MATHEMATICAL METHODS

FOURIER SERIES

I YEAR B.Tech

By

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SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

Name of the Unit	Name of the Topic
Unit-I **Solution of Linear systems**	**Matrices and Linear system of equations:** Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems.
Unit-II **Eigen values and Eigen vectors**	Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.
Unit-III **Linear Transformations**	Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermition and skew Hermition matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition.
Unit-IV **Solution of Non-linear Systems**	**Solution of Algebraic and Transcendental Equations**- Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method **Interpolation:**Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton's Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange's Interpolation formulae- B. Spline interpolation, Cubic spline.
Unit-V **Curve fitting & Numerical Integration**	**Curve Fitting:** Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. **Numerical Integration:** Numerical Differentiation-Simpson's 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.
Unit-VI **Numerical solution of ODE**	Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.
Unit-VII **Fourier Series**	Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.
Unit-VIII **Partial Differential Equations**	Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.

CONTENTS

UNIT-VII

FOURIER SERIES

FOURIER SERIES

Fourier Series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions.

Most of the single valued functions which occur in applied mathematics can be expressed in the form of Fourier series, which is in terms of sines and cosines.

Fourier series is to be expressed in terms of periodic functions- sines and cosines.

Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.

We know that, Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives. Further, because of the periodic nature, Fourier series constructed for one period is valid for all values.

Periodic Functions

A function $f(x)$ is said to be periodic function with period $T > 0$ if for all x , $f(x+T) = f(x)$, and T is the least of such values.

Ex: 1) $\sin x$, $\cos x$ are periodic functions with period 2π.

2) $\tan x$, $\cot x$ are periodic functions with period π.

Euler's Formulae

The Fourier Series for the function $f(x)$ in the interval $C \leq x \leq C + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + \ b_n \sin nx \right)$$

where $a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \ dx$

$$a_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx \, dx$$

These values a_0 , a_n , b_n are known as Euler's Formulae.

CONDITIONS FOR FOURIER EXPANSION *(Dirchlet Conditions)*

A function $f(x)$ defined in $[0, 2\pi]$ has a valid Fourier series expansion of the form

$$\frac{a_0}{2} + \sum_{n=0}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Where a_0, a_n, b_n are constants, provided

1) $f(x)$ is well defined and single-valued, except possibly at a finite number of point in the interval $[0, 2\pi]$.
2) $f(x)$ has finite number of finite discontinuities in the interval in $[0, 2\pi]$.
3) $f(x)$ has finite number of finite maxima and minima.

Note: The above conditions are valid for the function defined in the Intervals $[-\pi, \pi][0, 2l], [-l, l]$.

- $\{1, \cos 1x, \cos 2x, \cos 3x, \ \dots, \cos nx, \dots \ , \sin 1x, \sin 2x, \sin 3x, \dots, \sin nx, \dots\}$
  Consider any two, All these have a common period 2π. Here 1 $= \cos 0x$
- $\left\{1, \cos\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \cos\frac{3\pi x}{l}, \ \dots, \cos\frac{n\pi x}{l}, \dots \ , \sin\frac{\pi x}{l}, \sin\frac{2\pi x}{l}, \sin\frac{3\pi x}{l}, \dots, \sin\frac{n\pi x}{l}, \dots\right\}$
  All these have a common period $2l$.
  These are called complete set of orthogonal functions.

Definition of Fourier series

- Let $f(x)$ be a function defined in $[0, 2\pi]$. Let $f(x + 2\pi) = f(x)\ \forall\ x$, then the Fourier Series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

These values a_0, a_n, b_n are called as Fourier coefficients of $f(x)$ in $[0, 2\pi]$.

- Let $f(x)$ be a function defined in $[-\pi, \pi]$. Let $f(x + 2\pi) = f(x)\ \forall\ x$, then the Fourier Series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

These values a_0, a_n, b_n are called as Fourier coefficients of $f(x)$ in $[-\pi, \pi]$.

