MATHEMATICAL METHODS

SOLUTION OF LINEAR SYSTEMS

I YEAR B.Tech

By

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SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

Name of the Unit	Name of the Topic
Unit-I Solution of Linear systems	**Matrices and Linear system of equations:** Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems.
Unit-II Eigen values and Eigen vectors	Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.
Unit-III Linear Transformations	Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermition and skew Hermition matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition.
Unit-IV Solution of Non-linear Systems	**Solution of Algebraic and Transcendental Equations**- Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method **Interpolation:**Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton's Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange's Interpolation formulae- B. Spline interpolation, Cubic spline.
Unit-V Curve fitting & Numerical Integration	**Curve Fitting:** Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. **Numerical Integration:** Numerical Differentiation-Simpson's 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.
Unit-VI Numerical solution of ODE	Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.
Unit-VII Fourier Series	Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.
Unit-VIII Partial Differential Equations	Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.

CONTENTS

UNIT-I

SOLUTIONS OF LINEAR SYSTEMS

MATRICES

Matrix: The arrangement of set of elements in the form of rows and columns is called as Matrix. The elements of the matrix being Real (or) Complex Numbers.

Order of the Matrix: The number of rows and columns represents the order of the matrix. It is denoted by $m \times n$, where m is number of rows and n is number of columns.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is 2×3 matrix.

Note: Matrix is a system of representation and it does not have any Numerical value.

Types of Matrices

- Rectangular Matrix: A matrix is said to be rectangular, if the number of rows and number of columns are not equal.

  Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a rectangular matrix.

- Square Matrix: A matrix is said to be square, if the number of rows and number of columns are equal.

  Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a Square matrix.

- Row Matrix: A matrix is said to be row matrix, if it contains only one row.

  Ex: $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a row matrix.

- Column Matrix: A matrix is said to be column matrix, if it contains only one column.

  Ex: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column matrix

- Diagonal Matrix: A square matrix $A_{n \times n}$ is said to be diagonal matrix if $a_{ij} = 0 \ \forall \ i \neq j$

  (Or)

  A Square matrix is said to be diagonal matrix, if all the elements except principle diagonal elements are zeros.

  - The elements on the diagonal are known as principle diagonal elements.
  - The diagonal matrix is represented by $A = diag[a_{11} \ \ a_{22} \ \ \ldots \ a_{nn}]$

  Ex: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ then $A = diag[1 \ \ 2]$

- Trace of a Matrix: Suppose A is a square matrix, then the trace of A is defined as the sum of its diagonal elements.

  i.e. $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

  - $Tr(A + B) = Tr(A) + Tr(B)$

- Scalar Matrix: A Square matrix $A_{n\times n}$ is said to be a Scalar matrix if $a_{ij} = 0 \ \forall \ i \neq j$
$a_{ij} = k \ \forall \ i = j$

(Or)

A diagonal matrix is said to be a Scalar matrix, if all the elements of the principle diagonal are equal.

i.e. $a_{ij} = k \ \forall \ i = j$

  - Trace of a Scalar matrix is nk.

- Unit Matrix (or) Identity Matrix: A Square matrix $A_{n\times n}$ is said to be a Unit (or) Identity matrix if $a_{ij} = \begin{cases} 0 & \forall \ i \neq j \\ 1 & \forall \ i = j \end{cases}$

(Or)

A Scalar matrix is said to be a Unit matrix if the scalar $k = 1$

Ex: $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

  - Unit matrix is denoted by I.
  - The Trace of a Unit Matrix is n, where order of the matrix is $n \times n$.

- Transpose of a Matrix: Suppose A is a $m \times n$ matrix, then transpose of A is denoted by $A'(or)\ A^T$ and is obtained by interchanging of rows and columns of A.

