

MATHEMATICS PAPER IIB

COORDINATE GEOMETRY AND CALCULUS.

TIME : 3hrs

Max. Marks.75

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS.

10X2 =20

1. The equation of a circle having the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is
2. Show that the points $(-6, 1)$, $(2, 3)$ are Conjugate points with respect to the circle $x^2 + y^2 - 2x + 2y + 1 = 0$
3. Find the angle between the circles given by the equations
4. If the length of the major axis of an ellipse is three times the length of its minor axis then find the eccentricity of the ellipse.
5. The equation of the tangent at $(at^2, 2at)$ to the parabola is $y^2 = 4ax$ is $yt = x + at^2$.
6. Evaluate $\int \frac{1}{e^x - 1} dx$
7. Find $\int \left(1 - \frac{1}{x^2}\right) e^{\left(x + \frac{1}{x}\right)} dx$ on I where $I = (0, \infty)$.
8. Evaluate $\lim_{n \rightarrow \infty} \left(\frac{1 + 2^4 + 3^4 + \dots + n^4}{n^5} \right)$

9. Evaluate $\int_1^2 \frac{\log x}{x^2} dx$

10. Obtain the differential equation which corresponds to the following family of rectangular hyperbolas which have the coordinates axes as asymptotes.

SECTION B

SHORT ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING

5 X 4 = 20

11. Find the locus of P where the tangents drawn from to $x^2 + y^2 = a^2$ include an angle α

12. Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

13. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other then show that $fg = fg'$.

14. Find the equation of the ellipse referred to its major and minor axes as the coordinate axes X, Y – respectively with latus rectum of length 4 and distance between foci is $4\sqrt{2}$.

15. If e, e_1 be the eccentricity of a hyperbola and its conjugate hyperbola then $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.

16. Find the area of the region bounded by the curves

$$y = \sin 2x, y = \sqrt{3} \sin x, x = 0, x = \frac{\pi}{6}.$$

17. solve the equation $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$

SECTION C

LONG ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING

5 X 7= 35

18. Find the locus of the point whose polars with respect to the circles $x^2 + y^2 - 4x - 4y - 8 = 0$ and $x^2 + y^2 - 2x + 6y - 2 = 0$ are mutually perpendicular.

19) Find the equation of the circle which touches $x^2 + y^2 - 4x + 6y - 12 = 0$ (-1, -1) internally with a radius of 2.

20. Prove that the orthocenter of the triangle formed by any three tangents to a parabola lies on the directrix of the parabola.

21 Evaluate $\int \frac{2x^2 + x + 1}{(x + 3)(x - 2)^2} dx$

22. Evaluate $\int \frac{1}{\sin x + \sqrt{3} \cos x} dx$

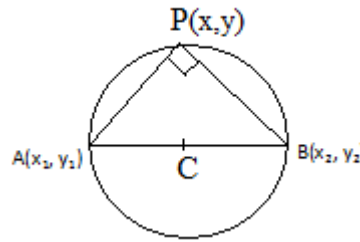
23. Show that

$$\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$$

24. solve the equation $y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0$

SOLUTIONS:

1. The equation of a circle having the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$



as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Let $P(x, y)$ be any point on the circle. Given points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Now $\angle APB = \frac{\pi}{2}$. (angle in a semi circle.)

Slope of AP. Slope of BP = -1

$$\Rightarrow \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow y - y_2 \quad y - y_1 = - \quad x - x_2 \quad x - x_1$$

$$\Rightarrow x - x_2 \quad x - x_1 + y - y_2 \quad y - y_1 = 0$$

2) Show that the points $(-6, 1), (2, 3)$ are conjugate points with respect to the circle

$$x^2 + y^2 - 2x + 2y + 1 = 0$$

Sol. $S = x^2 + y^2 - 2x + 2y + 1 = 0$

Points are $(-6, 1), (2, 3)$

$$\text{Now } S_{12} = -6 \cdot 2 + 1 \cdot 3 - (-6 + 2) + (1 + 3) + 1$$

$$= -12 + 3 + 4 + 4 + 1 = 0.$$

Therefore given points are conjugate points.

