## MATHEMATICS PAPER IIA

TIME : 3hrs
Max. Marks. 75
Note: This question paper consists of three sections $A, B$ and $C$.

## SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. 10X2 =20
1.Prove that roots of $(x-a)(x-b)=h^{2}$ are always real.
2.If $\alpha, \beta$ and $\gamma$ are the roots of $x^{3}+p x^{2}+q x+r=0$, then find $\sum \frac{1}{\alpha^{2} \beta^{2}}$
3.Show that the points in the Argand diagram represented by the complex numbers $2+2 i,-2-2 i,-2 \sqrt{3}+2 \sqrt{3} i$ are the vertices of an equilateral triangle.
4.If $z=2-3 i$, then show that $z^{2}-4 z+13=0$.
5.If $1, \omega, \omega^{2}$ are the cube roots of units then prove that $\frac{1}{2+\omega}+\frac{1}{1+2 \omega}=\frac{1}{1+\omega}$
6.If ${ }^{(n+1)} P_{5}:{ }^{n} P_{6}=2: 7$ find $n$
7.Find the number of ways of arranging the letters of the word INDEPENDENCE
8. Find the set of $x$ for which the binomial expansion $(2+5 x)^{-1 / 2}$ Is valid
9. Find the maximum number of times a fair coin must be tossed so that the probability of getting atleast one head is atleast 0.8 .
10. Find the variance for the discrete data given below.
$6,7,10,12,13,4,8,12$

## SECTION B <br> SHORT ANSWER TYPE QUESTIONS. ANSWER ANY FIVE OF THE FOLLOWING

11. Let be the real roots of $a x^{2}+b x+c=0$ and $\alpha<\beta$. Then prove that i) $x \in \boldsymbol{R}$, $\alpha<x<\beta \Rightarrow$ $a x^{2}+b x+c$ and $a$ have the opposite signs
ii) $\quad x \in R, \quad x<\alpha \quad$ or $\quad x>\beta \Rightarrow a x^{2}+b x+c<$ and $a$ have the same sign.
$\therefore$
12. If n is an integer and $z=c i s \theta$ then show that $\frac{z^{2 n}-1}{z^{2 n}+1}=i \tan n \theta$
13.Find the number of 4 -digit numbers which can be formed using the digits 0,2 , 5, 7,8 that are divisible by (i) 2 (ii) 4 when repetition is allowed.
13. prove that ${ }^{n+1} c_{r}={ }^{n} c_{r-1}+{ }^{n} c_{r}$
14. Resolve $\frac{x+3}{(1-x)^{2}\left(1+x^{2}\right)}$ in to partial fractions.
15. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, find the probability that (i) you both enter the same section (iii) you both enter the different sections.
16. The probability that Australia wins a match against India in a cricket game is given to be $1 / 3$. If India and Australia play 3 matches, what is the probability that,
i) Australia will loose all the three matches?
ii) Australia will win atleast one match?

## SECTION C <br> LONG ANSWER TYPE QUESTIONS. ANSWER ANY FIVE OF THE FOLLOWING

18. Given that the roots of $x^{3}+3 p x^{2}+3 p x+r=0$ are in (i) A.P., show that $2 p^{2}-3 q p+r=0$
(ii) G.P., show that $p^{3} r=q^{3} \quad$ (iii) H.P., show that $2 q^{3}=r(3 p q-r)$
19. If $\cos \alpha+\cos \beta+\cos \vartheta=0=\sin \alpha+\sin \beta+\sin \vartheta=0$ then show that
(I) $\cos (2 \alpha-\beta-\vartheta)+\cos \{2 \beta-\vartheta-\alpha\}+\sin (2 \vartheta-\alpha-\beta)=3$
(II) $\sin (2 \alpha-\beta-\vartheta)+\sin (2 \beta-\vartheta-\alpha)+\sin (2 \vartheta-\alpha-\beta)=0$
(iii) $\cos (\alpha+\beta)+\cos (\beta+\vartheta)+\cos (\vartheta+\alpha)=0$
(iv) $\sin (\alpha+\beta)+\sin (\beta+\vartheta)+\sin (\vartheta+\alpha)=0$
20. Using
binomial
theorem,
prove
that $5^{4 n}+52 n-1$ is divisible by 676 for all positive integers $n$.
21. For $\mathrm{n}=0,1,2,3, \ldots \mathrm{n}$, prove that $\mathrm{C}_{0} \cdot \mathrm{C}_{\mathrm{r}}+\mathrm{C}_{1} \cdot \mathrm{C}_{\mathrm{r}+1}+\mathrm{C}_{2} \cdot \mathrm{C}_{\mathrm{r}+2}+\ldots+\mathrm{C}_{\mathrm{n}-\mathrm{r}} \cdot \mathrm{C}_{\mathrm{n}}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}+\mathrm{r}}$ and hence deduce that
i) $\mathrm{C}_{0}^{2}+\mathrm{C}_{1}^{2}+\mathrm{C}_{2}^{2}+\ldots+\mathrm{C}_{\mathrm{n}}^{2}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}}$
ii) $\mathrm{C}_{0} \cdot \mathrm{C}_{1}+\mathrm{C}_{1} \cdot \mathrm{C}_{2}+\mathrm{C}_{2} \cdot \mathrm{C}_{3}+\ldots+\mathrm{C}_{\mathrm{n}-1} \mathrm{C}_{\mathrm{n}}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}+1}$
22. If $A, B, C$ are three independent events such that $P\left(A \cap B^{C} \cap C^{C}\right)=\frac{1}{4}$ $\mathrm{P}\left(\mathrm{A}^{\mathrm{C}} \cap \mathrm{B} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{8}, \mathrm{P}\left(\mathrm{A}^{\mathrm{C}} \cap \mathrm{B}^{\mathrm{C}} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4}$ then find $\mathrm{P}(\mathrm{A}), \mathrm{P}(\mathrm{B})$ and $\mathrm{P}(\mathrm{C})$.
23. The probability of a bomb hitting a bridge is $1 / 2$ and three direct hits (not necessarily consecutive) are needed to destroy it. Find the minimum number of bombs required so that the probability of the bridge being destroyed is greater than 0.9.
24. The arithmetic mean and standard deviation of a set of 9 items are 43 and 5 respectively. If an item of value 63 is added to that set, find the new mean and standard deviation of 10 item set given.

