MATHEMATICS PAPER IIB

COORDINATE GEOMETRY AND CALCULUS.

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. **10X2 =20**

1. If $x^2 + y^2 - 4x + 6y + c = 0$ represents a circle with radius 6 then find the value of c.

2. Discuss the relative position of the pair of circles $x^2 + y^2 - 4x - 6y - 12 = 0$ $x^2 + y^2 + 6x + 18y + 26 = 0$.

3. Find 'k' if the pair of circles $x^2+y^2 + 2by-k = 0$, $x^2+y^2+2ax+8=0$ are orthogonal.

4. Find the equation of the parabola whose focus is s(1, –7) and vertex is A(1, –2).

5. If the lines $3x - 4y = 12$ and $3x + 4y = 12$ meets on a hyperbola S = 0 then find the eccentricity of the hyperbola S = 0.

6. Evaluate $\int e^x \frac{x+2}{(x+3)^2} dx$ on $I \subset R \setminus \{-3\}$

7. Evaluate $\int \frac{dx}{1+\cos^2 x}$

8. Evaluate $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

9. Evaluate $\int_{0}^{4} |2-x|\, dx$

10. Obtain the differential equation which corresponds to each of the following family of the circles which touch the Y-axis at the origin.

SECTION B

SHORT ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING **5 X 4 = 20**

11. Show that the tangent at (-1,2) the Circle $x^2 + y^2 - 4x - 8y + 7 = 0$ touches the Circle $x^2 + y^2 + 4x + 6y = 0$ and also Find its point of contact.

12. Find the equation of a circle which passes through (2,-3)and (-4,5) and having the centre on
$4x + 3y + 1 = 0$

13. If d is the distance between the centers of two intersecting circles with radii r_1, r_2 and θ is the angle between the circles then $\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$.

14. Find the length of major axis, minor axis, latus rectum, eccentricity, coordinates of the centre, foci and equations of directrices of the ellipse.
$9x^2 + 16y^2 = 144$

15. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel (ii) perpendicular to the line $x + 2y = 0$.

16. Find the area of the region $x, y \ / \ x^2 - x - 1 \le y \le -1$

17. Solve the differential equation $\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$

SECTION C

LONG ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING **5 X 7= 35**

18. Find the pair of tangents drawn from (1,3) to the circle $x^2+y^2 - 2x + 4y - 11 = 0$

19.Find the equation of the circle passing through (-1, 0) and touching $x + y - 7=0$ at (3,4)

20. The equation of a parabola in the standard form is $y^2 = 4ax$.

21.Evaluate $\int a^x \cos 2x \, dx$ on R(a > 0 and a ≠ 1).

22.Evaluate $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$

23. Evaluate $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

24. Solve $1+e^{x/y}\, dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$.

SOLUTIONS

1. If $x^2 + y^2 -4x + 6y + c = 0$ represents a circle with radius 6 then find the value of c.

sol.

Centre = (-g, -f) = (2,-3)

$r = \sqrt{g^2 + f^2 - c}$; g = - 2, f = 3

$\Rightarrow 6 = \sqrt{4 + 9 - c}$

$36 = 13 - c \Rightarrow c = -23$

. Discuss the relative position of the following pair of circles.

2. $x^2 + y^2 - 4x - 6y - 12 = 0$

$x^2 + y^2 + 6x + 18y + 26 = 0.$

Sol. Centers of the circles are A (2,3), B(-3, -9)

Radii are $r_1 = \sqrt{4+9+12} = 5$

$r_2 = \sqrt{9+81-26} = 8$

$AB = \sqrt{(2+3)^2 + (3+9)^2}$

$= \sqrt{25+144} = 13 = r_1 + r_2$

$\therefore$ The circle touches externally.

3 Find 'k' if the following pair of circles are orthogonal.

$x^2+y^2 + 2by-k = 0,\ x^2+y^2+2ax+8=0$

Sol. Given circles are $x^2+y^2 + 2by-k = 0,\ x^2+y^2+2ax+8=0$

from above equations $g_1 = 0$; $f_1 = b$; $c_1 = -k$

$g_2 = a$; $f_1 = 0$; $c_1 = 8$

since the circles are orthogonal ,

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2(0)(a) + 2(b)(0) = -k + 8$

$0 = -k + 8$

$K = 8$

4. Find the equation of the parabola whose focus is s(1, –7) and vertex is A(1, –2).

Sol.