3. Find the angle between the circles given by the equations

$$i) \quad x^2 + y^2 - 12x - 6y + 41 = 0; \quad x^2 + y^2 - 4x + 6y - 59 = 0$$

$$\text{Sol.} \quad \sqrt{x^2 + y^2 - 12x - 6y + 41} = 0$$

$$\text{Centre } C_1 = (6, 3) \quad \text{radius} \quad r_1 = \{36 + 9 - 41\}^{1/2} = 2$$

$$x^2 + y^2 - 4x + 6y - 59 = 0, \text{ then centre } C_2 = (-2, -3)$$

$$\text{radius } r_2 = \{4 + 9 - 59\}^{1/2} = \{72\}^{1/2} = 6\sqrt{2}$$

$$C_1 C_2 = d = \sqrt{(6 + 2)^2 + (3 + 3)^2} = \sqrt{64 + 36} = 10$$

$$\text{Let } \theta \text{ be the angle between the circles, then } \cos \theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1 r_2} = \frac{100 - 4 - 72}{2 \times 2 \times 6\sqrt{2}} =$$

$$\frac{24}{4 \times 6\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

4. If the length of the major axis of an ellipse is three times the length of its minor axis then find the eccentricity of the ellipse.

Sol: Let the ellipse in the standard form be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(1)$$

Length of major axis is 'a' and length of minor axis is 'b'. Given that $a = 3b$

$$\Rightarrow a^2 = 9b^2 \Rightarrow a^2 = 9a^2(1 - e^2)$$

$$\Rightarrow 1 - e^2 = \frac{1}{9} \Rightarrow e^2 = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

$$\therefore \text{Eccentricity of the ellipse} = \frac{2\sqrt{2}}{3}.$$

5. The equation of the tangent at $(at^2, 2at)$ to the parabola is $y^2 = 4ax$ is $yt = x + at^2$.

Proof:

Equation of the parabola is $y^2 = 4ax$.

Equation of the tangent at $(at^2, 2at)$ is $S_1 = 0$.

$$\Rightarrow (2at)y - 2a(x + at^2) = 0$$

$$\Rightarrow 2aty = 2a(x + at^2) \Rightarrow yt = x + at^2.$$

6. $\int \frac{1}{e^x - 1} dx$

Sol. $\int \frac{1}{e^x - 1} dx = \frac{e^x - (e^x - 1)}{e^x - 1} dx = \int \frac{e^x dx}{e^x - 1} - \int dx$

$$= \log(e^x - 1) - x = \log(e^x - 1) - \log e^x + C$$

$$= \log \left| \frac{e^x - 1}{e^x} \right| + C$$

7. Find $\int \left(1 - \frac{1}{x^2}\right) e^{\left(x + \frac{1}{x}\right)} dx$ on I where

$$I = (0, \infty).$$

Sol: Let $x + \frac{1}{x} = t$ then $\left(1 - \frac{1}{x^2}\right) dx = dt$

$$\begin{aligned} \therefore \int \left(1 - \frac{1}{x^2}\right) e^{\left(x + \frac{1}{x}\right)} dx &= \int e^t dt \\ &= e^t + c = e^{\left(x + \frac{1}{x}\right)} + c. \end{aligned}$$

$$8. \lim_{n \rightarrow \infty} \left(\frac{1 + 2^4 + 3^4 + \dots + n^4}{n^5} \right)$$

$$\text{Sol: } \lim_{n \rightarrow \infty} \left(\frac{1 + 2^4 + 3^4 + \dots + n^4}{n^5} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{i^4}{n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^4$$

$$= \int_0^1 x^4 dx = \left[\frac{x^5}{5} \right]_0^1 = \frac{1}{5}.$$

$$9. \int_1^2 \frac{\log x}{x^2} dx$$

$$\text{Sol. } \int_1^2 \frac{\log x}{x^2} dx = \int_1^2 \log x \cdot \frac{1}{x^2} dx$$

$\uparrow f \quad \uparrow g$

$$= \int_1^2 \left[\log x \int \frac{dx}{x^2} - \int \frac{d}{dx} (\log x) \int \frac{dx}{x^2} dx \right]$$

$$= \int_1^2 \left[-\frac{\log x}{x} - \int \frac{1}{x} \left(-\frac{1}{x}\right) \right] dx = \int_1^2 \left[\frac{\log x}{x} - \frac{1}{x} \right]$$

$$= \left(\frac{-\log 2 - 1}{2} \right) + (1) = \frac{1}{2} (1 - \log 2)$$

10. Obtain the differential equation which corresponds to the following family of rectangular hyperbolas which have the coordinates axes as asymptotes.

