MATHEMATICAL METHODS

INTERPOLATION

I YEAR B.Tech

By

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SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

Name of the Unit	Name of the Topic
Unit-I Solution of Linear systems	**Matrices and Linear system of equations:** Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems.
Unit-II Eigen values and Eigen vectors	Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.
Unit-III Linear Transformations	Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermition and skew Hermition matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition.
Unit-IV Solution of Non-linear Systems	**Solution of Algebraic and Transcendental Equations**- Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method **Interpolation:**Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton's Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange's Interpolation formulae- B. Spline interpolation, Cubic spline.
Unit-V Curve fitting & Numerical Integration	**Curve Fitting:** Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. **Numerical Integration:** Numerical Differentiation-Simpson's 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.
Unit-VI Numerical solution of ODE	Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.
Unit-VII Fourier Series	Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.
Unit-VIII Partial Differential Equations	Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.

CONTENTS

UNIT-IV(b)

INTERPOLATION

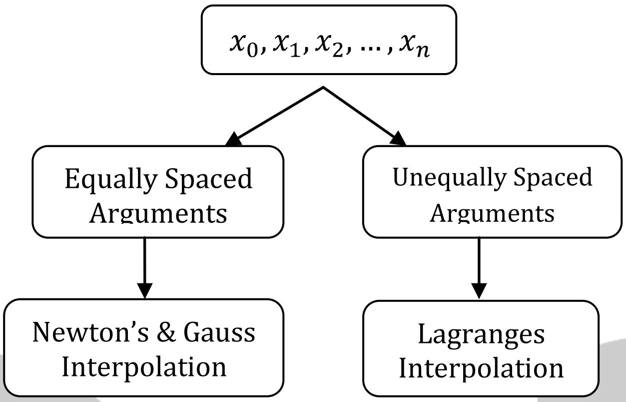
INTERPOLATION

The process of finding the curve passing through the points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is called as Interpolation and the curve obtained is called as Interpolating curve.

Interpolating polynomial passing through the given set of points is unique.

Let $x_0, x_1, x_2, \dots, x_n$ be given set of observations and $y = f(x)$ be the given function, then the method to find $f(x_m)\ \forall\ x_0 \leq x_m \leq x_n$ is called as an Interpolation.

If x_m is not in the range of x_0 and x_n , then the method to find (x_m) is called as Extrapolation.



The Interpolation depends upon finite difference concept.

If $x_0, x_1, x_2, \dots, x_n$ be given set of observations and let $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$ be their corresponding values for the curve $y = f(x)$, then $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ is called as finite difference.

When the arguments are equally spaced i.e. $x_i - x_{i-1} = h\ \forall\ i$ then we can use one of the following differences.

- Forward differences
- Backward differences
- Central differences

Forward Difference

Let us consider $x_0, x_1, x_2, \dots, x_n$ be given set of observations and let $y_0, y_1, y_2, \dots, y_n$ are corresponding values of the curve $y = f(x)$, then the Forward difference operator is denoted by Δ and is defined as $\Delta y_0 = y_1 - y_0,\ \Delta y_1 = y_2 - y_1, \dots, \Delta y_{n-1} = y_n - y_{n-1}$.

In this case $\Delta y_0, \Delta y_1, \dots, \Delta y_n$ are called as First Forward differences of y.

The difference of first forward differences will give us Second forward differences and it is denoted by Δ^2 and is defined as $\Delta^2 y_0 = \Delta(\Delta y_0)$

$$= \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 - y_0$$

Similarly, the difference of second forward differences will give us third forward difference and it is denoted by Δ^3.

Forward difference table

x	$y = f(x)$	First Forward differences Δy	Second Forward differences $\Delta^2 y$	Third Forward differences $\Delta^3 y$	Fourth differences $\Delta^4 y$
x_0	y_0	$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$			
x_2	y_2			$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_2$	
			$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		
x_3	y_3	$\Delta y_2 = y_3 - y_2$			$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
				$\Delta^3 y_1 = \Delta^2 y_3 - \Delta^2 y_2$	
.	.		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
.	.	.			
.	.	.			
		.			
x_{n-1}	y_{n-1}				
x_n	y_n	$\Delta y_{n-1} = \Delta y_n - \Delta y_{n-1}$			

Note: If h is common difference in the values of x and $y = f(x)$ be the given function then $\Delta f(x) = f(x + h) - f(x)$.

