x^2 + y^2 - 4x - 2y + 3 = 0$

13. If S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.

Sol.



Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Foci are S(ae, 0), T(–ae, 0)

B(0, b) is the end of the minor axis

STB is an equilateral triangle

$SB = ST \Rightarrow SB^2 = ST^2$

$a^2e^2 + b^2 = 4a^2e^2$

$b^2 = 3a^2e^2$

$a^2(1 - e^2) = 3a^2e^2$

$1 - e^2 = 3e^2$

$4e^2 = 1 \Rightarrow e^2 = \frac{1}{4}$

Eccentricity of the ellipse : $e = \frac{1}{2}$.

14. Find the centre eccentricity, foci directrices and length of the latus rectum of the hyperbola $4x^2 - 9y^2 - 8x - 32 = 0$,

Sol. i) $4x^2 - 9y^2 - 8x - 32 = 0$

$4(x^2 - 2x) - 9y^2 = 32$

$4(x^2 - 2x + 1) - 9y^2 = 36$

$$\frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$$

Centre of the hyperbola is (1, 0)

$a^2 = 9, b^2 = 4 \Rightarrow a = 3, b = 2$

$$e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{9+4}{9}} = \frac{\sqrt{13}}{3}$$

Foci are $\left(1 \pm 3 \cdot \frac{\sqrt{13}}{3}, 0\right) = (1 \pm \sqrt{13}, 0)$

Equations of directrices are :

$$x = 1 \pm \frac{3 \cdot 3}{\sqrt{13}} \Rightarrow x = 1 \pm \frac{9}{\sqrt{13}}$$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \cdot 4}{3} = \frac{8}{3}$

15. $\int \frac{\sec x}{(\sec x + \tan x)^2} dx$ on $I \subset R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$.

Sol: $\int \frac{\sec x}{(\sec x + \tan x)^2}$

$$= \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)^3} dx$$

Let $\sec x + \tan x = t$

then $(\sec x \tan x + \sec^2 x)\, dx = dt$

$\Rightarrow \sec x(\sec x + \tan x)dx = dt$

$$\therefore \int \frac{\sec x}{(\sec x + \tan x)^2} dx$$

$$= \int \frac{dt}{t^3} = \int t^{-3} dt = \frac{t^{-2}}{-2}$$

$$= -\frac{1}{2t^2} = -\frac{1}{2(\sec x + \tan x)^2} + c$$

16. $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$ on $I \subset R \setminus \left(\left\{a+\frac{(2n+1)\pi}{2} : n \in Z\right\} \cup \left\{b+(2n+1)\frac{\pi}{2} : n \in Z\right\}\right)$

Sol. $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(\overline{x-b}-\overline{x-a})}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}$$

$$\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)} \int \tan(x-b)-\tan(x-a)\ dx$$

$$= \frac{1}{\sin(a-b)}\ \log|\sec(x-b)| - \log|\sec(x-a)|\ + C$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\sec(x-b)}{\sec(x-a)}\right| + C$$

17. $(x^2 + y^2)dy = 2xy\ dx$

Sol. $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$ which is a homogeneous d.e.

Put $y = vx$

$$\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

$$v + x\cdot\frac{dv}{dx} = \frac{2x(vx)}{x^2+v^2x^2} = \frac{2v}{1+v^2}$$

$$x\cdot\frac{dv}{dx} = \frac{2v}{1+v^2} - v = \frac{2v-v-v^3}{1+v^2} = \frac{v-v^3}{1+v^2}$$

$$\int \frac{1+v^2}{v(1-v^2)}dv = \int \frac{dx}{x}$$

Let $\frac{1+v^2}{v(1-v^2)} = \frac{A}{v} + \frac{B}{1+v} + \frac{C}{1-v}$

$$1+v^2 = A(1-v^2) + BV(1-v) + CV(1+v)$$

$$v = 0 \Rightarrow 1 = A$$

$$v = 1 \Rightarrow 1+1 = C(2) \Rightarrow c = 1$$

$$v = -1 \Rightarrow 1+1 = B(-1)(2) \Rightarrow 2 = -2B \Rightarrow B = -1$$

$$\int \frac{1+v^2}{v(1-v^2)} dv = \int \frac{dv}{v} - \int \frac{dv}{1+v} + \int \frac{dv}{1-v}$$

$$= \log v - \log(1+v) - \log(1-v) = \log \frac{v}{1-v^2}$$

$$\therefore \log \frac{v}{1-v^2} = \log x + \log c = \log cx$$

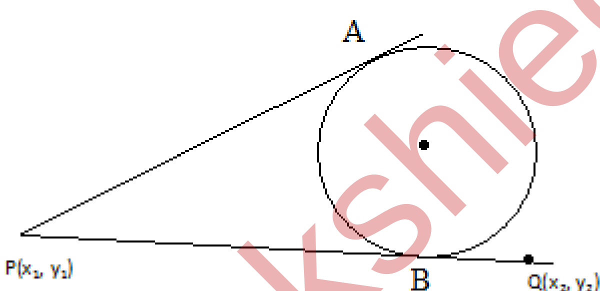
$$\frac{v}{1-v^2} = cx \Rightarrow v = cx(1-v^2)$$

$$\frac{y}{x} = cx\left(1-\frac{y^2}{x^2}\right) \Rightarrow \frac{y}{x} = cx\frac{(x^2-y^2)}{x^2}$$

Solution is : $y = c(x^2 - y^2)$

18. The equation to the pair of tangents to the circle S = 0 from P(x_1, y_1) is $S_1^2 = S_{11}S$.

Proof:



Let the tangents from P to the circle S=0 touch the circle at A and B.

