MATHEMATICS-I

APPLICATIONS OF INTEGRATION

I YEAR B.Tech

By

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UNIT-4
APPLICATIONS OF INTEGRATION

Riemann Integrals:

Let us consider an interval $[a, b]$ with $a < b$

If $= x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$, then a finite set $\{x_0, x_1, x_2, \dots , x_{n-1}, x_n\}$ is called as a partition of $[a, b]$ and it is denoted by P.

The sub intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots , [x_{n-1}, x_n]$ are called $n-$ segments (or) $n-$ sub intervals.

The r^{th} sub interval in this process is $I_r = [x_{r-1}, x_r]$ and its length is given by $\delta_r = x_r - x_{r-1}$

Note: For every interval $[a, b]$, it is possible to define infinitely many partitions.

Norm (or) Mesh of the partition: The maximum of the lengths of the sub intervals w.r.t the partition P is called as Norm of the partition P (or) Mesh of the partition P and it is denoted by $\|P\|$ or μ_P

Refinement: If P and P are two partitions of $[a, b]$ and if $\subset P$, then P is called as Refinement of P.

Lower and Upper Reimann Sum's

Let $f: [a, b] \longrightarrow \mathbb{R}$ is bounded and $P = \{x_0, x_1, x_2, \dots , x_{n-1}, x_n\}$ be a partition on $[a, b]$, then r^{th} sub interval is given by $I_r = [x_{r-1}, x_r]$ and its length is given by $\delta_r = x_r - x_{r-1}$

If f is bounded on $[a, b]$, then f is bounded on I_r

let m_r and M_r be Infimum and supremum of f on I_r , then

- The sum $\sum_{r=1}^{n} m_r\, \delta_r$ is called as lower Reimann sum and it is denoted by $L(P, f)$
- The sum $\sum_{r=1}^{n} M_r\, \delta_r$ is called as upper Reimann sum and it is denoted by $U(P, f)$

Note: Always, $U(P, f) \geq L(P, f)$

Problem

1) If $f(x) = x \ \forall\ x \in [0, 1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0, 1]$ then compute $U(P, f), L(P, f)$

Sol: Given $f(x) = x$ defined on $[0,1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0,1]$

Here, let $I_1 = \left[0, \frac{1}{3}\right]$, $I_2 = \left[\frac{1}{3}, \frac{2}{3}\right]$, $I_3 = \left[\frac{2}{3}, 1\right]$

And, $\delta_1 = \frac{1}{3} - 0 = \frac{1}{3}$, $\delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$, $\delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$

Let m_r and M_r be Infimum and supremum of f on I_r , then

$$m_1 = 0 , M_1 = \frac{1}{3}$$

$$m_2 = \frac{1}{3} , M_2 = \frac{2}{3}$$

$$m_3 = \frac{2}{3} , M_2 = 1$$

Hence, $U(P, f) = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 = \frac{2}{3}$

Also, $L(P, f) = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 = \frac{1}{3}$

Lower Reimann Integral: Let $f: [a, b] \longrightarrow \mathbb{R}$ be a bounded function and P is a partition of $[a, b]$, then supremum of $\{L(P, f) | P \in \phi[a, b]\}$ is called as Lower Reimann integral on $[a, b]$ and it is denoted by $\underline{\int_a^b} f(x)\, dx$

Upper Reimann Integral: Let $f: [a, b] \longrightarrow \mathbb{R}$ be a bounded function and P is a partition of $[a, b]$, then Infimum of $\{U(P, f) | P \in \phi[a, b]\}$ is called as Upper Reimann integral on $[a, b]$ and it is denoted by $\overline{\int_a^b} f(x)\, dx$

Riemann Integral

If $f: [a, b] \longrightarrow \mathbb{R}$ be a bounded function and P is a partition of $[a, b]$ and if $\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx$, then f is said to be Riemann integrable on $[a, b]$ and it is denoted by $\int_a^b f(x)\ dx$