Backward Difference

Let us consider $x_0, x_1, x_2, \dots, x_n$ be given set of observations and let $y_0, y_1, y_2, \dots, y_n$ are corresponding values of the curve $y = f(x)$, then the Backward difference operator is denoted by ∇ and is defined as $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$.

In this case $\nabla y_0, \nabla y_1, \dots, \nabla y_n$ are called as First Backward differences of y.

The difference of first Backward differences will give us Second Backward differences and it is denoted by ∇^2 and is defined as $\nabla^2 y_2 = \nabla(\nabla y_2)$

$$= \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 - y_0$$

Similarly, the difference of second backward differences will give us third backward difference and it is denoted by ∇^3.

Backward difference table

x	$y=f(x)$	First Backward differences Δy	Second Backward differences $\Delta^2 y$	Third Backward differences $\Delta^3 y$	Fourth differences $\Delta^4 y$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
		$\nabla y_2 = y_2 - y_1$			
x_2	y_2			$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
			$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		
					$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
x_3	y_3	$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
.	.	.			
.	.	.	$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
.	.	.	.		
x_{n-1}	y_{n-1}				
x_n	y_n	$\nabla y_n = \nabla y_n - \nabla y_{n-1}$			

Note: If h is common difference in the values of x and $y = f(x)$ be the given function then $\nabla f(x+h) = f(x+h) - f(x)$.

Central differences

Let us consider $x_0, x_1, x_2, \dots, x_n$ be given set of observations and let $y_0, y_1, y_2, \dots, y_n$ are corresponding values of the curve $y = f(x)$, then the Central difference operator is denoted by δ and is defined as

- If n is odd : $\delta^n y_{r-\frac{1}{2}} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1,2,3, \dots$
- If n is even : $\delta^n y_r = \delta^{n-1} y_{r+\frac{1}{2}} - \delta^{n-1} y_{r-\frac{1}{2}}, r = 1,2,3, \dots$

  and $\delta^0 y_r = y_r$

The Central difference table is shown below

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{\frac{1}{2}}$			
			$\delta^2 y_1$		
x_1	y_1				
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$	
x_3	y_3				
			$\delta^2 y_3$		
		$\delta y_{\frac{7}{2}}$			
x_4	y_4				

Note: Let h be common difference in the values of x and $y = f(x)$ be given function then $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$

Symbolic Relations and Separation of Symbols

Average Operator: The average operator μ is defined by the equation

$$\mu y_r = \frac{1}{2}\left(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right)$$

(Or)

Let h is the common difference in the values of x and $y = f(x)$ be the given function, then the average operator is denoted by μ and is defined as $\mu f(x) = \frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}$

Shift Operator: The Shift operator E is defined by the equation $Ey_r = y_{r+1}$

Similarly, $E^n y_r = y_{n+r}$

(Or)

Let h is the common difference in the values of x and $y = f(x)$ be the given function, then the shift operator is denoted by E and is defined as $Ef(x) = f(x+h)$

Inverse Operator: The Inverse Operator E^{-1} is defined as $E^{-1}y_r = y_{r-1}$