## SOLUTIONS

1.Prove that roots of $(x-a)(x-b)=h^{2}$ are always real.

Sol: $\quad(x-a)(x-b)=h^{2}$

$$
x^{2}-(a+b) x+\left(a b-h^{2}\right)=0
$$

Discriminant $=(a+b)^{2}-4\left(a b-h^{2}\right)=0$

$$
\begin{aligned}
& =(a+b)^{2}-4 a b+4 h^{2} \\
& =(a-b)^{2}+4 h^{2} \\
& =(a-b)^{2}+(2 h)^{2}>0
\end{aligned}
$$

$\therefore$ Roots are real.
2.If $\alpha, \beta$ and $\gamma$ are the roots of $x^{3}+p x^{2}+q x+r=0$, then find $\sum \frac{1}{\alpha^{2} \beta^{2}}$

Sol: $\quad \alpha, \beta$ and $\gamma$ are the roots of $x^{3}+p x^{2}+q x+r=0$.

$$
\begin{aligned}
& \alpha+\beta+\gamma=-p . \\
& \alpha \beta+\beta \gamma+\gamma \alpha=q \\
& \alpha \beta \gamma=-r \\
& \sum \frac{1}{\alpha^{2} \beta^{2}} \\
& \sum \frac{1}{\alpha^{2} \beta^{2}}=\frac{1}{\alpha^{2} \beta^{2}}+\frac{1}{\beta^{2} \gamma^{2}}+\frac{1}{\gamma^{2} \alpha^{2}} \\
& \frac{\alpha^{2}+\beta^{2}+\gamma^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}} \\
& =\frac{(\alpha+\beta+\gamma)^{2}-2(\alpha \beta+\beta \gamma+\gamma \alpha)}{(\alpha \beta \gamma)^{2}} \\
& =\frac{(-p)^{2}-2 q}{(-r)^{2}}=\frac{p^{2}-2 q}{r^{2}}
\end{aligned}
$$

3.Show that the points in the Argand diagram represented by the complex numbers $2+2 \mathrm{i},-2-2 \mathrm{i},-2 \sqrt{3}+2 \sqrt{3} \mathrm{i}$ are the vertices of an equilateral triangle.
Sol: $\quad \mathrm{A}(2,2), \mathrm{B}(-2,-2), \mathrm{C}(-2 \sqrt{3}, 2 \sqrt{3})$

$$
\begin{aligned}
& \mathrm{AB}=\sqrt{(2+2)^{2}+(2+2)^{2}}=4 \sqrt{2} \\
& \mathrm{BC}=\sqrt{(-2+2 \sqrt{3})^{2}+(-2-2 \sqrt{3})^{2}} \\
& \mathrm{BC}=\sqrt{4+12-8 \sqrt{3}+4+12+8 \sqrt{3}}=4 \sqrt{2} \\
& \mathrm{AC}=\sqrt{(2+2 \sqrt{3})^{2}+(2-2 \sqrt{3})^{2}}=4 \sqrt{2} \\
& \mathrm{AB}=\mathrm{AC}=\mathrm{BC}
\end{aligned}
$$

$\triangle \mathrm{ABC}$ is equilateral.
4.If $z=2-3 i$, then show that $z^{2}-4 z+13=0$.

Sol: $\quad z=2-3 i \Rightarrow z-2=-3 i \Rightarrow(z-2)^{2}=(-3 i)^{2}$

$$
\begin{aligned}
& \Rightarrow z^{2}+4-4 z=-9 \\
& \Rightarrow z^{2}-4 z+13=0
\end{aligned}
$$

5.If $1, \omega, \omega^{2}$ are the cube roots of units then prove that $\frac{1}{2+\omega}+\frac{1}{1+2 \omega}=\frac{1}{1+\omega}$

Solution : -

$$
\begin{aligned}
& \text { L.H.S } \frac{1}{2+\omega}+\frac{1}{1+2 \omega} \\
& \frac{1+2 \omega+2+\omega}{(2+\omega)(1+2 \omega)}=\frac{3(1+\omega)}{2+4 \omega+\omega+2 \omega^{2}} \\
& =\frac{3(1+\omega)}{2\left(1+\omega^{2}\right)+5 \omega} \\
& =\frac{3\left(-\omega^{2}\right)}{-2 \omega+5 \omega} \because 1+\omega=-\omega^{2} \\
& 1+\omega^{2}=\omega \\
& =\frac{-3 \omega^{2}}{3 \omega}=-\omega \\
& =-\frac{1}{\omega^{2}}=\frac{1}{1+\omega}
\end{aligned}
$$