- Let $f(x)$ be a function defined in $[0, 2l]$. Let $f(x+2l) = f(x)\ \forall\ x$, then the Fourier Series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\ dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

These values a_0 , a_n , b_n are called as Fourier coefficients of $f(x)$ in $[0, 2l]$.

- Let $f(x)$ be a function defined in $[-l, l]$. Let $f(x+2\pi) = f(x)\ \forall\ x$, then the Fourier Series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\ dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

These values a_0 , a_n , b_n are called as Fourier coefficients of $f(x)$ in $[-l, l]$.

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know that if $f(x)$ be a function defined in $[-\pi, \pi]$. Let $f(x+2\pi) = f(x)\ \forall\ x$, then the Fourier Series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

These values a_0 , a_n , b_n are called as Fourier coefficients of $f(x)$ in $[-\pi, \pi]$.

Case (i): When $f(x)$ is an even function

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx$

Since $\cos nx$ is an even function, $f(x)$ is an even function $\Longrightarrow$ Product of two even functions is even

$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx \quad (\because\ Integrand\ is\ even)$

Now, $\sin nx$ is an odd function, $f(x)$ is an even function $\Longrightarrow$ Product of odd and even is odd

$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \;\; (\because\ Integrand\ is\ odd)$

Thus, if a function $f(x)$ is even in $[-\pi, \pi]$, its Fourier series expansion contains only cosine terms.

Hence Fourier Series is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx, n = 0, 1, 2, 3, \ldots$

Case (ii): When $f(x)$ is an Odd Function

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = 0$

Since $\cos nx$ is an even function, $f(x)$ is an odd function $\Longrightarrow$ Product of even and odd is even

$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad (\because\ Integrand\ is\ odd)$

Now, $\sin nx$ is an odd function, $f(x)$ is an odd function $\Longrightarrow$ Product of two odd functions is even

$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx \qquad (\because\ Integrand\ is\ even)$

Thus, if a function $f(x)$ is Odd in $[-\pi, \pi]$, its Fourier series expansion contains only sine terms.

Hence, if $f(x)$ is odd function defined in $[-\pi, \pi]$, $f(x)$ can be expanded as a series of the form

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$

HALF RANGE FOURIER SERIES

Half Range Fourier Sine Series defined in $[0, \pi]$: The Fourier half range sine series in $[0, \pi]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n\ sin\, nx,$$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, sin\, nx\ dx$

This is Similar to the Fourier series defined for odd function in $[-\pi, \pi]$

Half Range Fourier Cosine Series defined in $[0, \pi]$: The Fourier half range Cosine series in $[0, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

This is Similar to the Fourier series defined for even function in $[-\pi, \pi]$

Half Range Fourier Sine Series defined in $[0, l]$: The Fourier half range sine series in $[0, \pi]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $b_n = \frac{2}{l}\int_0^{l} f(x) \sin\frac{n\pi x}{l}\, dx$

This is Similar to the Fourier series defined for odd function in $[-l, l]$

Half Range Fourier Cosine Series defined in $[0, l]$: The Fourier half range Cosine series in $[0, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx$

$a_n = \frac{2}{l}\int_0^{l} f(x) \cos\frac{n\pi x}{l}\, dx$

This is Similar to the Fourier series defined for even function in $[-l, l]$

Important Formulae

- $\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^{a} f(x)\, dx & if\ f(2a-x) = f(x) \\ 0 & if\ f(2a-x) = -f(x) \end{cases}$
- $\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^{a} f(x)\, dx & if\ f\ is\ even \\ 0 & if\ f\ is\ odd \end{cases}$

  Here Even function means: If $f(-x) = f(x)$, then $f(x)$ is called as even function