Rectification: The process of finding the length of the arc of the curve is called as Rectification

Length of the arc of the curve

Equation of the curve	Arc Length
Cartesian Form	
(i) $y = f(x)$ and $x = a$ and $x = b$	$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
(ii) $x = f(y)$ and $y = a$ and $y = b$	$s = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$
Parametric Form	
$x = x(\theta), y = y(\theta)$ and $\theta = \theta_1$ and $\theta = \theta_2$	$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$
Polar Form	
(i) $r = f(\theta)$ and $\theta = \alpha$ and $\theta = \beta$	$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
(ii) $\theta = f(r)$ and $r = r_1$ and $r = r_2$	$s = \int_{r_1}^{r_2} \sqrt{1 + \left(\frac{d\theta}{dr}\right)^2}\, dr$

Problems on length of the arc of the curve

1) Find the length of the arc of the curve $y = \log\left(\frac{e^x-1}{e^x+1}\right)$ from $x = 1$ to $x = 2$.

Solution: We know that, the equation of the length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad .... \ (1)$$

Given $y = \log\left(\frac{e^x-1}{e^x+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{e^x - 1}{e^x + 1}\right)} \frac{d}{dx}\left(\frac{e^x - 1}{e^x + 1}\right)$$

$$= \frac{2e^x}{e^{2x}-1}$$

∴ The required length of the arc of the curve is given by $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$\Rightarrow S = \int_1^2 \sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1} dx$$

$$= \int_1^2 \frac{e^x(e^x+e^{-x})}{e^x(e^x-e^{-x})} dx$$

$$= \int_1^2 \frac{(e^x+e^{-x})}{(e^x-e^{-x})} dx \quad = [\log(e^x-e^{-x})]_1^2$$

$$= \left[\log\left(\frac{e^{2x}-1}{e^x}\right)\right]_1^2$$

$$= [\log(e^{2x}-1)-\log e^x]_1^2$$

$$= \log(e^4-1)-\log e^2-\log(e^2-1)+\log e^1$$

$$= \log\left(\frac{e^4-1}{e^2-1}\right)-\log\left(\frac{e^2}{e}\right) = \log(e^2+1)-\log e$$

$$= \log\left(\frac{e^2+1}{e}\right) = \log\left(e+\frac{1}{e}\right)$$

2) Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$

Solution: We know that, the equation of the length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad .... \quad (1)$$

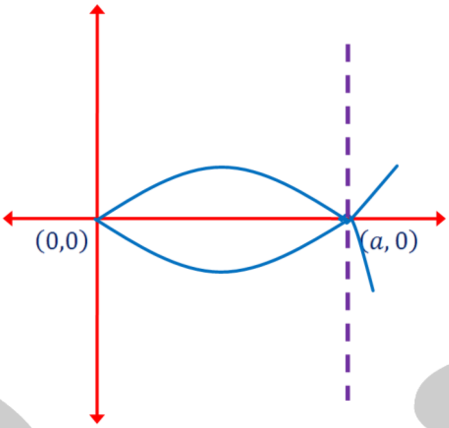
Given $3ay^2 = x(x-a)^2$

$$\Longrightarrow y = \frac{1}{\sqrt{3a}}\sqrt{x}\,(x-a)$$

$$= \frac{1}{\sqrt{3a}}\left(x^{\frac{3}{2}} - ax^{\frac{1}{2}}\right)$$

Now, $\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left(\frac{3}{2}x^{\frac{1}{2}} - a\frac{1}{2}x^{-\frac{1}{2}}\right)$

$$= \frac{1}{\sqrt{3a}}\left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right) = \frac{1}{\sqrt{3a}}\left(\frac{3x-a}{2\sqrt{x}}\right)$$



Here the curve is symmetrical about the $X - axis$. Hence the length of the arc will be double that of the arc of the loop about the $X - axis$.