6.If ${ }^{(n+1)} P_{5}:{ }^{n} P_{6}=2: 7$ find $n$

Sol: $\quad{ }^{(n+1)} P_{5}:{ }^{n} P_{6}=2: 7 \Rightarrow \frac{(n+1)_{p_{5}}}{n_{p_{6}}}=\frac{2}{7}$

$$
\Rightarrow \frac{(n+1) n(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)(n-4)(n-5)}=\frac{2}{7}
$$

$$
\begin{aligned}
& \Rightarrow 7(n+1)=2(n-4)(n-5) \\
& \Rightarrow 7 n+7=2 n^{2}-18 n+40 \\
& \Rightarrow 2 n^{2}-25 n+33=0 \\
& \Rightarrow 2 n(n-1)-3(n-11)=0 \\
& \Rightarrow(n-11)(2 n-3)=0 \Rightarrow n=11 \text { or } \frac{3}{2}
\end{aligned}
$$

Since n is a positive integer, $\mathrm{n}=11$
7.Find the number of ways of arranging the letters ofthe word INDEPENDENCE

Sol: The word INDEPENDENCE contains 12 letters in
which there are 3 N's are alike, 2 D's are alike, 4 E's are alike and rest are different
$\therefore$ The number of required arrangements $=\frac{(12)!}{4!3!2!}$
8. Find the set of $x$ for which the binomial expansion $(2+5 x)^{-1 / 2}$ Is valid

Sol.

$$
(2+5 x)^{-1 / 2}=2^{-1 / 2}\left(1+\frac{5 x}{2}\right)^{-1 / 2}
$$

The binomial expansion of $(2+5 x)^{-1 / 2}$ is valid when $\left|\frac{5 x}{3}\right|<1 \Rightarrow|x|<\frac{2}{5}$ i.e. $\mathrm{E}=\left(\frac{-2}{5}, \frac{2}{5}\right)$
9. Find the maximum number of times a fair coin must be tossed so that the probability of getting atleast one head is atleast 0.8 .
Sol. Let n be number of times a fair coin tossed x denotes the number of heads getting $x$ follows binomial distribution with parameters $n$ and $p=1 / 2$ given $p(x$ $\geq 1) \geq 0.8$

$$
\Rightarrow 1-\mathrm{p}(\mathrm{x}=0) \geq 0.8 \Rightarrow \mathrm{p}(\mathrm{x}=0) \leq 0.2
$$

$$
\Rightarrow{ }^{\mathrm{n}} \mathrm{C}_{0}\left(\frac{1}{2}\right)^{\mathrm{n}} \leq 0.2 \Rightarrow\left(\frac{1}{2}\right)^{\mathrm{n}} \leq \frac{1}{5}
$$

The maximum value of $n$ is 3 .
10. Find the variance for the discrete data given below.
$6,7,10,12,13,4,8,12$

Sol. Mean $\overline{\mathrm{x}}=\frac{6+7+10+12+13+4+8+12}{8}=\frac{72}{8}=9$

| $x_{i}$ | $x_{i}-\bar{x}$ | $\left(x_{i}-\bar{x}\right)^{2}$ |
| :---: | :---: | :---: |
| 6 | -3 | 9 |
| 7 | -2 | 4 |
| 10 | 1 | 1 |
| 12 | 3 | 9 |
| 13 | 4 | 16 |
| 4 | -5 | 25 |
| 8 | -1 | 1 |
| 12 | 3 | 9 |

Variance $\left(\sigma^{2}\right)=\frac{\sum_{i=1}^{8}\left(x_{i}-\bar{x}\right)^{2}}{n}=\frac{74}{8}=9.25$.
11. Let be the real roots of $a x^{2}+b x+c=0$ and $\alpha<\beta$. Then prove that i) $x \in \boldsymbol{R}$, $\alpha<x<\beta \Rightarrow$
$a x^{2}+b x+c$ and $a$ have the opposite signs
ii) $\quad x \in R, \quad x<\alpha \quad$ or $\quad x>\beta \Rightarrow a x^{2}+b x+c \quad$ and $a$ have the same sign.

## Proof:

$\alpha, \beta$ are the roots of $a x^{2}+b x+c=0$
$a x^{2}+b x+c=a(x-\alpha)(x-\beta)$
$\frac{a x^{2}+b x+c}{a}=(x-\alpha)(x-\beta)$
i) Suppose $x \in R, \alpha<x<\beta$

$$
\Rightarrow x-\alpha>0, x-\beta<0
$$

$$
\begin{aligned}
& \Rightarrow(x-\alpha)(x-\beta)<0 \Rightarrow \frac{a x^{2}+b x+c}{a}<0 \\
& \quad \Rightarrow a x^{2}+b x+c, a \text { have opposite sign }
\end{aligned}
$$

ii) Suppose $x \in R, x<\alpha$

$$
x<\alpha<\beta \text { then } x-\alpha<0, x-\beta<0
$$

$$
\Rightarrow(x-\alpha)(x-\beta)>0
$$

$$
\Rightarrow \frac{a x^{2}+b x+c}{a}>0
$$

$$
\Rightarrow a x^{2}+b x+c, \text { a have same sign }
$$

$$
\text { suppose } \quad x \in R, x>\beta
$$

$$
\begin{aligned}
& \quad x-\alpha>0, x-\beta>0 \\
\Rightarrow & (x-\alpha)(x-\beta)>0 \\
\Rightarrow & \frac{a x^{2}+b x+c}{a}>0 \\
\Rightarrow & a x^{2}+b x+c, a \text { have same sign }
\end{aligned}
$$

$$
\therefore \quad x \in R, x<\alpha \quad \text { or } x>\beta \Rightarrow a x^{2}+b x+c \text { and } \quad a \text { have the same sign. }
$$