In general, $E^{-n}y_r = y_{r-n}$

Properties

1) *Prove that* $E = 1 + \Delta$

Sol: Consider R.H.S: $(1+\Delta)y_n = y_n + \Delta y_n$

$= y_n + (y_{n+1} - y_n)$

$= y_{n+1}$

$= E^1 y_n \quad (\because E^n y_r = y_{n+r})$

$\therefore E = 1 + \Delta$

2) *Prove that* $\nabla = 1 - E^{-1}$

Sol: Consider L.H.S: $\nabla y_n = y_n - y_{n-1}$

$= y_n - E^{-1}y_n$

$= (1 - E^{-1})y_n$

$\therefore \nabla = 1 - E^{-1}$

3) *Prove that* $\Delta = E\nabla = \nabla E$

Sol: Case (i) Consider $(E\nabla)y_n = E(\nabla y_n)$

$= E(y_n - y_{n-1})$

$= Ey_n - Ey_{n-1}$

$= y_{n+1} - y_n$

$= \Delta y_n$

$\therefore \Delta = E\nabla$

Case (ii) Consider $(\nabla E)y_n = \nabla(Ey_n)$

$= \nabla(y_{n+1})$

$= y_{n+1} - y_n$

$= \Delta y_n$

$\therefore \Delta = \nabla E$

Hence from these cases, we can conclude that $\Delta = E\nabla = \nabla E$

4) *Prove that* $(1+\Delta)(1-\nabla) = 1$

Sol: Consider $(1+\Delta)(1-\nabla)y_n = (1+\Delta)(y_n - \nabla y_n)$

$= (1+\Delta)\,(y_n - \{y_n - y_{n-1}\}$

$= (1+\Delta)y_{n-1}$

$$= (y_{n-1} + \{y_n - y_{n-1}\})$$
$$= y_n$$

Hence $(1+\Delta)(1-\nabla) = 1$

5) Prove that $\Delta = \nabla(1-\nabla)^{-1}$ (Hint: Consider $\Delta(1-\nabla)$)

6) Prove that $(1+\Delta) = (E-1)\nabla^{-1}$

7) Prove that $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Sol: We know that $\delta y_r = y_{r+\frac{1}{2}} - y_{r-\frac{1}{2}}$

$$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) y_r$$

Hence the result $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

8) Prove that $\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$

Sol: We know that $\mu y_r = \frac{1}{2}\left(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right)$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) y_r$$

Hence proved that $\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$

9) Prove that $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Sol: We know that $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Squaring on both sides, we get $\delta^2 = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2$

$$\text{L.H.S} \Longrightarrow 1 + \frac{1}{4}\delta^2 = 1 + \frac{1}{4}(E^1 + E^{-1} - 2)$$

$$= \frac{1}{4}(E^1 + E^{-1} + 2) = \mu^2$$

Hence the result

Relation between the operator D and E

Here Operator $D \equiv \frac{d}{dx}$

We know that $Ef(x) = f(x+h)$

Expanding using Taylor's series , we get

$$Ef(x) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \ \dots$$

$$= [1 + hD + h^2D^2 + \ \dots]f(x)$$

$$= e^{hD}f(x)$$

$$\Longrightarrow E = e^{hD}$$

Newton’s Forward Interpolation Formula

Statement: If $x_0, x_1, x_2, \dots, x_n$ are given set of observations with common difference h and let $y_0, y_1, y_2, \dots, y_n$ are their corresponding values, where $y = f(x)$ be the given function then

$$f(x) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\,\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \dots + \frac{p\,(p-1)(p-2)\dots(p-(n-1))}{n!}\,\Delta^n y_0$$

where $p = \frac{x-x_0}{h}$

Proof: Let us assume an n^{th} degree polynomial

$$f(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + \dots + A_n\,(x-x_0)(x-x_1)\,\dots(x-x_{n-1}) \text{ ---> (i)}$$

Substitute $x = x_0$ in (i), we get $f(x_0) = A_0 \implies y_0 = A_0$

Substitute $x = x_1$ in (i), we get $f(x_1) = A_0 + A_1(x_1 - x_0) \implies y_1 = y_0 + A_1 h$

$$\implies A_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

Substitute $x = x_2$ in (i), we get $f(x_2) = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$

$$\implies y_2 = y_0 + A_1(2h) + A_2(2h)(h)$$

$$\implies y_2 = y_0 + 2h\left(\frac{\Delta y_0}{h}\right) + 2h^2 A_2$$

$$\implies A_2 = \frac{1}{2h^2}\Delta^2 y_0$$

Similarly, we get $A_n = \frac{1}{nh^2}\Delta^n y_0$

Substituting these values in (i), we get

$$f(x) = y_0 + (x-x_0)\frac{1}{h}\Delta y_0 + (x-x_0)(x-x_1)\frac{1}{2h^2}\Delta^2 y_0 + \dots + (x-x_0)(x-x_1)\,\dots(x-x_{n-1})\frac{1}{nh^2}\Delta^n y_0$$