  Odd function means : If $f(-x) = -f(x)$, then $f(x)$ is called as odd function.
- $\int_a^{b} f(x)\, dx = -\int_b^{a} f(x)\, dx$
- $\int_a^{b} f(x)\, dx = \int_a^{c} f(x)\, dx + \int_c^{b} f(x)\, dx\ ; a < c < b$
- $\int_0^{a} f(x)\, dx = \int_0^{a} f(a-x)\, dx$, Also $\int_a^{b} f(x)\, dx = \int_a^{b} f(a+b-x)\, dx$

Problems on Fourier Series

1) Find the Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Here, $f(x) = x^2$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[(2\pi)^3 - 0] = \frac{8}{3}\pi^2$$

$$\Longrightarrow \boxed{a_0 = \frac{8}{3}\pi^2}$$

Again, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx}_{v}\,dx$

$$= \frac{1}{\pi}\left[x^2\int \cos nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \cos nx\,dx\right)dx\right\}\right]$$

$$\left[\because \int uv\,dx = u\int v\,dx - \left\{\int \frac{du}{dx}.\left(\int v\,dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \left\{\int 2x\left(\frac{\sin nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left\{\int \underbrace{x}_{u}\,\underbrace{\sin nx}_{v}\,dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \int 1.\frac{\cos nx}{n}\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right]$$

$$\Rightarrow \boxed{a_n = \frac{4}{n^2}}$$

Again, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\underbrace{\sin nx}_{v}\, dx$

$$= \frac{1}{\pi}\left[x^2 \int \sin nx\, dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \sin nx\, dx\right)dx\right\}\right]$$

$$\left[\because \int uv\, dx = u\int v\, dx - \left\{\int \frac{du}{dx}.\left(\int v\, dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \left\{\int 2x\left(-\frac{\cos nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left\{\int \underbrace{x}_{u}\underbrace{\cos nx}_{v}\, dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \int 1.\frac{\sin nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\int \sin nx\, dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right]$$

$$\Rightarrow \boxed{b_n = -\frac{4\pi}{n}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

This is the Fourier series for the function $f(x) = x^2$

Hence the result

2) Find the Fourier series of the periodic function defined as $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; \ 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots \ = \frac{\pi^2}{8}$

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Here, $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; \ 0 < x < \pi \end{cases}$

Now, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right]$

$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx \right]$

$= \frac{1}{\pi} \left[(-\pi) \int_{-\pi}^{0} dx + \int_{0}^{\pi} x\, dx \right]$

$= \frac{1}{\pi} \left[(-\pi)[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi} \right] = \frac{1}{\pi} \left[(-\pi)(\pi) + \frac{\pi^2}{2} \right]$

$= \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right] = -\frac{\pi}{2}$

$$\Rightarrow \boxed{a_0 = -\frac{\pi}{2}}$$

Also, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx \right]$

$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx \right]$

$= \frac{1}{\pi} \left[-\pi \int_{-\pi}^{0} (\cos nx)\, dx + \int_{0}^{\pi} x \cos nx\, dx \right]$

$= \frac{1}{\pi} \left[-\pi \left(\frac{\sin nx}{n} \right)_{-\pi}^{0} + \left\{ x \left(\frac{\sin nx}{n} \right) - \int 1 \left(\frac{\sin nx}{n} \right) dx \right\}_0^{\pi} \right]$

$= \frac{1}{\pi} \left[-\frac{\pi}{n} (\sin nx)_{-\pi}^{0} + \left\{ \frac{x \sin nx}{n} - \frac{1}{n} \int \sin nx\, dx \right\}_0^{\pi} \right]$