Ex: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ -2 & -5 \\ 3 & 4 \end{bmatrix}$

  - If A is of Order $m \times n$, then A^T is of Order $n \times m$
  - If A is a square matrix, then $Tr(A) = Tr(A^T)$
  - $(A \pm B)^T = A^T \pm B^T$
  - $(AB)^T = B^T A^T$
  - $(kA)^T = kA^T$
  - If A is a scalar matrix then $A^T = A$
  - $(A^T)^T = A$
  - $I^T = I$

- Upper Triangular Matrix: A matrix $A_{m\times n}$ is said to be an Upper Triangular matrix, if $a_{ij} = 0 \ \forall \ i > j$.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 2 & 1 \\ 0 & 0 & 8 & 0 \end{bmatrix}$ is Upper Triangular matrix

  - In a square matrix, if all the elements below the principle diagonal are zero, then it is an Upper Triangular Matrix

    Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ is a Upper Triangular matrix.

- Lower Triangular Matrix: A matrix $A_{m\times n}$ is said to be an Lower Triangular matrix, if $a_{ij} = 0 \ \forall \ i < j$.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 9 & 8 & 0 \end{bmatrix}$ is Upper Triangular matrix

- ❖ In a square matrix, if all the elements above the principle diagonal are zero, then it is an Lower Triangular Matrix

  Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 7 & 7 \end{bmatrix}$ is a Lower Triangular matrix.

- ❖ Diagonal matrix is Lower as well as Upper Triangular matrix.

- **Equality of two matrix:** Two matrices $A_{m\times n}, B_{m\times n}$ are said to be equal if $a_{ij} = b_{ij}\ \forall\ i, j$
- **Properties on Addition and Multiplication of Matrices**
  - ❖ Addition of Matrices is Associative and Commutative
  - ❖ Matrix multiplication is Associative
  - ❖ Matrix multiplication need not be Commutative
  - ❖ Matrix multiplication is distributive over addition

    i.e. $A(B+C) = AB + AC$ (Left Distributive Law)

    $(B+C)A = BA + CA$ (Right Distributive Law)
  - ❖ Matrix multiplication is possible only if the number of columns of first matrix is equal to the number of rows of second matrix.
- **Symmetric Matrix:** A Square matrix $A_{n\times n}$ is said to be symmetric matrix if $A^T = A$

  i.e. $a_{ij} = a_{ji}\ \forall\ i, j$
  - ❖ Identity matrix is a symmetric matrix.
  - ❖ Zero square matrix is symmetric. i.e. $O_{n\times n}$.
  - ❖ Number of Independent elements in a symmetric matrix are $\frac{n(n+1)}{2}$, n is order.
- **Skew Symmetric Matrix:** A Square matrix $A_{n\times n}$ is said to be symmetric matrix if $A^T = -A$

  i.e. $a_{ij} = -a_{ji}\ \forall\ i, j$

  It is denoted by A'
  - ❖ Zero square matrix is symmetric. i.e. $O_{n\times n}$.
  - ❖ The elements on the principle diagonal are zero.
  - ❖ Number of Independent elements in a skew symmetric matrix are $\frac{n(n-1)}{2}$, n is order.

Ex: 1) $A = \begin{bmatrix} 1 & 3 \\ -3 & 0 \end{bmatrix}$ is not a skew symmetric matrix

2) $A = \begin{bmatrix} 0 & -3 & -5 \\ 3 & 0 & -9 \\ 5 & 9 & 0 \end{bmatrix}$ is a skew symmetric matrix.

Theorem

Every Square matrix can be expressed as the sum of a symmetric and skew-symmetric matrices.

Sol: Let us consider A to be any matrix.

Now, $A = \frac{1}{2}(2A)$

$$= \frac{1}{2}(A + A)$$

$$= \frac{1}{2}[(A + A^T) + (A - A^T)]$$

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

This is in the form of $A = B + C$, where $B = \frac{1}{2}(A + A^T)$, $C = \frac{1}{2}(A - A^T)$

Now, we shall prove that one is symmetric and other one is skew symmetric.