∴ The required length of the loop is $=2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\int_0^a \sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \left(\frac{3x}{\sqrt{x}} + \frac{a}{\sqrt{x}}\right)dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \left(3\sqrt{x} + \frac{a}{\sqrt{x}}\right)dx$$

$$= \frac{2}{\sqrt{12a}}\left[3.\frac{2}{3}x^{\frac{3}{2}} + a.2\sqrt{x}\right]_0^a$$

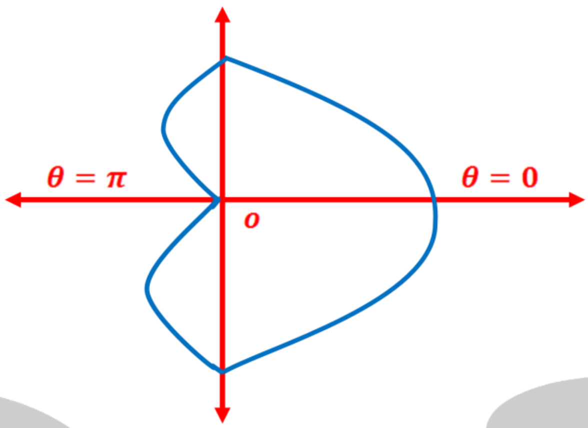
$$= \frac{2}{\sqrt{3a}}\left[x^{\frac{3}{2}} + a\sqrt{x}\right]_0^a$$

$$= \frac{4a}{\sqrt{3}}\text{ units}$$

3) Find the perimeter of the cardioids $r = a(1 + \cos\theta)$.

Solution: We know that the length of the arc of the curve $r = f(\theta)$ and $\theta = \alpha, \theta = \beta$ is given by

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



Given $r = a(1 + \cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta$$

The cardioids is symmetrical about the initial line and passes through the pole.

Hence the length of the arc will be double that of the arc of the loop about the *pole*.

∴ The required length of the loop is $=2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2 sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2a^2(1+\cos\theta)}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2 cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 8a$$

4) Find the perimeter of the curve $x^2 + y^2 = r^2$.

Solution: Given equation of the curve is $x^2 + y^2 = r^2$ ... (1)

The given curve is an equation of circle with radius $'a'$.

We know that the length of the arc of the curve $y = f(x)$ between the absissae $x = a$ and $x = b$ is given by

$$S = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now, let us find the length of the arc AB of the given curve.

Differentiating (1) w.r.t $'x'$, we get

$$\Longrightarrow 2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Now, the length of the curve AB is

$$S = \int_{x=0}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Longrightarrow S = \int_{x=0}^{a} \sqrt{1 + \left(-\frac{x}{y}\right)^2}\, dx$$

$$\Longrightarrow S = \int_{x=0}^{a} \sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= \int_{x=0}^{a} \sqrt{\frac{y^2 + x^2}{y^2}}\, dx$$

$$= \int_{x=0}^{a} \sqrt{\frac{a^2}{y^2}}\, dx \quad = a \int_{x=0}^{a} \frac{1}{y}\, dx$$

$$= a \int_{x=0}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$= a \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$$

$$= \frac{\pi a}{2}$$

$\therefore$ Required Perimeter of the sphere $= 4 \times S = 4 \times \frac{\pi a}{2} = 2\pi a$