$= \frac{1}{\pi} \left[-\frac{\pi}{n} (\sin nx)_{-\pi}^{0} + \left\{ \frac{x \sin nx}{n} - \frac{1}{n} \frac{(-\cos nx)}{n} \right\}_0^{\pi} \right]$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(\sin nx)_{-\pi}^{0} + \left\{\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}[0-\sin(-n\pi)] + \left\{\left(\frac{\pi\sin n\pi}{n} + \frac{1}{n^2}\cos n\pi\right) - \left(0 + \frac{1}{n^2}\cos n0\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi + \left\{\left(\frac{\pi\sin n\pi}{n} + \frac{1}{n^2}\cos n\pi\right) - \frac{1}{n^2}.1\right\}\right] \qquad \left[\because \begin{array}{c}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = \sin\theta\end{array}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[-\frac{\pi}{n}(0) + \left\{\left(\frac{\pi(0)}{n} + \frac{1}{n^2}(-1)^n\right) - \frac{1}{n^2}\right\}\right] \qquad \left[\because \begin{array}{c}\sin n\pi = \ \ 0 \\ \& \\ \cos n\pi = (-1)^n\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi n^2}[(-1)^n - 1]}$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_0^{\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\, dx + \int_0^{\pi} x\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0}(\sin nx)\, dx + \int_0^{\pi} x\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - \int 1\left(-\frac{\cos nx}{n}\right)dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{1}{n}\int\cos nx\, dx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{1}{n}\frac{(\sin nx)}{n}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}[1-\cos(-n\pi)] + \left\{\left(-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin n\pi\right) - \left(-0 + \frac{1}{n^2}\sin n0\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}[1-\cos n\pi] - \frac{\pi\cos n\pi}{n}\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$$\left[\because \begin{array}{c}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = \sin\theta\end{array} \& \begin{array}{c}\sin n\pi = \ \ 0 \\ \& \\ \cos n\pi = (-1)^n\end{array}\right]$$

$$\Rightarrow \boxed{b_n = \frac{1}{n}(1 - 2\cos n\pi)}$$

Hence, the Fourier series for given $f(x)$ is given by

$$f(x) = \frac{-\frac{\pi}{2}}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n^2}[(-1)^n - 1]\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right)$$

$$\Rightarrow f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n^2}[(-1)^n - 1]\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right)$$

$$\Rightarrow f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ \dots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ \dots\right)$$

Deduction: Put $x = 0$ in the above function $f(x)$, we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots\right)$$

Since, $f(x)$ is discontinuous at $x = 0$, $\begin{matrix} f(0-0) = -\pi \\ f(0+0) = \ \ 0 \end{matrix}$

$$\Rightarrow f(0) = \frac{1}{2}[f(0-0) + f(0+0)]$$

$$\Rightarrow f(0) = \frac{1}{2}(-\pi) = -\frac{\pi}{2}$$

Hence, $f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots\right)$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ \dots\ = \frac{\pi^2}{8}$$

Hence the result

3) Expand the function $f(x) = x^2$ as Fourier series in $[-\pi, \pi]$.

Hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots = \frac{\pi^2}{6}$

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Here, $f(x) = x^2$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\, dx$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$

$$\Longrightarrow \boxed{a_0 = \frac{2\pi^2}{3}}$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \qquad \left[\because f(x)\ is\ even \Longrightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2x\sin nx}{n^3}\right]_0^{\pi} = \frac{4}{n^2}(-1)^n$$

$$\Longrightarrow \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= 0 \qquad \left[\because f(x)\ is\ odd \Longrightarrow \int_{-a}^{a} f(x)dx = 0\right]$$

Hence, the Fourier series for given $f(x)$ is given by

$$f(x) = x^2 = \frac{\left(\frac{2\pi^2}{3}\right)}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$\Longrightarrow x^2 = \frac{\pi^2}{3} + 4\left(-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \ \dots\right)$$

Deduction: Put $x = \pi$ in the above equation, we get

$$\Longrightarrow \pi^2 = \frac{\pi^2}{3} + 4\left(-\cos\pi + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} - \ \dots\right)$$

$$\Longrightarrow \pi^2 - \frac{\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots\right)$$

$$\Longrightarrow \frac{2\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots\right)$$

$$\Longrightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \dots$$

Hence the Result

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