----(ii)

But given $p = \frac{x-x_0}{h}$

$\implies x - x_0 = ph \implies x = x_0 + h$

$\implies x - x_1 = x - (x_0 + h)$

$= (x - x_0) - h$

$= ph - h = (p-1)h$

Similarly, $x - x_2 = (p-2)h$,

$\vdots$

$x - x_{n-1} = (p-(n-1))h$

Substituting in the Equation (ii), we get

$$f(x) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\,\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \dots + \frac{p\,(p-1)(p-2)\dots(p-(n-1))}{n!}\,\Delta^n y_0$$

Newton's Backward Interpolation Formula

Statement: If $x_0, x_1, x_2, \dots, x_n$ are given set of observations with common difference h and let $y_0, y_1, y_2, \dots, y_n$ are their corresponding values, where $y = f(x)$ be the given function then

$$f(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\nabla^3 y_0 + \dots + \frac{p\,(p+1)(p+2)\dots(p+(n-1))}{n!}\nabla^n y_0$$

where $p = \frac{x-x_0}{h}$

Proof: Let us assume an n^{th} degree polynomial

$$f(x) = A_0 + A_1(x-x_n) + A_2(x-x_n)(x-x_{n-1}) + \dots + A_n\,(x-x_n)(x-x_{n-1})\dots(x-x_1) \quad \text{--> (i)}$$

Substitute $x = x_n$ in (i), we get $f(x_n) = A_0 \implies y_n = A_0$

Substitute $x = x_{n-1}$ in (i), we get $f(x_{n-1}) = A_0 + A_1(x_{n-1} - x_n) \implies y_{n-1} = y_n - A_1 h$

$$\implies A_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_0}{h}$$

Substitute $x = x_{n-2}$ in (i), we get $f(x_{n-2}) = A_0 + A_1(x_{n-2} - x_n) + A_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$

$$\implies y_{n-2} = y_n + A_1(-2h) + A_2(-2h)(-h)$$

$$\implies y_{n-2} = y_n - 2h\left(\frac{\nabla y_n}{h}\right) + 2h^2 A_2$$

$$\implies A_2 = \frac{1}{2h^2}\nabla^2 y_n$$

Similarly, we get $A_n = \frac{1}{nh^2}\nabla^n y_n$

Substituting these values in (i), we get

$$f(x) = y_n + (x-x_n)\frac{1}{h}\nabla y_n + (x-x_n)(x-x_{n-1})\frac{1}{2h^2}\nabla^2 y_n + \dots$$

$$+(x-x_n)(x-x_{n-1})\dots(x-x_1)\frac{1}{nh^2}\nabla^n y_n \quad \text{---- (ii)}$$

But given $p = \frac{x-x_n}{h}$

$\implies x - x_n = ph \implies x = x_n + h$

$\implies x - x_{n-1} = x - (x_n - h)$

$= (x - x_n) + h$

$= ph + h = (p+1)h$

Similarly, $x - x_{n-2} = (p+2)h$,

⋮

$x - x_1 = (p + (n-1))h$

Substituting in the Equation (ii), we get

$$f(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\Delta^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\Delta^3 y_n + \dots + \frac{p\,(p+1)(p+2)\dots(p+(n-1))}{n!}\Delta^n y_n$$

Gauss forward central difference formula

Statement: If $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$ are given set of observations with common difference h and let $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ are their corresponding values, where $y = f(x)$ be the given function then $y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p\,(p-1)(p+1)\,(p-2)}{4!}\Delta^4 y_{-2} + \dots$ where $p = \frac{x-x_0}{h}$.