Let us consider $\boldsymbol{B} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$

$\Rightarrow B^T = \left[\frac{1}{2}(A + A^T)\right]^T$

$$= \frac{1}{2}(A + A^T)^T$$

$$= \frac{1}{2}[A^T + (A^T)^T]$$

$[\because (A + B)^T = A^T + B^T]$

$$= \frac{1}{2}[A^T + A] = B$$

$\Rightarrow \boldsymbol{B}^T = \boldsymbol{B}$

$\therefore B$ is Symmetric Matrix

Again, let us consider $\boldsymbol{C} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$

$\Rightarrow C^T = \left[\frac{1}{2}(A - A^T)\right]^T$

$$= \frac{1}{2}(A - A^T)^T$$

$$= \frac{1}{2}[A^T - (A^T)^T]$$

$[\because (A - B)^T = A^T - B^T]$

$$= \frac{1}{2}[A^T - A] = -\frac{1}{2}(A - A^T) = -C$$

$\Rightarrow \boldsymbol{C}^T = -\boldsymbol{C}$

$\therefore C$ is Skew-Symmetric Matrix

Hence, every square matrix can be expressed as sum of symmetric and skew-symmetric matrices.

- Conjugate Matrix: Suppose A is any matrix, then the conjugate of the matrix A is denoted by $\bar{A}$ and is defined as the matrix obtained by taking the conjugate of every element of A.
    - Conjugate of $a + ib$ is $a - ib$
    - $\overline{(\bar{A})} = A$
    - $\overline{A.B} = \bar{A}.\bar{B}$
    - $\overline{A + B} = \bar{A} + \bar{B}$

    Ex: If $A = \begin{bmatrix} 1 & 2+3i \\ 3-4i & -2i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1 & 2-3i \\ 3+4i & 2i \end{bmatrix}$

- Conjugate Transpose of a matrix (or) Transpose conjugate of a matrix: Suppose A is any square matrix, then the transpose of the conjugate of A is called Transpose conjugate of A. It is denoted by $A^\theta = (\bar{A})^T = \overline{(A^T)}$.

    Ex: If $A = \begin{bmatrix} 1-i & -2i \\ 4-3i & 5-4i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 1+i & 2i \\ 4+3i & 5+4i \end{bmatrix}$

    Now, $(\bar{A})^T = \begin{bmatrix} 1+i & 4+3i \\ 2i & 5+4i \end{bmatrix} = A^\theta$

- $(A^\theta)^\theta = A$
- $(A+B)^\theta = A^\theta + B^\theta$
- $(AB)^\theta = B^\theta A^\theta$

- Orthogonal Matrix: A square matrix A is said to be Orthogonal if $A\,A^T = A^T A = I$

  Ex: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

  - If A is orthogonal, then A^T is also orthogonal.
  - If A, B are orthogonal matrices, then AB is orthogonal.

Elementary Row Operations on a Matrix

There are three elementary row operations on a matrix. They are

- Interchange of any two Rows.
- Multiplication of the elements of any row with a non-zero scalar (or constant)
- Multiplication of elements of a row with a scalar and added to the corresponding elements of other row.

Note: If these operations are applied to columns of a matrix, then it is referred as elementary column operation on a matrix.

- Elementary Matrix: A matrix which is obtained by the application of any one of the elementary operation on Identity matrix (or) Unit matrix is called as Elementary Matrix

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a Elementary matrix. $(\because\ R_1 \leftrightarrow R_2)$

- To perform any elementary Row operations on a matrix A, pre multiply A with corresponding elementary matrix.
- To perform any elementary column operation on a matrix A, post multiply A with corresponding elementary matrix.

Determinant of a Matrix

Determinant of a Matrix: For every square matrix, we associate a scalar called determinant of the matrix.

- If A is any matrix, then the determinant of a Matrix is denoted by $|A|$
- The determinant of a matrix is a function, where the domain set is the set of all square matrices and Image set is the set of scalars.

- Determinant of 1×1 matrix: If $A = [a]_{1\times 1}$ matrix then $|A| = a$
- Determinant of 2×2 matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$
- Determinant of 3×3 matrix: If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

- Minor of an Element: Let $A = (a_{ij})_{n\times n}$ be a matrix, then minor of an element a_{ij} is denoted by M_{ij} and is defined as the determinant of the sub-matrix obtained by Omitting i^{th} row and j^{th} column of the matrix.
- Cofactor of an element: Let $A = (a_{ij})_{n\times n}$ be a matrix, then cofactor of an element a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j}\, M_{ij}$
- Cofactor Matrix: If we find the cofactor of an element for every element in the matrix, then the resultant matrix is called as Cofactor Matrix.
- Determinant of a $n \times n$ matrix: Let $A = (a_{ij})_{n\times n}$ be a matrix, then the determinant of the matrix is defined as the sum of the product of elements of i^{th} row (or) j^{th} column with corresponding cofactors and is given by