12. If n is an integer and $z=\operatorname{cis} \theta$ then show that $\frac{z^{2 n}-1}{z^{2 n}+1}=i \tan n \theta$

Solution : -

$$
\begin{aligned}
\frac{z^{2 n}-1}{z^{2 n}+1}= & \frac{(\cos \theta+i \sin \theta)^{2 n}-1}{(\cos \theta+i \sin \theta)^{2 n}+1} \\
& =\frac{\cos 2 n \theta+i \sin 2 n \theta-1}{\cos 2 n \theta+i \sin 2 n \theta+1} \\
& =\frac{-(1-\cos 2 n \theta)+i \sin 2 n \theta}{(1+\cos 2 n \theta)+i \sin 2 n \theta} \\
& =\frac{i^{2}\left(2 \sin ^{2} n \theta\right)+2 i \sin n \theta \cos n \theta}{2 \cos ^{2} n \theta+2 i \sin n \theta \cos n \theta}\left\{\because-1=i^{2}\right\}
\end{aligned}
$$

$$
=\frac{\not 2 i \sin n \theta\{\cos n \theta+i \sin n \theta\}}{\not 2 \cos n \theta\{\underline{\cos n \theta}+i \sin n \theta\}}=i \tan n \theta
$$

13. Find the number of 4-digit numbers which can be formed using the digits 0,2 ,
$5,7,8$ that are divisible by (i) 2 (ii) 4 when repetition is allowed.
Sol: Given digits are $0,2,5,7,8$.
i) Divisible by 2 :

The thousand's place of 4 digit number when repetition is allowed can be filled in 4 ways. (using non-zero digits)

The 4 -diti number is divisible by 2 , when the units place is an even digit. This can be done in 3 ways.

The remaining 2 places can be filled by 5 ways each i.e., $5^{2}=25$ ways.
$\therefore$ Number of 4 digit numbers which are divisible by 2 is $4 \times 3 \times 25=300$. ii) Divisible by 4 :

A number is divisible by 4 only when the number in last two places (ten's and unit's) is a multiple of 4 .

As repletion is allowed the last two places should be filled with one of the following :

$$
00,08,20,28,52,72,80,88
$$

This can be done is 8 ways.
Thousand's place is filled in 4 ways.
(i.e., using non-zero digits)

Hundred's placed can be filled in 5 ways.
$\therefore$ Total number of 4 digit numbers formed

$$
=8 \times 4 \times 5=160 \text {. }
$$

14. prove that ${ }^{n+1} c_{r}=^{n} c_{r-1}+{ }^{n} c_{r}$
15. Resolve $\frac{x+3}{(1-x)^{2}\left(1+x^{2}\right)}$ in to partial fractions.

Sol: Let $\frac{x+3}{(1-x)^{2}\left(1+x^{2}\right)}$

$$
\begin{gathered}
=\frac{\mathrm{A}}{(1-\mathrm{x})}+\frac{\mathrm{B}}{(1-\mathrm{x})^{2}}+\frac{\mathrm{Cx}+\mathrm{D}}{\left(1+\mathrm{x}^{2}\right)} \\
\Rightarrow \mathrm{x}+3=\mathrm{A}(1-\mathrm{x})\left(1+\mathrm{x}^{2}\right)+\mathrm{B}\left(1+\mathrm{x}^{2}\right) \\
+(\mathrm{Cx}+\mathrm{D})(1-\mathrm{x})^{2}
\end{gathered}
$$

Comparing the coefficients of like power of x , we get
$\mathrm{A}+\mathrm{B}+\mathrm{D}=3$
$-\mathrm{A}+\mathrm{C}-2 \mathrm{D}=1$
$\mathrm{A}+\mathrm{B}-2 \mathrm{C}+\mathrm{D}=0$
$-\mathrm{A}+\mathrm{C}=0$

Solving these equations, we get

$$
\begin{aligned}
\mathrm{A}=\frac{3}{2}, \mathrm{~B}=2, \mathrm{C}=\frac{3}{2}, \mathrm{D}=- & \frac{1}{2} \\
& \therefore \frac{\mathrm{x}+3}{(1-\mathrm{x})^{2}\left(1+\mathrm{x}^{2}\right)}= \\
& \frac{3}{2(1-\mathrm{x})}+\frac{2}{(1-\mathrm{x})^{2}}+\frac{3 \mathrm{x}-1}{2\left(1+\mathrm{x}^{2}\right)}
\end{aligned}
$$

16. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, find the probability that (i) you both enter the same section (ii) you both enter the different sections.
Sol. $n(S)={ }^{100} C_{40}$
i) You both enter the same section:

$$
\begin{aligned}
& \mathrm{n}(\mathrm{~A})={ }^{98} \mathrm{C}_{38}+{ }^{98} \mathrm{C}_{58} \\
& \mathrm{P}(\mathrm{~A})=\frac{\mathrm{n}(\mathrm{~A})}{\mathrm{n}(\mathrm{~S})}=\frac{{ }^{98} \mathrm{C}_{38}+{ }^{98} \mathrm{C}_{58}}{{ }^{100} \mathrm{C}_{40}}=\frac{17}{33}
\end{aligned}
$$

ii) You both enter the different sections:

$$
\begin{aligned}
& \mathrm{n}(\mathrm{~A})={ }^{98} \mathrm{C}_{39}+{ }^{98} \mathrm{C}_{59} \\
& \mathrm{P}(\mathrm{~A})=\frac{\mathrm{n}(\mathrm{~A})}{\mathrm{n}(\mathrm{~S})}=\frac{{ }^{98} \mathrm{C}_{39}+{ }^{98} \mathrm{C}_{59}}{{ }^{100} \mathrm{C}_{40}}=\frac{16}{33}
\end{aligned}
$$

17. The probability that Australia wins a match against India in a cricket game is given to be $1 / 3$. If India and Australia play 3 matches, what is the probability that,
i) Australia will loose all the three matches?
ii) Australia will win atleast one match?

