MATHEMATICAL METHODS

Eigen Values & Eigen Vectors

I YEAR B.Tech

By

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SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

Name of the Unit	Name of the Topic
Unit-I Solution of Linear systems	**Matrices and Linear system of equations:** Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems.
Unit-II Eigen values and Eigen vectors	Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.
Unit-III Linear Transformations	Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermition and skew Hermition matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition.
Unit-IV Solution of Non-linear Systems	**Solution of Algebraic and Transcendental Equations-** Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method **Interpolation:**Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton's Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange's Interpolation formulae- B. Spline interpolation, Cubic spline.
Unit-V Curve fitting & Numerical Integration	**Curve Fitting:** Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. **Numerical Integration:** Numerical Differentiation-Simpson's 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.
Unit-VI Numerical solution of ODE	Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.
Unit-VII Fourier Series	Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.
Unit-VIII Partial Differential Equations	Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.

CONTENTS

UNIT-II

Eigen Values & Eigen Vectors

- **Properties of Eigen values and Eigen Vectors**
- **Theorems**
- **Cayley – Hamilton Theorem**
- **Inverse and powers of a matrix by Cayley – Hamilton theorem**
- **Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices**

Eigen Values and Eigen Vectors

Characteristic matrix of a square matrix: Suppose $A_{n\times n}$ is a square matrix, then $[A-\lambda I]$ is called characteristic matrix of A, where λ is indeterminate scalar (I.e. undefined scalar).

Characteristic Polynomial: $|A-\lambda I|$ is called as characteristic polynomial in λ.

- Suppose A is a $n\times n$ matrix, then degree of the characteristic polynomial is n

Characteristic Equation: $|A-\lambda I|=0$ is called as a characteristic equation of A.

Characteristic root (or) Eigen root (or) Latent root

The roots of the characteristic equation are called as Eigen roots.

- Eigen values of the triangular matrix are equal to the elements on the principle diagonal.
- Eigen values of the diagonal matrix are equal to the elements on the principle diagonal.
- Eigen values of the scalar matrix are the scalar itself.
- The product of the eigen values of A is equal to the determinant of A.
- The sum of the eigen values of A = Trace of A.
- Suppose A is a square matrix, then 0 is one of the eigen value of $A \Leftrightarrow A$ is singular.

  i.e. $|A-\lambda I|=0$, if $\lambda=0$ then $|A|=0 \implies A$ is singular.
- If λ is the eigen value of A, then λ^2 is eigen value of A^2.
- If λ is the eigen value of A, then λ^{-1} is eigen value of A^{-1}.
- If λ is the eigen value of A, then $k\lambda$ is eigen value of kA , k is non-zero scalar.
- If λ is the eigen value of A, then $\frac{|A|}{\lambda}$ is eigen value of $adj\ A$.
- If A & B are two non-singular matrices, then AB and BA will have the same Eigen values.
- If A & B are two square matrices of order n and are non-singular, then $A^{-1}B$ and $B^{-1}A$ will have same Eigen values.
- The characteristic roots of a Hermition matrix are always real.
- The characteristic roots of a real symmetric matrix are always real.
- The characteristic roots of a skew Hermition matrix are either zero (or) Purely Imaginary

Eigen Vector (or) Characteristic Vector (or) Latent Vector

Suppose A is a $n \times n$ matrix and λ is an Eigen value of A, then a non-zero vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be an eigen vector of A corresponding to a eigen value λ if $AX = \lambda X$ (or) $(A - \lambda I)X = 0$.

- ❖ Corresponding to one Eigen value, there may be infinitely many Eigen vectors.
- ❖ The Eigen vectors of distinct Eigen values are Linearly Dependent.