  $|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ \dots + \ a_{in}A_{in}$ (For i^{th} row)

  - $|A| = |A^T|$
  - If any two rows (or) columns are interchanged, then the determinant of resulting matrix is $-|A|$.
  - If any row (or) column is zero then $|A| = 0$.
  - If any row (or) column is a scalar multiple of other row (or) column, then $|A| = 0$.
  - If any two rows (or) columns are identical then $|A| = 0$.
  - If any row (or) column of A is multiplied with a non-zero scalar λ, then determinant if resulting matrix is $\lambda\,|A|$.
  - If $A_{n\times n}$ is multiplied with a non-zero scalar λ, then determinant of the resulting matrix is given by $\lambda^n\,|A|$.
  - Determinant of the diagonal matrix is product of diagonal elements.
  - Determinant of the Triangular matrix (Upper or Lower) = product of the diagonal elements.
  - $|AB| = |A||B|$
  - If any row (or) column is the sum of two elements type, then determinant of a matrix is equal to the sum of the determinants of matrices obtained by separating the row (or) column.

  Ex: $\begin{vmatrix} a & b & c+d \\ p & q & r+s \\ w & x & y+z \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ w & x & y \end{vmatrix} + \begin{vmatrix} a & b & d \\ p & q & r \\ w & x & z \end{vmatrix}$
- Adjoint Matrix: Suppose A is a square matrix of $n \times n$ order, then adjoint of A is denoted by $adjA$ and is defined as the Transpose of the cofactor matrix of A.

  Ex: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

  - $A(adjA) = (adjA)A = |A|I$

i.e. Every square matrix A and its adjoint matrix are commutative w.r.t multiplication.

- $|adjA| = |A|^{n-1}, |A| \neq 0$
- $|A\ adjA| = |A|^n$
- $adj(AB) = (adjB)(adjA)$
- If A is a 3×3 scalar matrix with scalar k, then $adj(A) = k^2 I$.

- Singular Matrix: A square matrix A is said to be singular if $|A| = 0$.
- Non-singular Matrix: A square matrix A is said to be non-singular if $|A| \neq 0$.
- Inverse of a Matrix: A square matrix $A_{n\times n}$ is said to be Invertible if there exist a matrix $B_{n\times n}$ such that $AB = BA = I$, where B is called Inverse of A.
  - Necessary and sufficient condition for a square matrix A to be Invertible is that $|A| \neq 0$.
  - If A, B are two invertible matrices, then AB is also Invertible.
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - If $k \neq 0$ is a scalar, A is an Invertible matrix, then $(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$
  - Addition of two Invertible matrices need not be Invertible.
  - If A, B are two non-zero matrices such that $AB = 0$, then A, B are singular.
  - If A is orthogonal matrix, then Inverse of A is A^T. $(\because AA^T = A^TA = I)$
  - If A is Unitary matrix, then A^θ is Inverse of A. $\left(\because AA^\theta = A^\theta A = I\right)$
  - $(A^T)^{-1} = (A^{-1})^T$
  - $\left(A^\theta\right)^{-1} = (A^{-1})^\theta$
  - Inverse of an Identity matrix is Identity Itself.
  - If A is a non-singular matrix, then $A^{-1} = \frac{adj\ A}{|A|}$
  - If A is a non-singular matrix, then $AB = AC \Longrightarrow B = C$
  - If $AB = I$ then $BA = I$

Procedure to find Inverse of a Matrix

In order to find the determinant of a 3×3 matrix, we have to follow the procedure given below.

Let us consider the given matrix to be A

Step 1:Find determinant of A i.e. if $|A| \neq 0$ then only Inverse exists. Otherwise not (I.e. $|A| = 0$)

Step 2: Find Minor of each element in the matrix A.

Step 3: Find the Co-factor matrix.