Sol. Suppose A is the event of Australia winning the match.
Given $P(A)=\frac{1}{3}$
$\therefore \mathrm{P}(\overline{\mathrm{A}})=1-\mathrm{P}(\mathrm{A})=1-\frac{1}{3}=\frac{2}{3}$
i) Probability that Australia will loose the all three matches.

$$
=[\mathrm{P}(\overline{\mathrm{~A}})]^{3}=\left(\frac{2}{3}\right)^{3}=\frac{8}{27}
$$

ii) Probability that Australia will win atleast one match

$$
=1-[\mathrm{P}(\overline{\mathrm{~A}})]^{3}=1-\frac{8}{27}=\frac{19}{27}
$$

18. Given that the roots of $x^{3}+3 p x^{2}+3 p x+r=0$ are in (i) A.P., show that $2 p^{2}-3 q p+r=0$
(ii) G.P., show that $p^{3} r=q^{3}$
(iii) H.P., show that $2 q^{3}=r(3 p q-r)$

Sol: Given equation is $x^{3}+3 p x^{2}+3 p x+r=0$
(i) The roots are in A.P.

Suppose $a-d, a, a+d=-3 p$
$3 a=-3 p \Rightarrow a=-p$
$\because a^{\prime} x^{3}+3 p x^{2}+3 q x+r=0$
$\Rightarrow a^{3}+3 p a^{2}+3 q a+r=0$
But $a=-p$
$\Rightarrow-p^{3}+3 p(-p)^{2}+3 q(-p)+r=0$
$\Rightarrow 2 p^{3}-3 p q+r=0$ is the required condition
(ii) The roots are in G.P.

Suppose the roots be $\frac{a}{R}, a, a R$
Given $\left(\frac{a}{R}\right)(a)(a R)=-r$
$\Rightarrow a^{3}=-r$
$\Rightarrow a=(-r)^{1 / 3}$
$\because \mathrm{a}^{\prime}$ is a root of $x^{3}+3 p x^{2}+3 q x+r=0$
$\Rightarrow\left(-r^{1 / 3}\right)^{3}+3 p\left(-r^{1 / 3}\right)^{2}+3 q\left(-r^{1 / 3}\right)+r=0$
$\Rightarrow-r+3 p r^{2 / 3}-3 q r^{1 / 3}+r=0$
$p r^{2 / 3}=q r^{1 / 3}$
$\Rightarrow p r^{1 / 3}=q$
$\Rightarrow p^{3} r=q$ is the required condition
(iii) The roots of $x^{3}+3 p x^{2}+3 q x+r=0 \quad$ (1) are in H.P.

Let $y=\frac{1}{x}$ so that $\frac{1}{y^{3}}+\frac{3 p}{y^{2}}+\frac{3 q}{y}+r=0 \quad$ (2) are in A.P.
Suppose $a-d, a, a+d$ be the roots of (2)
Sum $=a-d, a, a+d=-\frac{3 q}{r}$
$3 a=-\frac{3 q}{r}$
$a=-\frac{q}{r}$
$\because \quad$ 'a' is root of $r y^{3}+3 q y^{2}+3 p y+1=0$
$\Rightarrow r a^{3}+3 q a^{2}+3 p a+1=0$
But $a=-\frac{q}{r}$
$\Rightarrow r\left(-\frac{q}{r}\right)^{3}+3 q\left(-\frac{q}{r}\right)^{2}+3 p\left(-\frac{q}{r}\right)+1=0$
$\frac{-q^{3}}{r^{2}}+\frac{3 q^{3}}{r^{2}}-\frac{3 p q}{r}+1=0$
$\Rightarrow-q^{3}+3 q^{3}-3 p q r+r^{2}=0$
$\Rightarrow 2 q^{3}=r(3 p q-r)$ is the required condition.
19. If $\cos \alpha+\cos \beta+\cos \vartheta=0=\sin \alpha+\sin \beta+\sin \vartheta=0$ then show that
(I ) $\cos (2 \alpha-\beta-\vartheta)+\cos \{2 \beta-\vartheta-\alpha\}+\sin (2 \vartheta-\alpha-\beta)=3$
(II) $\sin (2 \alpha-\beta-\vartheta)+\sin (2 \beta-\vartheta-\alpha)+\sin (2 \vartheta-\alpha-\beta)=0$
(iii) $\cos (\alpha+\beta)+\cos (\beta+\vartheta)+\cos (\vartheta+\alpha)=0$
(iv) $\sin (\alpha+\beta)+\sin (\beta+\vartheta)+\sin (\vartheta+\alpha)=0$

Solution : -
Let $x=\cos \alpha+i \sin \alpha \quad y=\cos \beta+i \sin \beta: z=\cos \vartheta+i \sin \vartheta$
$x+y+z=(\cos \alpha+\cos \beta+\cos \vartheta)+1(\sin \alpha+\sin \beta+\sin \vartheta)$
$x+y+z=0 \Rightarrow x^{3}+y^{3}+z^{3}=3 x y z$

$$
\begin{aligned}
& \frac{x^{3}+y^{3}+z^{3}}{x y z}=3 \Rightarrow \frac{x^{2}}{y z}+\frac{y^{2}}{z x}+\frac{z^{2}}{x y}=3 \\
& \frac{\operatorname{cis} 2 \alpha}{\operatorname{cis} \beta \operatorname{cis} \vartheta}+\frac{\operatorname{cis} 2 \beta}{\operatorname{cis} \vartheta \cdot \operatorname{cis} \alpha}+\frac{\operatorname{cis} 2 \vartheta}{\operatorname{cis} \alpha \operatorname{cis} \beta}=3 \\
& \operatorname{cis}(2 \alpha-\beta-\vartheta)+\operatorname{cis}(2 \beta-\vartheta-\alpha)+\cos (2 \vartheta-\alpha-\beta)=3 \\
& \{\cos (2 \alpha-\beta-\vartheta)+i \sin (2 \alpha-\beta-\vartheta)\}+\cos (2 \beta-\vartheta-\alpha)+1 \sin (2 \beta-\vartheta-\alpha) \\
& +\cos (2 \vartheta-\alpha-\beta)+1 \sin (2 \vartheta-\alpha-\beta)=3
\end{aligned}
$$