Sol. Equation of the rectangular hyperbola is $xy=c^2$ where c is arbitrary constant.

Differentiating w.r.t. x

$$x \frac{dy}{dx} + y = 0$$

11. Find the locus of P where the tangents drawn from to $x^2 + y^2 = a^2$ include an angle α

Sol. Equation of the circle is $S = x^2 + y^2 = a^2$
radius = a

let (x_1, y_1) be any point $\Rightarrow S_{11} = x_1^2 + y_1^2 - a^2$

let $2\theta (= \alpha)$ be the angle between the tangents. Then

$$\Rightarrow \tan \theta = \frac{r}{\sqrt{S_{11}}} = \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}$$

$$\Rightarrow \cos 2\theta = \frac{1 - \frac{a^2}{x_1^2 + y_1^2 - a^2}}{1 + \frac{a^2}{x_1^2 + y_1^2 - a^2}}$$

$$\Rightarrow \cos \alpha = \frac{x_1^2 + y_1^2 - 2a^2}{x_1^2 + y_1^2}$$

$$\Rightarrow x_1^2 + y_1^2 \cos \alpha = x_1^2 + y_1^2 - 2a^2 \text{ Locus of } (x_1, y_1) \text{ is}$$

$$(x^2 + y^2) \cos \alpha = x^2 + y^2 - 2a^2$$

$$2a^2 = (x^2 + y^2) (1 - \cos \alpha)$$

$$2a^2 = (x^2 + y^2) (2 \sin^2 \alpha/2)$$

$$x^2 + y^2 = \frac{a^2}{\sin^2 \alpha/2} = a^2 \operatorname{cosec}^2 \frac{\alpha}{2}$$

12. Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

Sol. let $S = x^2 + y^2 + 2gx + 2fy + c = 0$

$S=0$ is passing through $(0, 0)$

$$\Rightarrow 0 + 0 + 2g \cdot 0 + 2f \cdot 0 + c = 0 \Rightarrow c = 0$$

$$x^2 + y^2 + 2gx + 2fy = 0$$

Centre $(-g, -f)$ is on $x + y = 4$

$$\therefore -g - f = 4 \text{-----(1)}$$

$S=0$ is orthogonal to

$$x^2 + y^2 - 4x + 2y + 4 = 0$$

$$\Rightarrow -4g + 2f = 4 + 0$$

$$\Rightarrow f - 2g = 2 \text{-----(2)}$$

Solving (1) and (2) we get

$$-3g = 6 \Rightarrow g = -2$$

$$f = -2$$

Equation of circle is $x^2 + y^2 - 4x - 4y = 0$

13. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other then show that $f'g = fg'$.

Sol. $S = x^2 + y^2 + 2gx + 2fy = 0$

Centre $C_1 = (-g, -f)$, radius $r_1 = \sqrt{g^2 + f^2}$

$$S^1 = x^2 + y^2 + 2g'x + 2f'y = 0$$

$C_2 = (-g', -f')$, $r_2 = \sqrt{g'^2 + f'^2}$

Given circles are touching circles,

$$\therefore C_1C_2 = r_1 + r_2$$

$$\Rightarrow (C_1C_2)^2 = (r_1 + r_2)^2$$

$$(g' - g)^2 + (f' - f)^2 = g^2 + f^2 + g'^2 + f'^2 + 2\sqrt{g^2 + f^2}\sqrt{g'^2 + f'^2}$$

$$-2(gg' + ff') = 2\{g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2\}^{1/2}$$

$$\Rightarrow (gg' + ff')^2 = g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2$$

$$g^2g'^2 + f^2f'^2 + 2gg'ff' = g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2$$

$$\Rightarrow 2gg'ff' = g^2f'^2 + f^2g'^2$$

$$\Rightarrow g^2f'^2 + f^2g'^2 - 2gg'ff' = 0$$

$$\Rightarrow (gf' - fg')^2 = 0 \Rightarrow gf' = fg'$$

14. Find the equation of the ellipse referred to its major and minor axes as the coordinate axes X, Y – respectively with latus rectum of length 4 and distance between foci is $4\sqrt{2}$.