Step 4: Transpose of the co-factor matrix, which is known as $adj\ A$

Step 5: Inverse of A: $A^{-1} = \frac{adj\ A}{|A|}$

Calculation of Inverse using Row operations

Procedure: If A is a $n \times n$ square matrix such that $|A| \neq 0$, then calculation of Inverse using Row operation is as follows:

- Consider a matrix $[A \,/\, I]$ and now convert A to I using row operations. Finally we get a matrix of the form $[I \,/\, B]$, where B is called as Inverse of A.

Row reduced Echelon Form of matrix

Suppose A is a $n \times n$ matrix, then it is said to be in row reduced to echelon form, if it satisfies the following conditions.

- The number of zeros before the first non-zero element of any row is greater than the number of zeros before the first non-zero element of preceding (next) row.
- All the zero rows, if any, are represented after the non-zero rows.
  - Zero matrix and Identity matrix are always in Echelon form.
  - Row reduced echelon form is similar to the upper triangular matrix.
  - In echelon form, the number of non-zero rows represents the Independent rows of a matrix.
  - The number of non-zero rows in an echelon form represents Rank of the matrix.

Theorem

Prove that the Inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof: Let us consider A to be the square matrix.

Now, given that A is Orthogonal $\Rightarrow A\,A^T = A^T A = I$

Now, we have to prove " Inverse of an orthogonal matrix is orthogonal "

For that, consider $A\,A^T = I$

$\Rightarrow (A\,A^T)^{-'} = I^{-'}$

$\Rightarrow A^{-'}\,(A^T)^{-'} = I$

$\Rightarrow A^{-'}\,(A^{-'})^T = I$

$\Rightarrow A^{-'}$ is Orthogonal

FOR CONFIRMATION

If A is Orthogonal $\Rightarrow A\,A^T = A^T A = I$

If $A^{-'}$ is Orthogonal $\Rightarrow A^{-'}\,(A^{-'})^T = (A^{-'})^T A^{-'} = I$

Now, **let us prove transpose of an orthogonal matrix is orthogonal**

Given that A is Orthogonal $\Rightarrow A\,A^T = A^T A = I$

Consider $A\,A^T = I$

Now, $(A\,A^T)^T = I^T$

$\Rightarrow (A^T)^T A^T = I$

$\Rightarrow A^T$ is orthogonal.

FOR CONFIRMATION

If A is Orthogonal $\Rightarrow A\,A^T = A^T A = I$

If A^T is Orthogonal $\Rightarrow A^T\,(A^T)^T = (A^T)^T A^T = I$

Rank of the Matrix

If A is a non-zero matrix, then A is said to be the matrix of rank r, if

i. A has atleast one non-zero minor of order r, and

ii. Every $(r+1)^{\text{th}}$ order minor of A vanishes.

The order of the largest non-zero minor of a matrix A is called Rank of the matrix.

It is denoted by $\rho(A)$.

- When $A = 0$, then $\rho(A) = 0$.
- Rank of $I = n$, n is order of the matrix.
- If $|A| \neq 0$ for $A_{n\times n}$ matrix, then $\rho(A) = n$.
- For $A_{n\times n}$ matrix, $\rho(A) \leq n$.
- If $\rho(A) = r$, then the determinant of a sub-matrix, where order $> r$ is equal to zero.
- The minimum value of a Rank for a non-zero matrix is one.
- $\rho(AB) \leq \rho(A)$ & $\rho(AB) \leq \rho(B)$
- $\rho(A+B) \leq \rho(A) + \rho(B)$
- $\rho(A-B) \geq \rho(A) - \rho(B)$

Problem

Find the rank of the following matrix

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Sol: Let us consider $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_2 - 4R_1$, $R_4 \to R_4 - 4R_1$ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 3R_2$ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Therefore, the number of non-zero rows in the row echelon form of the matrix is 2.
Hence rank of the matrix is **2**.

Problem: Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ **into echelon form and hence find its rank.**

Sol: Let us consider given matrix to be A

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$$\begin{matrix} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 3R_1 \\ R_4 \longrightarrow R_4 - 6R_1 \end{matrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_4 \longrightarrow R_4 - R_2 \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_4 \longrightarrow R_4 - R_3 \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, this is in Echelon form and the number of non-zero rows is 3

Hence, $\rho(A) = 3$

Equallence of two matrices

Suppose A and B are two matrices, then B is said to be row equalent to A, if it is obtained by applying finite number of row operations on A. It is denoted by $B \overset{R}{\simeq} A$.