Comparing real and imaginary parts on both sides
$\cos (2 \alpha-\beta-\vartheta)+\cos (2 \beta-\vartheta-\alpha)+\cos (2 \vartheta-\alpha-\beta)=3$
$\sin (2 \alpha-\beta-\vartheta)+\sin (2 \beta-\vartheta-\alpha)+\sin (2 \vartheta-\alpha-\beta)=0$

We know that $x+y+z=0$
$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{\cos \alpha+1 \sin \alpha}+\frac{1}{\cos \beta+i \sin \beta}+\frac{1}{\cos \vartheta+i \sin \vartheta}$

$$
\begin{aligned}
& =\cos \alpha-1 \sin \alpha+\cos \beta-i \sin \beta+\cos \vartheta-i \sin \vartheta \\
& \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0 \\
& x+y+z=0 \Rightarrow(x+y+z)^{2}=0 \Rightarrow x^{2}+y^{2}+z^{2}+2 x y+2 y+2 z x=0 \\
& x^{2}+y^{2}+z^{2}+2 x y z\left\{\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right\}=0 \\
& \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0 \\
& \therefore y z+z x+x y=0 \\
& \therefore \operatorname{cis} \alpha \operatorname{cis} \beta+\operatorname{cis} \beta \operatorname{cis} \vartheta+\operatorname{cis} \vartheta \operatorname{cis} \alpha=0 \\
& =\operatorname{cis}(\alpha+\beta)+\cos (\beta+\vartheta)+\operatorname{cis}(\vartheta+\alpha)=0 \\
& \{\cos (\alpha+\beta)+i \sin (\alpha+\beta)\}+\{\cos (\beta+\vartheta)\}+\{\cos (\vartheta+\alpha)+i \sin (\vartheta+\alpha)\}=0
\end{aligned}
$$

By comparing real and imaginary parts on both sides

$$
\begin{aligned}
& \cos (\alpha+\beta)+\cos (\beta+\vartheta)+\cos (\vartheta+\alpha)=0 \\
& \sin (\alpha+\beta)+\sin (\beta+\vartheta)+\sin (\vartheta+\alpha)=0
\end{aligned}
$$

20. Using binomial theorem, prove that $5^{4 n}+52 n-1$ is divisible by 676 for all positive integers $n$.
Sol. $5^{4 \mathrm{n}}+52 \mathrm{n}-1=\left(5^{2}\right)^{2 \mathrm{n}}+52 \mathrm{n}-1$

$$
=(25)^{2 n}+52 n-1=(26-1)^{2 n}+52 n-1
$$

$$
=\left[{ }^{2 \mathrm{n}} \mathrm{C}_{0}(26)^{2 \mathrm{n}}-{ }^{2 \mathrm{n}} \mathrm{C}_{1}(26)^{2 \mathrm{n}-1}+{ }^{2 \mathrm{n}} \mathrm{C}_{2}(26)^{2 \mathrm{n}-2}\right.
$$

$$
-\ldots . .+{ }^{2 n} C_{2 n-2}(26)^{2}-{ }^{2 n} C_{2 n-1}(26)+
$$

$$
\left.{ }^{2 n} C_{2 n}(1)\right]+52 n-1
$$

$$
={ }^{2 n} C_{0}(26)^{2 n}-{ }^{2 n} C_{1}(26)^{2 n-1}+{ }^{2 n} C_{2}(26)^{2 n-2}
$$

$$
-\ldots . .+{ }^{2 n} C_{2 n-2}-2 n(26)+1+52 n-1
$$

$$
=(26)^{2}\left[{ }^{2 n} C_{0}(26)^{2 \mathrm{n}-2}-{ }^{2 \mathrm{n}} \mathrm{C}_{1}(26)^{2 \mathrm{n}-3}\right.
$$

$$
\left.+{ }^{2 \mathrm{n}} \mathrm{C}_{2}(26)^{2 \mathrm{n}-4}+\ldots+{ }^{2 \mathrm{n}} \mathrm{C}_{2 \mathrm{n}-2}\right]
$$

is divisible by $(26)^{2}=676$
$\therefore 5^{4 n}+52 n-1$ is divisible by 676 , for all positive integers n .
21. For $n=0,1,2,3, \ldots n$, prove that $C_{0} \cdot C_{r}+C_{1} \cdot C_{r+1}+C_{2} \cdot C_{r+2}+\ldots+C_{n-r} \cdot C_{n}={ }^{2 n} C_{n+r}$ and hence deduce that
i) $\mathrm{C}_{0}^{2}+\mathrm{C}_{1}^{2}+\mathrm{C}_{2}^{2}+\ldots+\mathrm{C}_{\mathrm{n}}^{2}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}}$
ii) $\mathrm{C}_{0} \cdot \mathrm{C}_{1}+\mathrm{C}_{1} \cdot \mathrm{C}_{2}+\mathrm{C}_{2} \cdot \mathrm{C}_{3}+\ldots+\mathrm{C}_{\mathrm{n}-1} \mathrm{C}_{\mathrm{n}}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}+1}$