Proof:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$\vdots$	$\vdots$				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}				
			$\Delta^2 y_{-2}$		
		Δy_{-1}			
				$\Delta^3 y_{-2}$	
x_0	y_0				
					$\Delta^4 y_{-2}$
			$\Delta^2 y_{-1}$		
x_1	y_1				
		Δy_0		$\Delta^3 y_{-1}$	
			$\Delta^2 y_0$		
x_2	y_2				
		Δy_{-1}			
$\vdots$	$\vdots$				

Let us assume a polynomial equation by using the arrow marks shown in the above table.

Let $y_p = y_0 + G_1\,\Delta y_0 + G_2\,\Delta^2 y_{-1} + G_3\,\Delta^3 y_{-1} + G_4\,\Delta^4 y_{-2} + \dots$ ---- (1)

where $G_0, G_1, G_2, \dots$ are unknowns

$y_p = y_{p+0} = E^p y_0 = (1+\Delta)^p y_0 \qquad (\because E = 1+\Delta)$

$\Longrightarrow\ y_p = \left(1 + p_{C_1}\Delta + p_{C_2}\Delta^2 + p_{C_3}\Delta^3 + \dots + p_{C_p}\Delta^p\right) y_0$

$\Longrightarrow\ y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \dots$ --- (2)

Now, $y_{-1} = y_{-1+0} = E^{-1} y_0 = (1+\Delta)^{-1} y_0$

$= (1 - \Delta + \Delta^2 - \Delta^3 + \dots) y_0$

$\Longrightarrow\ y_{-1} = y_0 - \Delta y_0 + \Delta^2 y_0 - \dots$

Therefore, $\Delta^2\, y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \dots$ ----- (3)

and $\quad \Delta^3\, y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \dots$ ----- (4)

Substituting 2, 3, 4 in 1, we get

$$y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \dots = y_0 + G_1\,\Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \cdots) + G_3(\Delta^3 y_0 - \Delta^4 y_0 + \dots) + \dots$$

Comparing corresponding coefficients, we get

$G_1 = p$, $G_2 = \frac{p\,(p-1)}{2!}$, $-G_2 + G_3 = \frac{p\,(p-1)(p-2)}{3!}$ $\Rightarrow$ $G_3 = \frac{p\,(p-1)(p+1)}{3!}$

Similarly, $G_4 = \frac{p\,(p-1)(p+1)\,(p-2)}{4!}$

Substituting all these values of $G_0, G_1, G_2, \ldots$ in (1), we get

$$\boldsymbol{y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p\,(p-1)(p+1)\,(p-2)}{4!}\Delta^4 y_{-2} + \ldots}$$

Gauss backward central difference formula

Statement: If $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ are given set of observations with common difference h and let $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ are their corresponding values, where $y = f(x)$ be the given function then

$$y_p = y_0 + p\,\Delta y_{-1} + \frac{p\,(p+1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p\,(p+1)(p-1)\,(p+2)}{4!}\Delta^4 y_{-2} + \ldots$$

where $p = \frac{x - x_0}{h}$.

Proof:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$\vdots$	$\vdots$				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_{-1}			
x_2	y_2				
$\vdots$	$\vdots$				

Let us assume a polynomial equation by using the arrow marks shown in the above table.

Let $y_p = y_0 + G_1\,\Delta y_{-1} + G_2\,\Delta^2 y_{-1} + G_3\,\Delta^3 y_{-2} + G_4\,\Delta^4 y_{-2} + \ldots$ ---- (1)

where $G_0, G_1, G_2, \ldots$ are unknowns

$y_p = y_{p+0} = E^p y_0 = (1+\Delta)^p y_0$ $\quad (\because E = 1 + \Delta)$

$$\Rightarrow\ y_p = \left(1 + p_{C_1}\Delta + p_{C_2}\Delta^2 + p_{C_3}\Delta^3 + \ldots + p_{C_p}\Delta^p\right) y_0$$

$$\Rightarrow\ y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ldots \quad \text{--- (2)}$$