Problem

Find the characteristic values and characteristic vectors of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution: Let us consider given matrix to be $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Now, the characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$\Rightarrow (1-\lambda)[1 - 2\lambda + \lambda^2 - 1] - 1[1 - \lambda - 1] + 1[1 - 1 + \lambda] = 0$$

$$\Rightarrow (1-\lambda)[-2\lambda + \lambda^2] + \lambda + \lambda$$

$$\Rightarrow -2\lambda + \lambda^2 + 2\lambda^2 - \lambda^3 + 2\lambda = 0$$

$$\Rightarrow 3\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow \lambda^2(3 - \lambda) = 0$$

$$\Rightarrow \lambda = 0, 0, 3$$

In order to find Eigen Vectors:

Case(i): Let us consider $\lambda = 0$

The characteristic vector is given by $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitute $\lambda = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This is in the form of Homogeneous system of Linear equation.

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + y + z = 0$$

Let us consider $z = k_1, y = k_2$

$$\Rightarrow x = -k_1 - k_2$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_2 \\ k_1 \end{bmatrix}$$

Now, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} + \begin{bmatrix} -k_2 \\ k_2 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \forall\, k_1, k_2 \in \mathbb{R}$$

(Or)

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (-k_1) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (-k_2) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Therefore, the eigen vectors corresponding to $\lambda = 0$ are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Case(ii): Let us consider $\lambda = 3$

The characteristic vector is given by $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitute $\lambda = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This is in the form of Homogeneous system of Linear equation.

$R_2 \longrightarrow 2R_2 + R_1$

$R_3 \longrightarrow 2R_3 + R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + R_2$ $\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Longrightarrow -3y + 3z = 0 \Longrightarrow y = z = k \ (let) \quad \forall\, k \in \mathbb{R}$$

Also, $-2x + y + z = 0$

$$\Longrightarrow -2x + k + k = 0$$

$$\Longrightarrow -2x + 2k = 0$$

$$\Longrightarrow x = k$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \forall\, k \in \mathbb{R}$$

Therefore, the characteristic vector corresponding to the eigen value $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence, the eigen values for the given matrix are $0, 0, 3$ and the corresponding eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Theorem

Statement: The product of the eigen values is equal to its determinant.

Proof: we have, $|\boldsymbol{A} - \boldsymbol{\lambda I}| = (-\mathbf{1})^{\boldsymbol{n}} \boldsymbol{\lambda}^{\boldsymbol{n}} + \ \dots + (-\mathbf{1})^{\boldsymbol{n}} \boldsymbol{a_0}$, where a_0 *is the last term*.

Now, put $\lambda = 0 \Longrightarrow |A| = a_0$

Since $|A - \lambda I| = (-1)^n \lambda^n + \ \dots + (-1)^n a_0 = 0$ is a polynomial in terms of λ

By solving this equation we get roots (i.e. the values of λ)

$$\Longrightarrow Product\ of\ roots = \frac{(-1)^n a_0}{(-1)^n} = a_0 = |A|$$

Hence the theorem.

Example: Suppose $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$

Now, $|A - \lambda I| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$

$$= (a - \lambda)(d - \lambda) - bc$$

$$= (-1)^2 (\lambda - a)(\lambda - b) - bc$$

$$= (-1)^2[\lambda^2 - \lambda(a+b) + ab] - bc$$

$$= (-1)^2\lambda^2 - \cdots + (ad - bc)$$

$$= (-1)^2\lambda^2 - \cdots + (-1)^2(ad - bc)$$

This is a polynomial in terms of λ,

$$Product\ of\ roots = \frac{(-1)^2(ad-bc)}{(-1)^2} = ad - bc = |A|$$

$$\boldsymbol{i.e. Product\ of\ roots = \frac{constant\ term}{coefficient\ of\ highest\ power\ term}}$$

CAYLEY-HAMILTON THEOREM

Statement: Every Square matrix satisfies its own characteristic equation

Proof: Let A be any square matrix.

Let $|A - \lambda I| = 0$ be the characteristic equation.

Let $A(\lambda) = [A - \lambda I] = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ \dots + a_n)$

Let $B(\lambda) = adj\ [A - \lambda I] = B_0\lambda^{n-1} + B_2\lambda^{n-2} + \ \dots + B_{n-1}$, where $B_0, B_1, \dots, B_{n-1}$ are the matrices of order $(n-1)$.