Sol. We know that

$$
\begin{equation*}
(1+x)^{n}=C_{0}+C_{1} x+C_{2} x^{2}+\ldots+C_{n} x^{n} . . \tag{1}
\end{equation*}
$$

On replacing $x$ by $1 / x$ in the above equation,

$$
\begin{equation*}
\left(1+\frac{1}{x}\right)^{n}=C_{0}+\frac{C_{1}}{x}+\frac{C_{2}}{x^{2}}+\ldots+\frac{C_{n}}{x^{n}} . . \tag{2}
\end{equation*}
$$

From (1) and (2)

$$
\begin{gather*}
\left(1+\frac{1}{x}\right)^{n}(1+x)^{n}=\left(C_{0}+\frac{C_{1}}{x}+\frac{C_{2}}{x^{2}}+\ldots+\frac{C_{n}}{x^{n}}\right) \\
\left(C_{0}+C_{1} x+C_{2} x^{2}+\ldots+C_{n} x^{n}\right) \ldots(3) \tag{3}
\end{gather*}
$$

The coefficient of $x^{r}$ in R.H.S. of (3)
$=\mathrm{C}_{0} \mathrm{C}_{\mathrm{r}}+\mathrm{C}_{1} \mathrm{C}_{\mathrm{r}+1}+\mathrm{C}_{2} \mathrm{C}_{\mathrm{r}+2}+\ldots+\mathrm{C}_{\mathrm{n}-\mathrm{r}} \mathrm{C}_{\mathrm{n}}$
The coefficient of $\mathrm{x}^{\mathrm{r}}$ in L.H.S. of (3)
$=$ the coefficient of $x^{r}$ in $\frac{(1+x)^{2 n}}{x^{n}}$
$=$ the coefficient of $\mathrm{x}^{\mathrm{n+r}}$ is $(1+\mathrm{x})^{2 \mathrm{n}}$
$={ }^{2 n} \mathrm{C}_{\mathrm{n}+\mathrm{r}}$
From (3) and (4), we get
$\mathrm{C}_{0} \cdot \mathrm{C}_{1}+\mathrm{C}_{1} \cdot \mathrm{C}_{2}+\mathrm{C}_{2} \cdot \mathrm{C}_{3}+\ldots+\mathrm{C}_{\mathrm{n}-1} \mathrm{C}_{\mathrm{n}}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}+1}$
i) On putting $\mathrm{r}=0$ in (i), we get

$$
\mathrm{C}_{0}^{2}+\mathrm{C}_{1}^{2}+\mathrm{C}_{2}^{2}+\ldots+\mathrm{C}_{\mathrm{n}}^{2}={ }^{2 \mathrm{n}} \mathrm{C}_{\mathrm{n}}
$$

ii) On substituting $\mathrm{r}=1$ in (i) we get

$$
C_{0} \cdot C_{1}+C_{1} \cdot C_{2}+C_{2} \cdot C_{3}+\ldots+C_{n-1} C_{n}={ }^{2 n} C_{n+1}
$$

22. If $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are three independent events such that $\mathrm{P}\left(\mathrm{A} \cap \mathrm{B}^{\mathrm{C}} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4}$
$\mathrm{P}\left(\mathrm{A}^{\mathrm{C}} \cap \mathrm{B} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{8}, \mathrm{P}\left(\mathrm{A}^{\mathrm{C}} \cap \mathrm{B}^{\mathrm{C}} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4}$ then find $\mathrm{P}(\mathrm{A}), \mathrm{P}(\mathrm{B})$ and $\mathrm{P}(\mathrm{C})$.
Sol. Since $A, B, C$ are independent events.

$$
\begin{align*}
& \mathrm{P}\left(\mathrm{~A} \cap \mathrm{~B}^{\mathrm{C}} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4} \\
& \Rightarrow \mathrm{P}(\mathrm{~A}) \cdot \mathrm{P}\left(\mathrm{~B}^{\mathrm{C}}\right) \cdot \mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4} \tag{1}
\end{align*}
$$

$\mathrm{P}\left(\mathrm{A}^{\mathrm{C}} \cap \mathrm{B} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{8}$
$\Rightarrow \mathrm{P}\left(\mathrm{A}^{\mathrm{C}}\right) \cdot \mathrm{P}(\mathrm{B}) \cdot \mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=\frac{1}{8}$
$\mathrm{P}\left(\mathrm{A}^{\mathrm{C}} \cap \mathrm{B}^{\mathrm{C}} \cap \mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4}$

$$
\begin{aligned}
& \Rightarrow \mathrm{P}\left(\mathrm{~A}^{\mathrm{C}}\right) \cdot \mathrm{P}\left(\mathrm{~B}^{\mathrm{C}}\right) \cdot \mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4} \ldots(3) \\
& \frac{(1)}{(3)} \Rightarrow \frac{\mathrm{P}(\mathrm{~A})}{\mathrm{P}\left(\mathrm{~A}^{\mathrm{C}}\right)}=\frac{1 / 4}{1 / 4}=1 \\
& \Rightarrow \frac{\mathrm{P}(\mathrm{~A})}{1-\mathrm{P}(\mathrm{~A})}=1 \Rightarrow \mathrm{P}(\mathrm{~A})=1-\mathrm{P}(\mathrm{~A}) \\
& \Rightarrow 2 \mathrm{P}(\mathrm{~A})=1 \Rightarrow \mathrm{P}(\mathrm{~A})=\frac{1}{2} \\
& \frac{(2)}{(3)} \Rightarrow \frac{\mathrm{P}(\mathrm{~B})}{\mathrm{P}\left(\mathrm{~B}^{\mathrm{C}}\right)}=\frac{1 / 8}{1 / 4} \Rightarrow \frac{\mathrm{P}(\mathrm{~B})}{1-\mathrm{P}(\mathrm{~B})}=\frac{1}{2} \\
& \Rightarrow 2 \mathrm{P}(\mathrm{~B})=1-\mathrm{P}(\mathrm{~B}) \Rightarrow 3 \mathrm{P}(\mathrm{~B})=1 \\
& \therefore \mathrm{P}(\mathrm{~B})=\frac{1}{3}
\end{aligned}
$$