Similarly, B is said to be column equalent to A, if it is obtained by applying finite number of column operations on A. It is denoted by $B \overset{C}{\simeq} A$.

- For equalent matrices Rank does not Alter (i.e. does not change)
- Equallence of matrices is an Equallence relation
- Here Equallence $\Rightarrow$ following three laws should satisfy
    - Reflexive: $A \simeq A$
    - Symmetric: $A \simeq B \ \Rightarrow B \simeq A$
    - Transitive: $A \simeq B, B \simeq C \Rightarrow A \simeq C$

Normal Form of a Matrix

Suppose A is any matrix, then we can convert A into any one of the following forms

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ (Or) } \begin{bmatrix} I_r \\ 0 \end{bmatrix} \text{ (Or) } \begin{bmatrix} I_r & 0 \end{bmatrix}$$

These forms are called as Normal forms of the matrix A. (Or canonical forms)

Procedure to find Normal form of the matrix A.

Aim: Define two non-singular matrices P & Q such that PAQ is in Normal Form.

Here, I_m is pre-factor
I_n is post factor

Step 1: Let us consider A is the given matrix of order $m \times n$.

Step 2: Rewrite A as $A = I_m A\, I_n$

Step 3: Reduce the matrix A (L.H.S) in to canonical form using elementary operations provided every row operation which is applied on A (L.H.S), should be performed on pre-factor I_m(R.H.S). And every column operation which is applied on A (L.H.S), should be performed on post-factor I_n (R.H.S).

Step 4: Continue this process until the matrix A at L.H.S takes the normal form.

Step 5: Finally, we get $I_r = PAQ$, r is rank of the matrix A.

- The order of Identity sub-matrix of the Normal form of A represents Rank of the matrix of A.
- Every matrix can be converted into Normal form using finite number of row and column operations.
- If we convert the matrix A in to Normal form then $\exists$ two non-singular matrices P and Q such that $PAQ = Normal\ Form$, where P and Q are the product of elementary matrices.
- Every Elementary matrix is a non-singular matrix.

SYSTEM OF LINEAR EQUATIONS

The system of Linear equations is of two types.

- Non-Homogeneous System of Linear Equations
- Homogeneous System of Linear Equations.

Non-Homogeneous System of Linear Equations

The system of equations which are in the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

then, the above system of equations is known as Non-Homogeneous system of Linear equations and it is represented in the matrix form as follows:

The above system of equation can be represented in the form of $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Solution of $AX = B$

The set of values $\{x_1, x_2, \dots, x_n\}$ is said to be a solution to $AX = B$ if it satisfies all the equations.

Consistent system of equations

A system of equations $AX = B$ is said to be consistent if it has a solution.
Otherwise, it is called as Inconsistent (i.e. no solution).

Augmented Matrix

The matrix $[A / B]$ is called as an Augmented matrix.

Necessary and Sufficient condition for $AX = B$ to be consistent is that $\rho[A/B] = \rho[A]$.

- If $\rho[A/B] = \rho[A] = r = n$ (number of variables (or) unknowns), then $AX = B$ has unique solution.
- If $m = n$ (i.e. Number of equations = Number of unknowns) and $|A| \neq 0$, then $AX = B$ has Uniquely solution
- If $\rho[A/B] = \rho[A] = r < n$ (unknowns) and $|A| \neq 0$, then $AX = B$ has Infinitely many Solutions.
- If $m = n$ (i.e. Number of equations = Number of unknowns) and $|A| = 0$, then $AX = B$ has Infinitely many solutions.
- If $m > n$ (i.e. Number of equations > Number of unknowns), then $AX = B$ has Infinitely many solutions if $\rho[A/B] = \rho[A] = r < n$.

Procedure for solving $AX = B$

Let $AX = B$ is a non-homogeneous system of Linear equations, then the solution is obtained as follows:

Step 1: Construct an Augmented matrix $[A/B]$.