Equation of AB is S_1 =0.

i.e., $x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$ =====i)

let Q(x_2,y_2) be any point on these tangents. Now locus of Q will be the equation of the pair of tangents drawn from P.

the line segment PQ is divided by the line AB in the ratio $-S_{11}:S_{22}$

$$\Rightarrow \frac{PB}{QB} = \left|\frac{S_{11}}{S_{22}}\right|$$ ----ii)

BUT $PB = \sqrt{S_{11}}, QB = \sqrt{S_{22}} \Rightarrow \frac{PB}{QB} = \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}}$ ----iii)

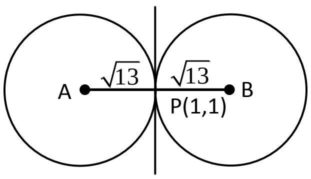
From ii) and iii) $\Rightarrow \frac{s^2{}_{11}}{s^2{}_{22}} = \frac{S_{11}}{S_{22}}$

$\Rightarrow S_{11}S_{22} = S^2{}_{12}$

Hence locus of $Q(x_2, y_2)$ is $S_{11}S = S^2{}_1$

19. Find the equations of circles which touch 2x – 3y + 1 = 0 at (1, 1) and having radius $\sqrt{13}$.

Sol. The centre of required circle lies on a line perpendicular to 2x – 3y + 1 = 0 and passing through (1, 1).



The equation of the line of centre can be taken as : 3x + 2y + k = 0.

This line passes through (1, 1)

3 + 2 + k = 0 ⇒ k = –5

Equation of AB is 3x + 2y – 5 = 0

The centres of A and B are situated on 3x + 2y – 5 = 0 at a distance $\sqrt{13}$ from (1, 1).

The centre B are given by

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

$\left(1+\sqrt{13}\left(-\frac{2}{\sqrt{13}}\right), 1+\sqrt{13}\cdot\frac{3}{\sqrt{13}}\right)$ and $\left(1-\sqrt{13}\left(-\frac{2}{\sqrt{13}}\right), 1-\sqrt{13}\cdot\frac{3}{\sqrt{13}}\right)$

i.e., (1 – 2, 1 + 3) and (1 + 2, 1 – 3)

(–1, 4) and (3, –2)

Case I.

Centre (–1, 4), $r = \sqrt{13}$

Equation of the circle is

$(x+1)^2+(y-4)^2=13$

$x^2+2x+1+y^2-8y+16-13=0$

$x^2+y^2+2x-8y+4=0$

Case II :

Centre (3, –2), $r=\sqrt{13}$

Equation of the circle is :

$(x-3)^2+(y+2)^2=13$

$x^2-6x+9+y^2+4y+4-13=0$

$x^2+y^2-6x+4y=0$

20.

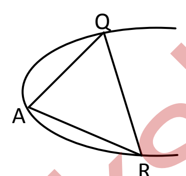
If a normal chord at a point t on the parabola $y^2 = 4ax$, subtends a right angle at vertex, then prove that $t=\pm\sqrt{2}$.

Sol. Equation of the parabola is $y^2 = 4ax$ …(1)

Equation of the normal at t is :

$tx + y = 2at + at^3$

$\frac{tx+y}{2at+at^3}=1$ …(2)



Homogenising (1) with the help of (2) combined equation of AQ, AR is

$y^2=\frac{4ax(tx+y)}{a(2t+t^3)}$

$y^2(2t+t^3)=4tx^2+4xy$

$4tx^2+4xy-(2t+t^3)y^2=0$

AQ, AR are perpendicular

Coefficient of x^2 + coefficient of y^2 = 0

$4t - 2t - t^3 = 0$

$2t - t^3 = 0$

$-t(t^2 - 2) = 0$

$t^2 - 2 = 0 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$

21. If $I_{m,n} = \int \sin^m x \cos^n x\,dx$, then show that $I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$ for a positive integer n and an integer $m \geq 2$.