Sol: Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$$

Length of the latus rectum

$$\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$$

Distance between foci, $S = (ae, 0)$ and $S' = (-ae, 0)$ is $2ae = 4\sqrt{2} \Rightarrow ae = 2\sqrt{2}$

$$\text{Also } b^2 = a^2(1 - e^2) \Rightarrow 2a = a^2 - (ae)^2 = a^2 - 8$$

$$\Rightarrow a^2 - 2a - 8 = 0$$

$$\Rightarrow (a - 4)(a + 2) = 0$$

$$\Rightarrow a = 4 (\because a > 0)$$

$$\therefore b^2 = 2a = 8$$

\therefore Equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{8} = 1 \text{ (or) } x^2 + 2y^2 = 16.$$

15. If e, e_1 be the eccentricity of a hyperbola and its conjugate hyperbola then

$$\frac{1}{e^2} + \frac{1}{e_1^2} = 1.$$

Sol. Equation of the hyperbola is

$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$e = \sqrt{\frac{a^2 + b^2}{a^2}} \Rightarrow e^2 = \frac{a^2 + b^2}{a^2}$$

$$\therefore \frac{1}{e^2} = \frac{a^2}{a^2 + b^2} \quad \dots(1)$$

Equation of the conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

$$e_1 = \sqrt{\frac{a^2 + b^2}{b^2}} \Rightarrow e_1^2 = \frac{a^2 + b^2}{b^2} \Rightarrow \frac{1}{e_1^2} = \frac{b^2}{a^2 + b^2} \quad \dots(2)$$

Adding (1) and (2)

$$\frac{1}{e^2} + \frac{1}{e_1^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$$

16. $y = \sin 2x, y = \sqrt{3} \sin x, x = 0, x = \frac{\pi}{6}.$

Sol;

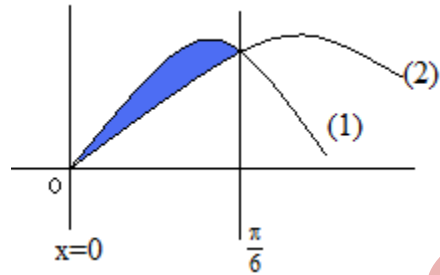
$$y = \sqrt{3} \sin x \quad \dots(2)$$

$$\text{Solving } \sin 2x = \sqrt{3} \sin x$$

$$\Rightarrow 2\sin x \cdot \cos x = \sqrt{3} \sin x$$

$$\Rightarrow \sin x = 0 \text{ or } 2 \cos x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x = 0, \cos x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{6}$$



$$\text{Required area} = \int_0^{\frac{\pi}{6}} \sin 2x - \sqrt{3} \sin x \, dx$$

$$= \left(-\frac{\cos 2x}{2} + \sqrt{3} \cos x \right)_0^{\frac{\pi}{6}}$$

$$= \left(-\frac{1}{4} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} \right) - \left(-\frac{1}{2} + \sqrt{3} \right)$$

$$= -\frac{1}{4} + \frac{3}{2} + \frac{1}{2} - \sqrt{3} = \frac{7}{4} - \sqrt{3} \text{ sq. units}$$

$$17. \frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$$

$$\frac{-dy}{y^2 + y + 1} = \frac{dx}{x^2 + x + 1}$$

Integrating both sides

$$-\int \frac{dy}{y^2 + y + 1} = \int \frac{dx}{x^2 + x + 1}$$

$$-\int \frac{dy}{\left(y + \frac{1}{2}\right)^2 + \frac{3}{4}} = \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$-\frac{2}{\sqrt{3}} \tan^{-1} \frac{(y+1/2)}{\sqrt{3/2}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{(x+1/2)}{\sqrt{3/2}} + c$$

$$\tan^{-1} \frac{2x+1}{\sqrt{3}} + \tan^{-1} \frac{2y+1}{\sqrt{3}} = c$$

18. Find the locus of the point whose polars with respect to the circles $x^2 + y^2 - 4x - 4y - 8 = 0$ and $x^2 + y^2 - 2x + 6y - 2 = 0$ are mutually perpendicular.

Sol. Equation of the circles is

$$S = x^2 + y^2 - 4x - 4y - 8 = 0 \quad (1)$$

$$S' = x^2 + y^2 - 2x + 6y - 2 = 0 \quad (2)$$

let P (x, y) be any position in the locus.