Focus s = (1, -7), vertex A(1, –2)

h = 1, k = –2, a = –2 + 7 = 5

since x coordinates of S and A are equal, axis of the parabola is parallel to y-axis.

And the y coordinate of S is less than that of A, therefore the parabola is a down ward parabola.

Let equation of the parabola be

$(x - h)^2 = -4a(y - k)$

$(x - 1)^2 = -20(y + 2)$

$x^2 - 2x + 1 = -20y - 40$

$\Rightarrow x^2 - 2x + 20y + 41 = 0$

5. If the lines 3x – 4y = 12 and 3x + 4y = 12 meets on a hyperbola S = 0 then find the eccentricity of the hyperbola S = 0.

Sol. Given lines 3x – 4y = 12 , 3x + 4y = 12

The combined equation of the lines is

(3x – 4y)(3x + 4y) = 144

$9x^2 - 16y^2 = 144$

$$\frac{x^2}{\frac{144}{9}} - \frac{y^2}{\frac{144}{16}} = 1 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$$

$a^2 = 16, b^2 = 9$

eccentricity $e = \sqrt{\frac{a^2 + b^2}{a^2}}$

$$= \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

6. $\int e^x \frac{x+2}{(x+3)^2} dx$ on $I \subset R \setminus \{-3\}$

Sol. $\int e^x \frac{x+2}{(x+3)^2} dx$

Hint : $\int e^x \, f(x) + f'(x) \, dx = e^x - f(x) + C$

$$= \int e^x \left\{\frac{x+3-1}{(x+3)^2}\right\} dx$$

$$= \int e^x \left\{\frac{1}{x+3} + \frac{(-1)}{(x+3)^2}\right\} dx = e^x \left(\frac{1}{x+3}\right) + C$$

7. $\int \frac{dx}{1+\cos^2 x}$

Sol. $\int \frac{dx}{1+\cos^2 x} = \int \frac{\sec^2 dx}{\sec^2 x + 1} = \int \frac{\sec^2 xdx}{\tan^2 x + 2}$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$= \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

8. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx \quad \ldots(i)$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(\pi/2 - \pi/2 - x)dx}{1+e^{-x}} \left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$ Adding (1) and (2) ,

$$= \int_{-\pi/2}^{\pi/2} \frac{e^x \cos xdx}{1+e^x} ----(2)$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos x(1+e^x)}{1+e^x} dx = \int_{-\pi/2}^{\pi/2} \cos xdx$$

$$2I = 2\int_0^{\pi/2} \cos x\, dx \quad \because \cos x \text{ is even function}$$

$$\Rightarrow I = \sin x \Big|_0^{\pi/2} \Rightarrow I = 1$$

9. $\int_0^4 |2-x|\, dx$

Sol. $\int_0^2 |2-x|\, dx + \int_2^4 |2-x|\, dx$

$$= \int_0^2 (2-x)dx + \int_2^4 (x-2)dx$$

$$= \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left(4 - \frac{4}{2}\right) - \left[(8-8) - \left(4 - \frac{4}{2}\right)\right]$$

$$= 2 - 0 + 2 = 4$$

10.Obtain the differential equation which corresponds to each of the family of The circles which touch the Y-axis at the origin.

Sol. Equation of the given family of circles is

$x^2 + y^2 + 2gx = 0$, g arbitrary const …(i)

$x^2 + y^2 = -2gx$

Differentiating w.r.t. x

$2x + 2yy_1 = -2g$ …(ii)

Substituting in (i)

$x^2 + y^2 = x(2x + 2yy_1)$ by (ii)

$= 2x^2 + 2xyy_1$

$yy^2 - 2xyy_1 - 2x^2 = 0$

$y^2 - x^2 = 2xy\frac{dy}{dx}$.

11 Show that the tangent at (-1,2) the Circle $x^2 + y^2-4x-8y+7=0$ touches the Circle $x^2 + y^2+4x+6y=0$ and also Find its point of contact.

Sol. $S\equiv x^2 + y^2 - 4x - 8y + 7 = 0$ equation of the tangent at (-1, 2) to S =0 is $S_1=0$

$\Rightarrow x(-1) + y(2) - 2(x-1) - 4(y+2) + 7 = 0$

$\Rightarrow -3x - 2y + 1 = 0 \Rightarrow 3x + 2y - 1 = 0$.