Now, $y_{-1} = y_{-1+0} = E^{-1} y_0 = (1+\Delta)^{-1} y_0$

$= (1 - \Delta + \Delta^2 - \Delta^3 + \ldots)\,y_0$

$\Rightarrow\ y_{-1} = y_0 - \Delta y_0 + \Delta^2 y_0 - \ldots$

Therefore, $\Delta y_{-1} = \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \ \dots$ ----- (3)

$\Delta^2\, y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \ \dots$ ----- (4)

Also $y_{-2} = y_{-2+0} = E^{-2} y_0 = (1+\Delta)^{-2} y_0$

$$= (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \ \dots) y_0$$

$$\Longrightarrow\ y_{-2} = y_0 - 2\Delta y_0 + 3\Delta^2 y_0 - \ \dots$$

Now, $\Delta^3\, y_{-2} = \Delta^3 y_0 - 2\Delta^4 y_0 + \ \dots$ ----- (5)

Substituting 2, 3, 4, 5 in 1, we get

$$y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ \dots\ = y_0 + G_1\,(\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \ \dots\ ) + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \cdots\ ) + G_3(\Delta^3 y_0 - 2\Delta^4 y_0 + \ \dots) + \ \dots$$

Comparing corresponding coefficients, we get

$G_1 = p\,, -G_1 + G_2 = \frac{p\,(p-1)}{2!}\ \Longrightarrow\ G_2 = \frac{p\,(p+1)}{2!}$

Also, $G_1 - G_2 + G_3 = \frac{p\,(p-1)(p-2)}{3!}\ \Longrightarrow G_3 = \frac{p\,(p+1)(p-1)}{3!}$

Similarly, $G_4 = \frac{p\,(p+1)(p-1)\,(p+2)}{4!}, \dots$

Substituting all these values of $G_0, G_1, G_2, \dots$ in (1), we get

$$\boldsymbol{y_p = y_0 + p\,\Delta y_{-1} + \frac{p\,(p+1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p\,(p+1)(p-1)\,(p+2)}{4!}\Delta^4 y_{-2} + \ \dots\quad, p = \frac{x - x_0}{h}}$$

Stirling's Formulae

Statement: If $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$ are given set of observations with common difference h and let $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ are their corresponding values, where $y = f(x)$ be the given function then

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\ \Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!}\ \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ \dots \text{ where } p = \frac{x - x_0}{h}$$

Proof: Stirling's Formula will be obtained by taking the average of Gauss forward difference formula and Gauss Backward difference formula.

We know that, from Gauss forward difference formula

$$y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p\,(p-1)(p+1)\,(p-2)}{4!}\Delta^4 y_{-2} + \ \dots \text{ ---- > (1)}$$

Also, from Gauss backward difference formula

$$y_p = y_0 + p\,\Delta y_{-1} + \frac{p\,(p+1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p\,(p+1)(p-1)\,(p+2)}{4!}\Delta^4 y_{-2} + \ \dots \text{ ---- > (2)}$$

Now, $Stirling's\ Formula = \frac{1}{2}(Gauss\ forward\ formula + Gauss\ backward\ formula)$

$$\therefore\ y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\ \Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!}\ \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ \dots$$

Lagrange's Interpolation Formula

Statement: If $x_0, x_1, x_2, \dots, x_n$ are given set of observations which are need not be equally spaced and let $y_0, y_1, y_2, \dots, y_n$ are their corresponding values, where $y = f(x)$ be the given function then $f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \cdots \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$

Proof: Let us assume an n^{th} degree polynomial of the form

$$f(x) = A_0(x-x_1)(x-x_2)\dots(x-x_n) + A_1(x-x_0)(x-x_2)\dots(x-x_n) + \dots + A_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \quad \text{---- (1)}$$

Substitute $x = x_0$, we get $f(x_0) = A_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$

$$\Rightarrow y_0 = A_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

$$\Rightarrow A_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

Again, $x = x_1$, we get $f(x_1) = A_1(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$

$$\Rightarrow y_1 = A_1(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$$

$$\Rightarrow A_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

Proceeding like this, finally we get, $A_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$

Substituting these values in the Equation (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \cdots \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

Note: This Lagrange's formula is used for both equally spaced and unequally spaced arguments.

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