Equation of the polar of p w.r.to circle (1) is

$$xx_1 + yy_1 - 2(x + x_1) - 2(y + y_1) - 8 = 0$$

$$x(x_1 - 2) + y(y_1 - 2) - (2x_1 + 2y_1 + 8) = 0 \quad (3)$$

Polar of P w.r. to circle (2) is

$$xx_1 + yy_1 - 1(x + x_1) - 3(y + Y_1) - 2 = 0$$

$$x_1 + yy_1 - x - x_1 + 3y + 3y_1 - 2 = 0$$

$$x(x_1 - 1) + y(y_1 + 3) - (x_1 + 3y_1 + 2) = 0$$

(3) and (4) are perpendicular

$$\Rightarrow a_1 a_2 + b_1 b_2 = 0$$

$$(x_1 - 2)(x_1 - 1) + (y_1 - 2)(y_1 + 3) = 0$$

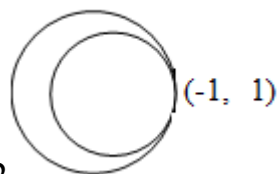
$$\Rightarrow x_1^2 + y_1^2 - 3x_1 + y_1 - 6 = 0$$

$$\text{Locus of } p(x_1, y_1) \text{ is } x^2 + y^2 - 3x + y - 4 = 0$$

19) Find the equation of the circle which touches $x^2 + y^2 - 4x + 6y - 12 = 0$ (-1, -1) internally with a radius of 2.

Sol. $x^2 + y^2 - 4x + 6y - 12 = 0$

$$C_1 = (2, -3), \quad r_1 = \sqrt{4 + 9 + 12} = 5$$



Radius of required circle is $r_2 = 2$

Let centre of the second circle be

$$C_2 = (h, k)$$

Point of contact $(-1, 1)$

Since the two circles touch internally, point of contact divides line of centres externally in the ratio 5:2

$$-1 = \frac{5h-4}{3} \quad 1 = \frac{5k+6}{3}$$

$$h = \frac{1}{5}, \quad k = \frac{3}{5}$$

centre = $(1/5, 3/5)$

Equation of a circle with centre $(\frac{1}{5}, \frac{3}{5})$ and radius 2 is given by

$$\left(x - \frac{1}{5}\right)^2 + \left(y + \frac{3}{5}\right)^2 = 4$$

$$5x^2 + 5y^2 - 2x + 6y - 18 = 0$$

20. Prove that the orthocenter of the triangle formed by any three tangents to a parabola lies on the directrix of the parabola.

Sol. Let $y^2 = 4ax$ be the parabola and , $A = (at_1^2, 2at_1), B = (at_2^2, 2at_2), C = (at_3^2, 2at_3)$ be any three points on it.

If P, Q, R are the points of intersection of tangents at A and B, B and C, C and A then

$$P = [at_1t_2, a(t_1 + t_2)], Q = [at_2t_3, a(t_2 + t_3)], R = [at_3t_1, a(t_3 + t_1)].$$

Consider the ΔPQR

then equation of \overline{QR} (Tangent at C) is $x - yt_3 + at_3^2 = 0$.

\therefore Altitude through P of ΔPQR is

$$t_3x + y = at_1t_2t_3 + a(t_1 + t_2) \quad \dots(1)$$

[\therefore Slope = $\frac{1}{t_3}$ and equation is

$$y - a(t_1 + t_2) = -t_3[x - at_1t_2]$$

$$\Rightarrow y + xt_3 = at_1t_2t_3 + a(t_1 + t_2)]$$

Similarly, the altitude through Q is

$$t_1x + y = at_1t_2t_3 + a(t_2 + t_3) \quad \dots(2)$$

Solving (1) and (2), we get

$$(t_3 - t_1)x = a(t_1 - t_3)$$

i.e., $x = -a$.