Equation of the second circle is $x^2 + y^2 + 4x + 6y = 0$

Centre C =(-2,-3) radius r $= \sqrt{g^2 + f^2 - c} = \sqrt{4+9} = \sqrt{13}$

Perpendicular distance from C to the line is d $= \left|\frac{3(-2)+2(-3)-1}{\sqrt{3^2+2^2}}\right| = \left|\frac{-13}{\sqrt{13}}\right| = \sqrt{13}$

d = r

Hence 3x + 2y – 1 =0 is also tangent to

$x^2 + y^2+4x +6y = 0$

Point of contact (foot of perpendicular)

Let (h, k) be foot of perpendicular from (-2,-3) to the line 3x+2y-1 =0

$\frac{h+2}{3} = \frac{k+3}{2} = \frac{|3(-2)+2(-3)-1|}{9+4} \Rightarrow \frac{h+2}{3} = 1$ and $\frac{k+3}{2} = 1$

h = 1, k = -1 therefore (1, -1) is point of contact.

12. Find the equation of a circle which passes through (2,-3)and (-4,5) and having the centre on

$4x + 3y + 1 = 0$

sol.

Let S(a,b) be the centre of the circle.

S(a,b) is a point on the line 4x + 3y +1 = 0

=> 4a + 3b +1 = 0 ----------(1)

A(2,-3)and B(-4,5) are two points on the circle.

Therefore, SA=SB => $SA^2 = SB^2$

=> $(a - 2)^2 + (b + 3)^2 = (a + 4)^2 + (b - 5)^2$

=> 3a-4b+7 = 0 ----(2)

Solving (1) and (2), we get

$(a,b) = (-1,1)$ = centre.

Raidus= $SA = \sqrt{(2+1)^2 + (-3-1)^2}$

$= 5$

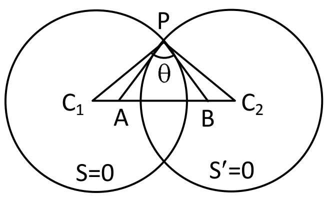
Equation of the circle is $(x+1)^2 + (y-1)^2 = 5^2$

: $x^2 + y^2 + 2X - 2y - 23 = 0$

13. If d is the distance between the centers of two intersecting circles with radii r_1, r_2 and θ is the angle between the circles then $\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$.

Proof :

Let C_1, C_2 be the centres of the two circles $S = 0$, $S' = 0$ with radii r_1, r_2 respectively. Thus $C_1C_2 = d$. Let P be a point of intersection of the two circles. Let PB, PA be the tangents of the circles $S = 0$, $S' = 0$ respectively at P.



Now $PC_1 = r_1$, $PC_2 = r_2$, $\angle APB = \theta$

Since PB is a tangent to the circle $S = 0$, $\angle C_1PB = \pi/2$

Since PA is a tangent to the circle $S' = 0$, $\angle C_2PA = \pi/2$

Now $\angle C_1PC_2 = \angle C_1PB + \angle C_2PA - \angle APB = \pi/2 + \pi/2 - \theta = \pi - \theta$

From ΔC_1PC_2, by cosine rule,

$$C_1^2C_2^2 = PC_1^2 + PC_2^2 - 2PC_1 \cdot PC_2 \cos\angle C_1PC_2 \Rightarrow$$

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\pi - \theta) \Rightarrow d^2 = r_1^2 + r_2^2 + 2r_1r_2\cos\theta$$

$$\Rightarrow 2r_1r_2\cos\theta = d^2 - r_1^2 - r_2^2 \Rightarrow \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

14. Find the length of major axis, minor axis, latus rectum, eccentricity, coordinates of the centre, foci and equations of directrices of the following ellipse.