From $(1) \Rightarrow \mathrm{P}(\mathrm{A}) \cdot \mathrm{P}\left(\mathrm{B}^{\mathrm{C}}\right) \cdot \mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4}$
$\Rightarrow\left(\frac{1}{2}\right)\left(1-\frac{1}{3}\right) \mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4}$
$\Rightarrow \mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=\frac{1}{4} \times 2 \times \frac{3}{2}=\frac{3}{4}$
$\therefore \mathrm{P}(\mathrm{C})=1-\mathrm{P}\left(\mathrm{C}^{\mathrm{C}}\right)=1-\frac{3}{4}=\frac{1}{4}$
$\therefore \mathrm{P}(\mathrm{A})=\frac{1}{2}, \mathrm{P}(\mathrm{B})=\frac{1}{3}, \mathrm{P}(\mathrm{C})=\frac{1}{4}$
23. The probability of a bomb hitting a bridge is $1 / 2$ and three direct hits (not necessarily consecutive) are needed to destroy it. Find the minimum number of bombs required so that the probability of the bridge being destroyed is greater than 0.9.
Sol. Let n be the minimum number of bombs required and x be the number of bombs that hit the bridge, then x follows binomial distribution with parameters n and $\mathrm{p}=1 / 2$.
Now $p(x \geq 3)>0.9$
$\Rightarrow 1-\mathrm{p}(\mathrm{x}<3)>0.9$
$\Rightarrow \mathrm{p}(\mathrm{x}<3)<0.1$
$\Rightarrow \mathrm{p}(\mathrm{x}=0)+\mathrm{p}(\mathrm{x}=1)+\mathrm{p}(\mathrm{x}=2)<0.1$

$$
\begin{aligned}
\Rightarrow{ }^{\mathrm{n}} \mathrm{C}_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{\mathrm{n}} & +{ }^{\mathrm{n}} \mathrm{C}_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{\mathrm{n}-1} \\
& +{ }^{\mathrm{n}} \mathrm{C}_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{\mathrm{n}-2}<0.1
\end{aligned}
$$

$\Rightarrow 1 \cdot \frac{1}{2^{\mathrm{n}}}+\frac{\mathrm{n}}{2^{\mathrm{n}}}+\frac{\mathrm{n}(\mathrm{n}-1)}{2} \frac{1}{2^{2}}<\frac{1}{10}$
$\Rightarrow 1 . \frac{1}{2^{\mathrm{n}}}+\frac{\mathrm{n}}{2^{\mathrm{n}}}+\frac{\mathrm{n}^{2}-\mathrm{n}}{2.2^{\mathrm{n}}}<\frac{1}{10}$
$\Rightarrow \frac{1}{2^{\mathrm{n}}}\left(1+\mathrm{n}+\frac{\mathrm{n}^{2}-\mathrm{n}}{2}\right)<\frac{1}{10}$
$\Rightarrow \frac{1}{2^{\mathrm{n}}}\left(\frac{2+2 \mathrm{n}+\mathrm{n}^{2}-\mathrm{n}}{2}\right)<\frac{1}{10}$
$\Rightarrow 5\left(\mathrm{n}^{2}+\mathrm{n}+2\right)<2^{\mathrm{n}}$
By trial and error, we get $n \geq 9$
$\therefore$ The least value of n is 9
$\therefore \mathrm{n}=9$
24. The arithmetic mean and standard deviation of a set of 9 items are 43 and 5 respectively. If an item of value 63 is added to that set, find the new mean and standard deviation of 10 item set given.
Sol. $\bar{X}=\frac{\sum_{i=1}^{9} x_{i}}{n}$

$$
\begin{aligned}
& 43=\frac{\sum_{i=1}^{9} x_{i}}{9} \\
& \sum_{i=1}^{9} x_{i}=43 \times 9=387
\end{aligned}
$$

$$
\text { New Mean }=\frac{\sum_{i=1}^{10} x_{i}}{n}=\frac{\sum_{i=1}^{9} x_{i}+x_{10}}{10}=\frac{387+63}{10}=45
$$

$$
S . D^{2}=\frac{\sum_{i=1}^{9} x_{i}^{2}}{9}-(\bar{x})^{2} \Rightarrow 5^{2}=\frac{\sum_{i=1}^{9} x_{i}^{2}}{9}-(43)^{2}
$$

$$
\frac{\sum_{i=1}^{9} x_{i}^{2}}{9}=25+1849 \Rightarrow \frac{\sum_{i=1}^{9} x_{i}^{2}}{9}=1874
$$

$$
\sum_{i=1}^{9} x_{i}^{2}=1874 \times 9=16866
$$

$$
\sum_{\mathrm{i}=1}^{10} \mathrm{x}_{\mathrm{i}}^{2}=\sum_{\mathrm{i}=1}^{9} \mathrm{x}_{\mathrm{i}}^{2}+\mathrm{x}_{10}^{2}=16866+3969=20835
$$

New S.D. $=\sqrt{\frac{\sum_{i=1}^{10} x_{i}{ }^{2}}{10}-(\bar{x})^{2}}=\sqrt{\frac{20835}{10}-(45)^{2}}$
$=\sqrt{2083.5-2025}=\sqrt{58.5}=7.6485$.