We know that, $A\ (adj\ A) = |A|I$

Take $A \longrightarrow [A - \lambda I]$

$\Rightarrow [A - \lambda I]\ (adj\ [A - \lambda I]) = [A - \lambda I]\ I$

$\Rightarrow [A - \lambda I]\ B(\lambda) = A(\lambda)\ I$

$\Rightarrow [A - \lambda I]\ B(\lambda) = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ \dots + a_n]\ I$

$\Rightarrow [A - \lambda I]\ (B_0\lambda^{n-1} + B_2\lambda^{n-2} + \ \dots + B_{n-1}) = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ \dots + a_n]\ I$

Comparing the coefficients of like powers of λ ,

$\Rightarrow -B_0 = (-1)^n I$ $\qquad (\times A^n)$

$AB_0 - B_1 = (-1)^n\ a_1\ I$ $\qquad (\times A^{n-1})$

$AB_1 - B_2 = (-1)^n\ a_2\ I$ $\qquad (\times A^{n-2})$

$\vdots \qquad \vdots$

$AB_{n-1} = (-1)^n\ a_n\ I$ $\qquad (\times I)$

Now, Pre-multiplying the above equations by $A^n, A^{n-1}, \dots, I$ and adding all these equations, we get

$$0 = (-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + \ \dots + a_nI]$$

which is the characteristic equation of given matrix A.

Hence it is proved that "Every square matrix satisfies its own characteristic equation".

Application of Cayley-Hamilton Theorem

Let A be any square matrix of order n. Let $|A - \lambda I| = 0$ be the characteristic equation of A.

Now, $|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ \dots \ + a_n) = 0$

By Cayley-Hamilton Theorem, we have $A^n + a_1A^{n-1} + a_2A^{n-2} + \ \dots \ + a_nI = 0 \quad (\because \lambda \longrightarrow A)$

$\Rightarrow \ -a_nI = \ A^n + a_1A^{n-1} + a_2A^{n-2} + \ \dots \ + a_{n-1}A$

$\Rightarrow \ I = \dfrac{-1}{a_n}(A^n + a_1A^{n-1} + a_2A^{n-2} + \ \dots \ + a_{n-1}A\,)$

Multiplying with A^{-1}

$\Rightarrow \ A^{-1} = \frac{-1}{a_n}(A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ \dots \ + a_{n-1}I\,)$.

Therefore, this theorem is used to find Inverse of a given matrix.

Calculation of Inverse using Characteristic equation

Step 1: Obtain the characteristic equation i.e. $|A - \lambda I| = 0$

Step 2: Substitute A in place of λ

Step 3: Multiplying both sides with A^{-1}

Step 4: Obtain A^{-1} by simplification.

Similarity of Matrices: Suppose A & B are two square matrices, then A, B are said to be similar if $\exists$ a non-singular matrix P such that $B = PAP^{-1}$ (or) $P^{-1}AP$.

Diagonalization: A square matrix A is said to be Diagonalizable if A is similar to some diagonal matrix.

- Eigen values of two similar matrices are equal.

Procedure to verify Diagonalization:

Step 1: Find Eigen values of A

Step 2: If all eigen values are distinct, then find Eigen vectors of each Eigen value and construct a matrix $P = [X_1 \quad X_2 \quad \cdots \quad X_n]$, where $X_1, X_2, \ \dots, X_n$ are Eigen vectors, then

$$PAP^{-1} = D = diag\,[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n]$$

MODAL AND SPECTRAL MATRICES: The matrix P in $PAP^{-1} = D$, which diagonalises the square matrix A is called as the **Modal Matrix**, and the diagonal matrix D is known as **Spectral Matrix**. i.e. $PAP^{-1} = D$, then P is called as Modal Matrix and D is called as Spectral Matrix.

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