$9x^2 + 16y^2 = 144$

Sol. Given equation is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$\therefore$ $a = 4$, $b = 3$ where $a > b$

Length of major axis = 2a = 2 × 4 = 8

Length of minor axis = 2b = 2 × 3 = 6

Length of latus rectum = $\frac{2b^2}{a} = \frac{2\cdot 9}{4} = \frac{9}{2}$

Eccentricity = $\sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

Centre is C(0, 0)

Foci are (±ae, 0) = $(\pm\sqrt{7}, 0)$

Equations of the directrices are

$$x = \pm\frac{a}{e} \Rightarrow x = \pm 4\cdot\frac{4}{\sqrt{7}} = \pm\frac{16}{\sqrt{7}}$$

$$\Rightarrow \sqrt{7}x = \pm 16$$

$$\frac{x^2}{9} - \frac{y^2}{27} = 1 \Rightarrow 3x^2 - y^2 = 27$$

15. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel (ii) perpendicular to the line x + 2y = 0.

Sol. Equation of the hyperbola is $x^2 - 4y^2 = 4$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1 \Rightarrow a^2 = 4, b^2 = 1$$

i)

given line is x + 2y = 0

since tangent is parallel to x + 2y = 0, slope of the tangent is $m = -\frac{1}{2}$

$$c^2 = a^2m^2 - b^2 = 4\cdot\frac{1}{4} - 1 = 1 - 1 = 0$$

c = 0

Equation of the parallel tangent is :

$$y = mx + c = -\frac{1}{2}x$$

$\Rightarrow 2y = -x \Rightarrow x + 2y = 0$

ii) The tangent is perpendicular to x + 2y = 0

Slope of the tangent $m = \frac{-1}{(-1/2)} = 2$

$c^2 = a^2m^2 - b^2 = 4\cdot 4 - 1 = 15$

$c = \pm\sqrt{15}$

Equation of the perpendicular tangent is

$y = 2x \pm \sqrt{15}$.

16. Find the area of the region $\{(x, y) / x^2 - x - 1 \le y \le -1\}$

Sol. let the curves be $y = x^2 - x - 1$ ------(1)

and $y = -1$ --------(2)

$$y = x^2 - x - 1 = \left(x - \frac{1}{2}\right)^2 - \frac{5}{4}$$

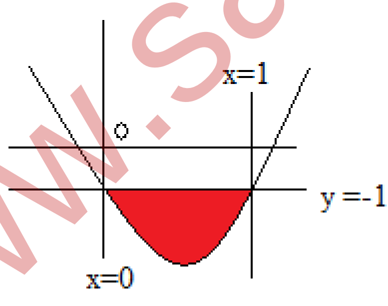
$y = \frac{5}{4} - \left(x - \frac{1}{2}\right)^2$ is a parabola with

vertex $\left(\frac{1}{2}, -\frac{5}{4}\right)$

from (1) and (2),

$$x^2 - x - 1 = -1 \Rightarrow x^2 - x = 0 \Rightarrow x = 0, x = 1$$

Given curves are intersecting at x=0 and x=1.



Required area = $\int_0^1 \left[y \text{ of } (1) - y \text{ of } (2) \right] dx$

$$A = \left| \int_0^1 x^2 - x - 1 \; dx - \int_0^1 -1 \; dx \right|$$

$$= \left| \int_0^1 \left(\frac{x^3}{3} - \frac{x^2}{2} - x \right) -_0^1 \; -x \right| = \frac{1}{6} \text{sq.units}$$

17. Solve the following differential equation.

$$\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$$

Sol. $\frac{dy}{dx} + \tan x \cdot y = \sec^3 x$ which is l.d.e in y

$$\text{I.F.} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$$

Sol is y.I.F. = $\text{y.I.F} = \int \text{Q. I.F. } dx$

$$y \cdot \sec x = \int \sec^4 x dx = \int (1 + \tan^2 x) \sec^2 x dx$$

$$= \tan x + \frac{\tan^3 x}{3} + c$$

18. Find the pair of tangents drawn from (1,3) to the circle $x^2+y^2 - 2x + 4y - 11 = 0$

Sol. $S = x^2+y^2 - 2x + 4y - 11 = 0$

Equation of pair of tangents from (3,2) to S=0 is $S.S_{11} = S_1^2$

$(x^2+y^2 - 2x + 4y - 11)(1+9-2+12-11)$

$= [x + 3y - 1(x+1) + 2(y + 3) - 11]^2$

$(x^2+y^2 - 2x + 4y - 11)\, 9 = (5y - 6)^2$

$9x^2+9y^2 - 18x + 36y - 99)$

$= 25y^2+36 - 60y$

$9x^2-16y^2 - 18x + 96y - 135 = 0$

Let θ be the angle between the pair of tangents. Then

$\text{Cos}\,\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}} = \frac{|9-16|}{\sqrt{(25)^2}}$