Volume of solid of Revolution

Region (R)	Axis	Volume of the solid generated
Cartesian form		
(i) $y = f(x)$, the $x - axis$ and the lines $x = a$ and $x = b$	$x - axis$	$V = \pi \int_a^b y^2\, dx$
(ii) $x = g(y)$, the $y - axis$ and the lines $y = c$ and $y = d$	$y - axis$	$V = \pi \int_c^d x^2\, dy$
(iii) $y = y_1(x), y = y_2(x), the\ x - axis\ and\ the\ ordinates\ x = a, x = b$	$x - axis$	$V = \pi \int_a^b (y_2^2 - y_1^2)\, dx$
(iv) $x = x_1(y), x = x_2(y), the\ y - axis\ and\ the\ ordinates\ y = a, y = b$	$y - axis$	$V = \pi \int_a^b (x_2^2 - x_1^2)\, dy$
Parametric form		
(i) $x = \phi(t), y = \psi(t),$ $the\ ordinates\ t = t_1, t = t_2$	$x - axis$	$V = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt}\, dt$
(ii) $x = \phi(t), y = \psi(t),$ $the\ abscissae\ t = t_1, t = t_2$	$y - axis$	$V = \pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt}\, dt$
Polar form		
(i) $r = f(\theta), the\ initial\ line\ \theta = 0\ and\ the\ radii\ vectors\ \theta = \alpha, \theta = \beta$	The initial line $\theta = 0$	$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin\theta\, d\theta$
(ii) $r = f(\theta), the\ line\ \theta = \frac{\pi}{2}\ \perp^r\ to\ \ the\ initial\ line\ and\ the\ ra$ $\alpha, \theta = \beta$	The line $\theta = \frac{\pi}{2}$	$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \cos\theta\, d\theta$
(iii) $r = f(\theta), the\ initial\ line\ \theta = r\ and\ the\ radii\ vectors\ \theta = \alpha, \theta = \beta$	The line $\theta = r$	$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin(\theta - r)\, d\theta$

Problems on Volume of solid of Revolution

1) Find the volume of the solid that result when the region enclosed by the curve $y = x^3$, $y = 0$, $x = 1$ is revolved about the $y - axis$.

Sol: We know that the volume of the solid generated by the revolution of the area bounded by the curve $x = f(y)$, the $y - axis$ and the lines $y = a, y = b$ is given by $V = \pi \int_a^b x^2\, dy$

Now, given curve $y = x^3$

∴ Required volume is given by $V = \pi \int_0^1 x^{\frac{2}{3}}\, dy$

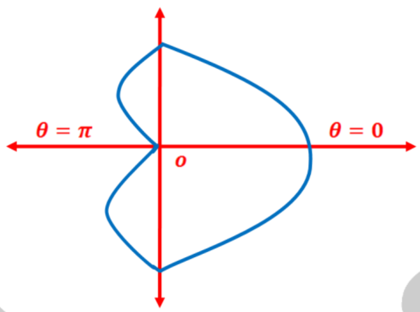
$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^1 = \frac{3\pi}{5}$$

2) Find the volume of the solid generated by the revolution of the cardioids $r = a(1+\cos\theta)$ about the initial line.

Sol: We know that the volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$, the initial line and $\theta = \alpha, \theta = \beta$ is given by

$$V = \frac{2}{3}\pi\int_0^{\pi} r^3 \sin\theta \; d\theta \quad \ldots(1)$$

Here, the given cardioids is symmetrical about the initial line. The upper half of the curve formed when θ varies from 0 to π.



$$V = \frac{2}{3}\pi\int_0^{\pi} r^3 \sin\theta \; d\theta$$

$$= \frac{2}{3}\pi\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta \; d\theta$$