Hence, the orthocenter of the triangle PQR with x coordinate as $-a$, lies on the directrix of the parabola.

$$21. \int \frac{2x^2 + x + 1}{(x+3)(x-2)^2} dx$$

$$\text{Sol. } \frac{2x^2 + x + 1}{(x+3)(x-2)^2} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$2x^2 + x + 1 = A(x-2)^2 + B(x+3)(x-2) + C(x+3)$$

$$x = 2 \Rightarrow 8 + 2 + 1 = C(2+3) = 5C \Rightarrow C = \frac{11}{5}$$

$$x = -3 \Rightarrow 18 - 3 + 1 = A(-5)^2 = 25A \Rightarrow A = \frac{16}{25}$$

Equating the coefficients of x^2

$$2 = A + B \Rightarrow B = 2 - A = 2 - \frac{16}{25} = \frac{34}{25}$$

$$\begin{aligned} \int \frac{2x^2 + x + 1}{(x+3)(x-2)^2} &= \frac{16}{25} \int \frac{dx}{x+3} + \frac{34}{25} \int \frac{dx}{x-2} \\ &\quad + \frac{11}{5} \int \frac{1}{(x-2)^2} dx \\ &= \frac{16}{25} \log|x+3| + \frac{34}{25} \log|x-2| - \frac{11}{5(x-2)} + C \end{aligned}$$

$$22. \int \frac{1}{\sin x + \sqrt{3} \cos x} dx$$

$$\text{Sol. Let } t = \tan \frac{x}{2} \text{ so that } dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{2 \frac{dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{\sqrt{3}(1-t^2)}{1+t^2}} = 2 \int \frac{dt}{\sqrt{3}(1-t^2) + 2t}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dt}{1-t^2 + \frac{2}{\sqrt{3}}t} = \frac{2}{\sqrt{3}} \int \frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2 - \left(t - \frac{1}{\sqrt{3}}\right)^2}$$

$$\begin{aligned}
 &= \frac{2}{\sqrt{3}} \frac{1}{\sqrt{3}} \log \left| \frac{\frac{2}{\sqrt{3}} + t - \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}} - t + \frac{1}{\sqrt{3}}} \right| + C \\
 &= \frac{1}{2} \log \left| \frac{t + \frac{1}{\sqrt{3}}}{\sqrt{3} - t} \right| + C = \frac{1}{2} \log \left| \frac{\sqrt{3}t + 1}{\sqrt{3}(\sqrt{3} - t)} \right| + C \\
 &= \frac{1}{2} \log \left| \frac{\sqrt{3} \tan \frac{x}{2} + 1}{\sqrt{3} \left(\sqrt{3} - \tan \frac{x}{2} \right)} \right| + c
 \end{aligned}$$

23. Show that

$$\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$$

Sol. Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x \right)}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} \left(\frac{x}{\sin x + \cos x} + \frac{(\pi/2 - x)}{\sin x + \cos x} \right) dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

Put $t = \tan \frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$

$$I = \frac{\pi}{4} \int_0^1 \frac{2 \frac{dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \frac{\pi}{2} \int_0^1 \frac{dt}{2t+1-t^2}$$

$$\begin{aligned}
 &= \frac{\pi}{2} \int_0^1 \frac{dt}{(\sqrt{2})^2 + (t-1)^2} = \frac{\pi}{2} \left(\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1} \right)_0^1 \\
 &= \frac{-\pi}{4\sqrt{2}} \left(\log \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) = \frac{\pi}{4\sqrt{2}} \log(\sqrt{2} + 1)^2 \\
 &= \frac{\pi}{4\sqrt{2}} 2 \log(\sqrt{2} + 1) = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)
 \end{aligned}$$

24. $y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0$

Sol.

$$y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0$$

$$\left(x - \frac{1}{y}\right) \frac{dy}{dx} = -y^2$$

$$\frac{dx}{dy} = \frac{x - 1/y}{-y^2} = -\frac{x}{y^2} + \frac{1}{y^3}$$

$$\frac{dx}{dy} + \frac{1}{y^2} \cdot x = \frac{1}{y^3} \text{ which is l.d.e in } x$$

$$\text{I.F.} = e^{\int \frac{1}{y^2} dy} = e^{-1/y}$$

$$\text{Sol is } x \cdot \text{I.F.} = \int \text{Q. I.F. } dy$$

$$x \cdot e^{-1/y} = \int \frac{e^{-1/y}}{y^3} dy \quad \dots(1)$$

$$\text{put } -\frac{1}{y} = z \Rightarrow \frac{1}{y^2} dy = dz$$

$$= \int z \cdot e^z dz = e^z(z-1)$$

$$x \cdot e^{-1/y} = -e^{-1/y} \left(-\frac{1}{y} - 1 \right) + c$$

$$\frac{x}{e^{1/y}} = \frac{1+y}{y \cdot e^{1/y}} + c$$

Hence solution is $xy = 1 + y + cy e^{1/y}$.