Sol. $I_{m,n} = \int \sin^m x \cos^n x\,dx$

$$= \int \sin^{m-1} x \cdot (\cos x)^n \sin x\,dx$$

$$= \int \sin^{m-1}(x)(\cos x)^n(-\sin x)dx$$

$$= -\left[\sin^{m-1}(x) \int (\cos x)^n (-\sin x)dx - \int \left\{\frac{d}{dx}\sin^{m-1}(x) \cdot \int \cos^n(x)(-\sin x)dx\right\}dx\right]$$

$$= -\left[\sin^{m-1}(x)\frac{\cos^{n+1}(x)}{n+1} - \int \left\{(m-1)\sin^{m-2}(x)\cos x \frac{\cos^{n+1} x}{n+1}\right\}dx\right]$$

$$= -\sin^{m-1}(x)\frac{\cos^{n+1}(x)}{n+1} + \frac{m-1}{n+1}\int \{\sin^{m-2}(x)\cos^n x \cos^2 x\}dx$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)\int \{\sin^{m-2}(x)\cos^n x - \sin^m(x)\cos^n(x)\}dx$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)\left[\int \sin^{m-2}(x)\cos^n x\,dx - \int \sin^m(x)\cos^n(x)dx\right]$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)I_{m-2,n} - \left(\frac{m-1}{n+1}\right)I_{m,n}$$

$$\therefore I_{m,n} + \left(\frac{m-1}{n+1}\right)I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right)I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\therefore \left(\frac{m+n}{n+1}\right)I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1}$$

$$+\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\therefore I_{m,n} = \frac{1}{m+n}(\sin^{m-1}(x)\cos^{n+1}(x))$$

$$+\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

22. $\int \frac{dx}{(1-x)\sqrt{3-2x-x^2}}$ on (–1, 3).

Sol: Put 1 – x = 1/t ⇒ x = $1-\frac{1}{t}$

$$\therefore dx = +\frac{1}{t^2}dt$$

Also $3-2x-x^2 = 3-\left(1-\frac{1}{t}\right)-\left(1-\frac{1}{t}\right)^2$

$$= 3+\frac{2}{t}-2-\left(\frac{1}{t^2}-\frac{2}{t}+1\right)$$

$$= \frac{4}{t}-\frac{1}{t^2} = \frac{4t-1}{t^2}$$

$$\therefore \int \frac{dx}{(1-x)\sqrt{3-2x-x^2}} = \int \frac{\left(\frac{1}{t^2}\right)dt}{\left(\frac{1}{t}\right)\sqrt{\frac{4t-1}{t^2}}}$$

$$= \int \frac{dt}{\sqrt{4t-1}} = \frac{2\sqrt{4t-1}}{4}+c$$

$$= \frac{1}{2}\sqrt{4t-1}+c$$

$$= \frac{1}{2}\sqrt{4\left(\frac{1}{1-x}\right)-1}+c$$

$$= \frac{1}{2}\sqrt{\frac{4-1+x}{1-x}}+c = \frac{1}{2}\sqrt{\frac{3+x}{1-x}}.$$

23. $\int_{-1}^{3/2} |x\sin\pi x|\,dx$

Sol. We know that |x·sin πx| = x· sin πx

where $-1 \le x \le 1$

and |x·sinπx| = –x sin πx where $1 < x \le 3/2$

$$\therefore \int_{-1}^{3/2} |x\sin\pi x|\,dx = \int_{-1}^{1}|x\sin\pi x|\,dx + \int_{1}^{3/2}|x\sin\pi x|\,dx$$

$$= \int_{-1}^{1} x\sin\pi x\ dx - \int_{1}^{3/2} x\sin\pi x\ dx$$

$$= \left(-\frac{x\cdot\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right)_{-1}^{1} - \left(-x\cdot\frac{\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right)_{1}^{3/2}$$

$$= \frac{1}{\pi} + \frac{(-1)(-1)}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right]$$

$$= \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi} = \frac{3}{\pi} + \frac{1}{\pi^2}$$

24. $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

Sol. $\frac{dy}{dx} - \frac{x-2}{x(x-1)}y = \frac{x^3(2x-1)}{x(x-1)}$

$$\text{I.F.} = e^{\int\frac{2-x}{x(x-1)}}dx \Rightarrow \frac{2-x}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$2-x = A(x-1) + Bx$$

$$x = 0 \Rightarrow 2 = -A \Rightarrow A = -2$$

$$x = 1 \Rightarrow 1 = B \Rightarrow B = 1$$

$$\frac{2-x}{x(x-1)} = \frac{-2}{x} + \frac{1}{x+1}$$

$$\int\frac{2-x}{x(x-1)}dx = -2\int\frac{dx}{x} + \int\frac{dx}{x-1}$$

$$= -\log x + \log(x-1) = \log\frac{x-1}{x^2}$$

$$\text{I.F.} = e^{\log\frac{x-1}{x^2}} = \frac{x-1}{x^2}$$

$$y\frac{x-1}{x^2} = \int\frac{x^3(2x-1)}{x(x-1)}\cdot\frac{x-1}{x^2}dx$$

$$= \int(2x-1)dx = x^2 - x + c$$

Solution is $y(x-1) = x^2(x^2 - x + c)$