$$= -\frac{2}{3}\pi\int_0^{\pi} a^3(1+\cos\theta)^3 \,(-\sin\theta)\, d\theta$$

$$= -\frac{2}{3}\pi a^3\left[\frac{(1+cos\,\theta)^4}{4}\right]_0^{\pi}$$

$$= \frac{8\pi a^3}{3} \text{ cu. units}$$

Surface area of solid of Revolution

Equation of the curve	Arc Length
Cartesian Form	
(i) $y = f(x)$ and $x = a$ and $x = b$, $X - axis$	$s = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$
(ii) $x = f(y)$ and $y = a$ and $y = b$, $Y - axis$	$s = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$
Parametric Form	
(i) $x = x(\theta), y = y(\theta)$ and $\theta = \theta_1$ and $\theta = \theta_2$ $X - axis$	$s = 2\pi \int_{\theta_1}^{\theta_2} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$
(ii) $x = x(\theta), y = y(\theta)$ and $\theta = \theta_1$ and $\theta = \theta_2$ $Y - axis$	$s = 2\pi \int_{\theta_1}^{\theta_2} x \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$
Polar Form	
(i) $r = f(\theta)$ and $\theta = \alpha$ and $\theta = \beta$	$s = 2\pi \int_{\alpha}^{\beta} r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$
(ii) $\theta = f(r)$ and $r = r_1$ and $r = r_2$	$s = 2\pi \int_{\alpha}^{\beta} r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

Problem

1) Find the surface area generated by the revolution of an arc of the catenary $y = c\cosh\frac{x}{c}$ about the $x - axis$.

Sol: We know that the surface area of the solid generated by the revolution of an arc $y = f(x)$ about the $-axis$, $x = a, x = b$ is given by

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi \int_0^c c\cosh\frac{x}{c} \sqrt{1 + \sinh^2\frac{x}{c}}\ dx$$

$$= 2\pi c \int_0^c \cosh^2 \frac{x}{c}\, dx$$

$$= 2\pi c \int_0^c \frac{1 + \cosh \frac{2x}{c}}{2}\, dx$$

$$= \pi c \int_0^c \left(1 + \cosh \frac{2x}{c}\right) dx$$

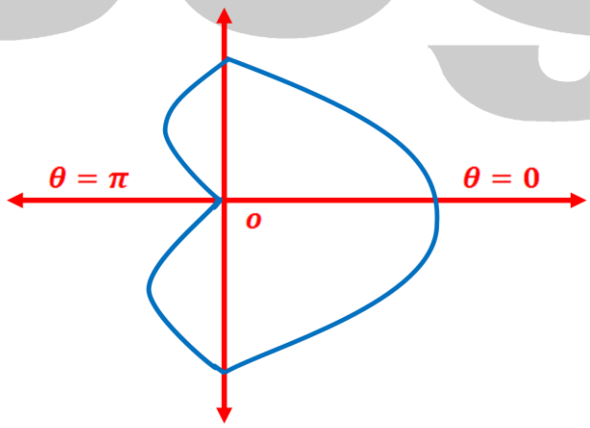
$$= \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]_0^c$$

$$= \pi c \left[c + \frac{c}{2} \sinh 2\right] = \pi c^2 \left(1 + \frac{1}{2} \sinh 2\right)$$

2) Find the surface area of the solid formed by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.

Sol: We know that the surface area of the solid formed by revolving the cardioid $r = f(\theta)$, the initial line and $\theta = \alpha, \theta = \beta$ is given by

$$S = 2\pi \int_\alpha^\beta r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



Given $r = a(1 + \cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = -a \sin\theta$$

The cardioid is symmetrical about the initial line and passes through the pole.

Hence, required surface area is given by

$$= 2\pi \int_0^\pi r \sin\theta \sqrt{a^2(1 + \cos\theta)^2 + a^2 sin^2\theta}\, d\theta$$

$$= 2\pi \int_{0}^{\pi} a(1+\cos\theta)\sin\theta \sqrt{2a^2(1+\cos\theta)}\ d\theta$$

$$= 2\sqrt{2}\pi a^2 \int_{0}^{\pi} (1+\cos\theta)^{\frac{3}{2}}\ \sin\theta\ d\theta$$

Let $(1+\cos\theta) = t$

$\Longrightarrow -\sin\theta\ d\theta = dt$

Lower Limit: $\theta = 0 \Longrightarrow t = 2$

Upper Limit: $\theta = \pi \Longrightarrow t = 0$

$\therefore$ Surface area $= 2\sqrt{2}\pi a^2 \int_{t=2}^{0} t^{\frac{3}{2}}\ (-dt)$

$$= 2\sqrt{2}\pi a^2 \int_{0}^{2} t^{\frac{3}{2}}\ dt$$

$$= \frac{32}{5}\pi a^2$$

Change of variables in Double Integral

Problem: **Evaluate the following integral by transforming into polar coordinates.**

$$I = \int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\ dx\,dy$$

Solution: Clearly, given coordinates are in Cartesian.