$= \frac{|-7|}{25} = \frac{7}{25}$

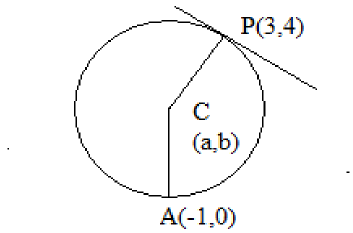
$\Rightarrow \theta = \cos^{-1}\left(\frac{7}{25}\right)$

19. Find the equation of the circle passing through (-1, 0) and touching x + y – 7=0 at (3,4)

Sol.

Let C(a,b) be the centre of the circle A(-1,0) and P(3,4)

Equation of the tangent is x+y-7 =0 -----------(i)



Now CA = CP

$\Rightarrow CA^2 = CP^2$

$\Rightarrow (a+1)^2 + b^2 = (a-3)^2 + (b-4)^2$

$\Rightarrow$ 8a +8b -24 =0 $\Rightarrow$a+b -3 = 0 ---(2)

Line CP is perpendicular to tangent (1)

$\therefore$ product of their slopes = -1

$\left(\frac{b-4}{a-3}\right)(-1) = -1 \Rightarrow$ a – b +1 =0 ----(3)

Solving (2) and (3) , a=1 and b=2.

Centre C =(1, 2)

radius r = CA = $\sqrt{(1+1)^2 + 2^2} = \sqrt{8}$

equation of the circle is $(x-1)^2 + (y-2)^2 = 8$

i.e., $x^2+y^2- 2x – 4y – 3 =0$

20. The equation of a parabola in the standard form is $y^2 = 4ax$.

Proof

Let S be the focus and L = 0 be the directrix of the parabola.

Let P be a point on the parabola.

Let M, Z be the projections of P, S on the directrix L = 0 respectively.

Let N be the projection of P on SZ.

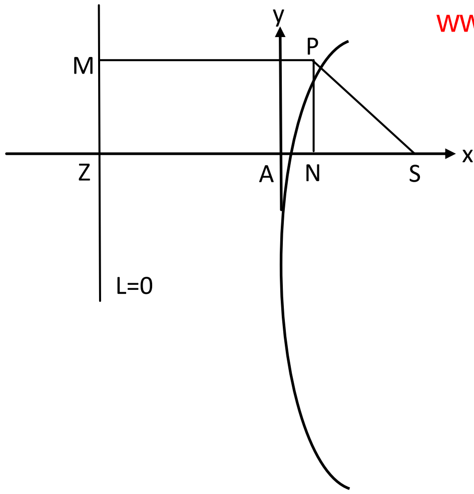
Let A be the midpoint of SZ.

Therefore, SA = AZ, $\Rightarrow$ A lies on the parabola. Let AS = a.

Let AS, the principal axis of the parabola as x-axis and Ay perpendicular to SZ as y-axis.

Then S = (a, 0) and the parabola is in the standard form.

Let P = (x_1, y_1).



Now PM = NZ = NA + AZ = x_1 + a

P lies on the parabola $\Rightarrow \frac{PS}{PM} = 1 \Rightarrow PS = PM$

$$\Rightarrow \sqrt{(x_1 - a)^2 + (y_1 - 0)^2} = x_1 + a$$

$$\Rightarrow (x_1 - a)^2 + y_1^2 = (x_1 + a)^2$$

$$\Rightarrow y_1^2 = (x_1 + a)^2 - (x_1 - a)^2 \Rightarrow y_1^2 = 4ax_1$$

The locus of P is $y^2 = 4ax$.

$\therefore$ The equation to the parabola is $y^2 = 4ax$.

21. $\int a^x \cos 2x\, dx$ on R(a > 0 and a ≠ 1).