Step 2: Convert $[A/B]$ into row reduced echelon form

Step 3: If $\rho[A/B] = \rho[A]$, then the system is consistent. Otherwise inconsistent.

Step 4: If $AX = B$ is consistent, then solution is obtained from the echelon form of $\rho[A/B]$.

Note: If $\rho[A/B] = \rho[A] = r$, then there will be $(n - r)$ variables which are Linearly Independent and remaining r variables are dependent on $(n - r)$ variables

Homogeneous system of Equations

The system of equations $AX = B$ is said to be homogeneous system of equations if $B = 0$ i.e. $AX = 0$.

To obtain solution of homogeneous system of equations the procedure is as follows:

Step 1: Convert $[A]$ into row reduced echelon form

Step 2: Depending on nature of $[A]$, we will solve further.

- $AX = 0$ is always consistent.
- $AX = 0$ has a Trivial solution always (i.e. Zero solution)
- If $\rho[A] = r = n$, (number of variables), then $AX = 0$ has Unique solution.(Trivial solution)
- If $m = n$ & $|A| \neq 0$ then $AX = 0$ has only Trivial solution i.e. Zero Solution
- If $\rho[A] = r < n$ (number of variables (or) unknowns), then $AX = 0$ has infinitely many solutions.
- If $m < n$, then $AX = 0$ has Infinitely many solutions.

Matrix Inversion Method

Suppose $AX = B$ is a non-homogeneous System of equations, such that $m = n$ and $|A| \neq 0$, then

$AX = B$ has unique solution and is given by $X = A^{-1}B$

Cramer's Rule

Suppose $AX = B$ is a non-homogeneous System of equations, such that $m = n$ and $|A| \neq 0$, then the solution of $AX = B$ is obtained as follows:

Step 1: Find determinant of A i.e. $|A| = \Delta$ (say)

Step 2: Now, $x_1 = \frac{\Delta_1}{\Delta}$, where Δ_1 is the determinant of A by replacing 1^{st} column of A with B.

Step 3: Now, $x_2 = \frac{\Delta_2}{\Delta}$, where Δ_1 is the determinant of A by replacing 2^{nd} column of A with B.

Step 4: Now, $x_3 = \frac{\Delta_3}{\Delta}$, where Δ_1 is the determinant of A by replacing 3rd column of A with B.

$\vdots \qquad\qquad\qquad \vdots$

Finally $x_i = \frac{\Delta_i}{\Delta}$, where Δ_i is the determinant of A by replacing ith column of A with B.

Gauss Elimination Method

Let us consider a system of 3 linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The augmented matrix of the corresponding matrix A is given by $[A|B]$

$$\text{i.e.} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Now, our aim is to convert augmented matrix to upper triangular matrix. (i.e. Elements below diagonal are zero).

In order to eliminate a_{21}, multiply with $-\frac{a_{21}}{a_{11}}$ to R_1 and add it to R_2

$$\text{i.e.} \left(-\frac{a_{21}}{a_{11}}\right) R_1 + R_2 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

Again, In order to eliminate a_{31}, multiply with $-\frac{a_{31}}{a_{11}}$ to R_1 and add it to R_3

$$\text{i.e.} \left(-\frac{a_{31}}{a_{11}}\right) R_1 + R_3 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a''_{32} & a''_{33} & b''_3 \end{array}\right]$$

This total elimination process is called as 1st stage of Gauss elimination method.

In the 2nd stage, we have to eliminate a''_{32}. For this multiply with $-\frac{a''_{31}}{a'_{22}}$ to R_2 and add it to R_3

$$\text{i.e.} \left(-\frac{a''_{32}}{a'_{22}}\right) R_2 + R_3 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a'''_{33} & b'''_3 \end{array}\right]$$

Now, above matrix can be written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$

$$a'''_{33}x_3 = b'''_3$$

From these 3 equations, we can find the value of x_3, x_2 and x_1 using backward substitution process.