Now, let us consider given integral to be $I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\ dx\,dy$

$$\Longrightarrow I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\ dy\,dx$$

$\Longrightarrow y = 0$ to $y = \sqrt{a^2-x^2}$ and $x = 0$ to $x = a$

$\Longrightarrow y = 0$ to $x^2+y^2 = a^2$

The given region is a circle with centre $O(0,0)$ and with radius a

Now, in order to change to polar coordinates, let us substitute

$x = r\cos\theta$
$y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$The\ limits\ for\ r: 0\ to\ a\ and\ for\ \theta: 0\ to\ \frac{\pi}{2}$

$$\therefore\ I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} (r\, sin\theta)\,.r.(r\, dr\, d\theta)$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^3\ sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left\{ \int_{r=0}^{a} r^3\, dr \right\} sin\theta\ d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{r=0}^{a} sin\theta\ d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\frac{\pi}{2}} sin\theta\ d\theta$$

$$= \frac{a^4}{4}\ [-cos\theta]_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{a^4}{4}$$

$$\therefore\ I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\ dx\, dy = \frac{a^4}{4}$$

2) By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy$ over the annular region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(b > a)$.

Solution: In order to change coordinates Cartesian to Polar, let us substitute

$$x = r\, cos\theta \qquad dx\, dy = r\, dr\, d\theta$$
$$y = r\, sin\theta$$

The limits are $r \longrightarrow a$ to b and $\theta \longrightarrow 0$ to 2π

$\therefore$ Let $I = \int\int \frac{x^2y^2}{x^2+y^2}\, dx\, dy$

$$\Rightarrow I = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{(r\cos\theta)^2 (r\, sin\theta)^2}{(r\cos\theta)^2 + (r\, sin\theta)^2}\ r\, dr\, d\theta$$

$$\Rightarrow I = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{r^4 \cos^2\theta \ \sin^2\theta}{r^2} \ r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r^3 \cos^2\theta \ \sin^2\theta \ dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\int_{r=a}^{b} r^3 dr \right] \cos^2\theta \ \sin^2\theta \ d\theta$$

$$= \frac{b^4 - a^4}{4} \int_{0}^{2\pi} \cos^2\theta \ \sin^2\theta \ d\theta$$

$$= \frac{b^4 - a^4}{4.4} \int_{0}^{2\pi} (2\sin\theta\cos\theta)^2 \, d\theta$$

$$= \frac{b^4 - a^4}{16} \int_{0}^{2\pi} (\sin 2\theta)^2 \, d\theta$$

$$= \frac{b^4 - a^4}{16} \int_{0}^{2\pi} \left(\frac{1 - \cos 4\theta}{2} \right) d\theta$$

$$\Rightarrow I = \frac{b^4 - a^4}{32} \int_{0}^{2\pi} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{b^4 - a^4}{32} \left[\theta - \frac{\sin 4\theta}{4} \right]_{\theta=0}^{2\pi}$$

$$= \frac{\pi}{32}(b^4 - a^4)$$

$$\therefore \ I = \int\int \frac{x^2 y^2}{x^2 + y^2} \, dx\,dy = \frac{\pi}{32}(b^4 - a^4)$$

Change of Order of Integration

Problem 1: Change the order of integration and evaluate

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\ dx$$

Solution: Let us consider

$$I = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\ dx$$

Hence, given limits are $x = 0$ to $x = 4a$ and $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$