Sol. $\int a^x \cos 2x\, dx$

$$= a^x \frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x \cdot a^x \log a\, dx$$

$$= \frac{a^x \cdot \sin 2x}{2} + \frac{\log a}{2}\int a^x(-\sin 2x)dx$$

$$= \frac{a^x \sin 2x}{2} + \frac{\log a}{2}\left(a^x \cdot \cos\frac{2x}{2} - \frac{1}{2}\int \cos 2x \cdot a^x \log a\, dx\right)$$

$$= \frac{a^x \sin 2x}{2} + \frac{a^x \log a \cos 2x}{4} - \frac{(\log a)^2}{4} I$$

$$\left(1 + \frac{(\log a)^2}{4}\right) I = \frac{a^x[2\sin 2x + (\log a)\cos 2x]}{4}$$

$$\frac{4 + (\log a)^2}{4} I = \frac{a^x[2\sin 2x + (\cos 2x)\log a]}{4}$$

$$\therefore I = \frac{2 \cdot a^x \cdot \sin 2x + (a^x \cdot \log a)\cos 2x}{(\log a)^2 + 4} + c$$

22. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$

Sol. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$

let 9cosx-sinx $= A \frac{d}{dx}\ 4\sin x + 5\cos x\ + B\ 4\sin x + 5\cos x$

9cosx-sinx $= A\ \ 4\cos x - 5\sin x\ + B\ 4\sin x + 5\cos x$

Comparing the coefficients of sin and cos , we get

9 = 4A+5B and -5 = -5A+4B

Solving these equations , A =1 and B=1.

$\therefore$ 9cosx-sinx $=1\ 4\cos x - 5\sin x\ +1\ 4\sin x + 5\cos x$

$=1\ 4\cos x - 5\sin x\ +1\ 4\sin x + 5\cos x$

$$\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx = \int \frac{(4\sin x + 5\cos x) + (4\cos x - 5\sin x)}{4\sin x + 5\cos x} dx$$

$$= \int dx + \int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x} dx$$

$$= x + \log|4\sin x + 5\cos x| + C$$

23. $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

Sol: Let

$$y = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n^2+1^2}{n^2}\right)\left(\frac{n^2+2^2}{n^2}\right)...\left(\frac{n^2+n^2}{n^2}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{n^2+i^2}{n^2}\right)^{1/n}$$

$$\therefore y = \lim_{n\to\infty}\left(\frac{n^2+i^2}{n^2}\right)^{1/n}$$

$$\therefore \log y = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{n^2+i^2}{n^2}\right)^{1/n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\log\left[1+\left(\frac{i}{n}\right)^2\right]$$

$$= \int_0^1 \log(1+x^2)dx$$

(using integration by parts)

$$= \left[x\log(1+x^2)\right]_0^1 - \int_0^1 x\frac{2x}{1+x^2}dx$$

$$= \log 2 - 2\int_0^1\left(\frac{x^2+1-1}{x^2+1}\right)dx$$

$$= \log 2 - 2\ \ x\ {}_0^1 + 2\left[\tan^{-1}x\right]_0^1$$

$$= \log 2 - 2 + 2(\tan^{-1}1)$$

$$= \log 2 - 2 + 2\frac{\pi}{4}$$

$$\therefore \log_e y = \log 2 - 2 + \frac{\pi}{2}$$

$$y = e^{\log 2 - 2 + \pi/2}$$

$$= e^{\log_e 2\cdot e^{\frac{\pi}{2}-2}}$$

$$= 2e^{\frac{\pi-4}{2}}$$

$$\therefore \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n} = 2e^{\frac{\pi-4}{2}}.$$

24. Solve $1+e^{x/y}\ \ dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$.

Sol. $1+e^{x/y}\ \ dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$$\Rightarrow \frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}}$$ which is a homogeneous d.e.

Put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$(1+e^{v})\frac{dx}{dy} + e^{v}(1-v) = 0$$

$$(1+e^{v})\left(v + y\frac{dv}{dy}\right) + e^{v}(1-v) = 0$$

$$v + ve^{v} + y(1+e^{v})\frac{dv}{dy} + e^{v} - ve^{v} = 0$$

$$y(1+e^{v})dv = -(v+e^{v})dy$$

$$\int\frac{1+e^{v}}{v+e^{v}}dv = -\int\frac{dy}{y}$$

$$\log(v+e^{v}) = -\log y + \log c \Rightarrow v + e^{v} = \frac{c}{y}$$

$$\frac{x}{y} + e^{x/y} = \frac{c}{y} \Rightarrow x + y\cdot e^{x/y} = c$$