Gauss Jordan Method

Let us consider a system of 3 linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The augmented matrix of the corresponding matrix A is given by $[A|B]$

$$\text{i.e.} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Now, our aim is to convert augmented matrix to upper triangular matrix.
In order to eliminate a_{21} , multiply with $-\frac{a_{21}}{a_{11}}$ to R_1 and add it to R_2

$$\text{i.e.} \left(-\frac{a_{21}}{a_{11}}\right)R_1 + R_2 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

Again, In order to eliminate a_{31} , multiply with $-\frac{a_{31}}{a_{11}}$ to R_1 and add it to R_3

$$\text{i.e.} \left(-\frac{a_{31}}{a_{11}}\right)R_1 + R_3 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a''_{32} & a''_{33} & b''_3 \end{array}\right]$$

This total elimination process is called as 1st stage of Gauss elimination method.
In the 2nd stage, we have to eliminate a''_{32}. For this, multiply with $-\frac{a''_{31}}{a'_{22}}$ to R_2 and add it to R_3

$$\text{i.e.} \left(-\frac{a''_{32}}{a'_{22}}\right)R_2 + R_3 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a'''_{33} & b'''_3 \end{array}\right]$$

In the 3rd stage, we have to eliminate a_{12}. For this, multiply with $-\frac{a_{12}}{a'_{22}}$ to R_2 and add it to R_1

$$\text{i.e.} \left(-\frac{a_{12}}{a'_{22}}\right)R_2 + R_1 \Rightarrow \left[\begin{array}{ccc|c} a_{11} & 0 & a''''_{13} & b''''_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a'''_{33} & b'''_3 \end{array}\right]$$

Now, above matrix can be written as

$$a_{11}x_1 \qquad + a''''_{13}x_3 = b''''_1$$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a'''_{33}x_3 = b_3'''$$

From these 3 equations, we can find the value of x_3, x_2 and x_1 using backward substitution process.

LU Decomposition (or) Factorization Method (or) Triangularization Method

This method is applicable only when the matrix A is positive definite (i.e. Eigen values are +ve)

Let us consider a system of 3 linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above system of equations can be represented in matrix as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This is in the form of $AX = B$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If A is positive definite matrix, then we can write $A = LU$, where

$$L = Lower\ triangular\ matrix = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \Rightarrow LY = B$$

$$U = Upper\ triangular\ matrix = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Here, Positive definite $\Longrightarrow$ Principle minors are non-zeros

Again, here Principle minors $\Longrightarrow$ Left most minors are called as Principle minors

i.e. $[a_{11}], \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ etc.

Now, $AX = B \Longrightarrow LUX = B \longrightarrow ①$

Let $UX = Y \longrightarrow ②$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$① \Longrightarrow LY = B \longrightarrow ③$

Using Forward substitutions, we get Y from equation ③.

Now, from 2 , R.H.S term Y is known.

Using Backward Substitution get X from ② which gives the required solution.

Solution of Tridiagonal System (Thomas Algorithm)

Let us consider a system of equations of the form $AX = B$, where

$$A = \begin{bmatrix} b_1 & c_1 & 0 & & 0 & 0 \\ a_1 & b_2 & c_2 & \cdots & 0 & 0 \\ 0 & a_2 & b_3 & & \ddots & 0 \\ & \vdots & & \ddots & & \\ 0 & 0 & \ddots & \cdots & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & & a_{n-1} & b_n \end{bmatrix}$$

Step 1: Take $\alpha_1 = b_1$

Calculate $\alpha_i = b_i - \frac{a_i c_{i-1}}{\alpha_{i-1}}, i = 2, 3, 4, \ldots$

Step 2: Take $\beta_1 = \frac{d_1}{b_1}$

Calculate $\beta_i = \frac{d_i - a_i \beta_{i-1}}{\alpha_i}, i = 2, 3, 4, \ldots$

Step 3: Take $x_n = \beta_n$ and

$$x_i = \beta_i - \frac{c_i x_{i+1}}{\alpha_i}, i = n-1, n-2, \ldots, 1$$

For Confirmation:

Let $A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_1 & b_2 & c_2 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{n-1} & c_n \end{bmatrix}$

Now, if we want to make a_1 as zero, then $R_2 \longrightarrow R_2 - \frac{a_2}{b_1} R_1$. Similarly, we get all other values.

* * *