Now, we have to convert y limits interms of

$constants\ and\ x\ limits\ in\ terms\ of\ y.$

Given $\int_{x \to const} \int_{y \to (x)} dy\, dx$. To find $\int_{y \to const} \int_{x \to (y)} dx\, dy$

$x = 0\ to\ x = 4a\ \ and\ y = \frac{x^2}{4a}\ to\ y = 2\sqrt{ax}$

$\Longrightarrow\ x^2 = 4ay$ to $y^2 = 4ax$

$\Longrightarrow x = 2\sqrt{ay}$ to $x = \frac{y^2}{4a}$

Also, if $x = 0 \Longrightarrow y = 0$

if $x = 4a \Longrightarrow y = 4a$

$$\therefore I = \int_{y=0}^{4a} \int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dy\ dx$$

$$= \int_{y=0}^{4a} \left[\int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dx \right] dy$$

$$= \int_{y=0}^{4a} [x]_{x=2\sqrt{ay}}^{y^2/_{4a}}\ dy = \frac{16a^2}{3}$$

2) Change the order of integration and evaluate

$$\int_{0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\;dy$$

Solution: Let us consider

$$I=\int_{x=0}^{a}\int_{y=\frac{x}{a}}^{\sqrt{x/a}} (x^2+y^2)\,dy\;dx$$

Hence, given limits are $x=0$ to $x=a$ and $y=\frac{x}{a}$ to $y=\sqrt{x/a}$

$\Longrightarrow\; x=ay$ to $y^2=\frac{x}{a}$

$\Longrightarrow x=ay$ to $x=ay^2$

Also, if $x=0 \Longrightarrow y=0$

if $x=a \Longrightarrow y=1$

$$\therefore I=\int_{y=0}^{1}\int_{x=ay^2}^{ay}(x^2+y^2)\,dy\;dx$$

$$=\int_{y=0}^{4a}\left[\int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dx\right] dy$$

$$=\frac{a^3}{28}+\frac{a}{20}$$

3) Change the order of integration and evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dx\;dy$ and hence evaluate the integral.

Solution: Let us consider

$$I=\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\;dx$$

Hence, given limits are $x=0$ to $x=1$ and $y=x^2$ to $y=2-x$

$\Longrightarrow\; x=ay$ to $y^2=\frac{x}{a}$

$\Longrightarrow x=ay$ to $x=ay^2$

Also, if $x = 0 \Rightarrow y = 0$

if $x = a \Rightarrow y = 1$

$$\therefore I = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2 + y^2) dy\ dx = \int_{y=0}^{4a} \left[\int_{x=2\sqrt{ay}}^{\frac{y^2}{4a}} dx \right] O(0,1)$$

$$\iint_R xy\,dx\ dy = \iint_{R_1} xy\,dx\ dy + \iint_{R_2} xy\,dx\,dy$$

Now, $x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$

$\Rightarrow x = -2, x = 1$

$$\therefore I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\ dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\ dy$$

$$= \frac{3}{8}$$

4) Change the order of integration and evaluate

$$\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$$

Sol: Let us consider $I = \int_{y=0}^{b} \int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$

Given limits are $x = 0, x = \frac{a}{b}\sqrt{b^2 - y^2}$ and $y = 0, y = b$

$\Rightarrow x = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Now, by changing the order of integration, we get

Here, x limits are in terms of y, y limits are constants $\Rightarrow$ y limits should be in terms of x, x limits should be constants

$Now, y = 0, y = \frac{b}{a}\sqrt{a^2 - x^2}$ and $x = 0, x = a$

$$\text{Now, } I = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dx\,dy = \int_{x=0}^{a} \left[\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy \right] dx$$

$$= \int_{x=0}^{a} x \left[\frac{y^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x \frac{b^2}{a^2}(a^2 - x^2)dx$$

$$= \frac{b^2}{2a^2}\int_{x=0}^{a} (xa^2 - x^3)\, dx = \frac{b^2a^2}{8}$$

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