MATHEMATICS-I

# LAPLACE TRANSFORMS

I YEAR B.Tech



**By**

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# SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

| Name of the Unit | Name of the Topic |
|---|---|
| Unit-I<br>Sequences and Series | 1.1 Basic definition of sequences and series<br>1.2 Convergence and divergence.<br>1.3 Ratio test<br>1.4 Comparison test<br>1.5 Integral test<br>1.6 Cauchy's root test<br>1.7 Raabe's test<br>1.8 Absolute and conditional convergence |
| Unit-II<br>Functions of single variable | 2.1 Rolle's theorem<br>2.2 Lagrange's Mean value theorem<br>2.3 Cauchy's Mean value theorem<br>2.4 Generalized mean value theorems<br>2.5 Functions of several variables<br>2.6 Functional dependence, Jacobian<br>2.7 Maxima and minima of function of two variables |
| Unit-III<br>Application of single variables | 3.1 Radius , centre and Circle of curvature<br>3.2 Evolutes and Envelopes<br>3.3 Curve Tracing-Cartesian Co-ordinates<br>3.4 Curve Tracing-Polar Co-ordinates<br>3.5 Curve Tracing-Parametric Curves |
| Unit-IV<br>Integration and its applications | 4.1 Riemann Sum<br>4.3 Integral representation for lengths<br>4.4 Integral representation for Areas<br>4.5 Integral representation for Volumes<br>4.6 Surface areas in Cartesian and Polar co-ordinates<br>4.7 Multiple integrals-double and triple<br>4.8 Change of order of integration<br>4.9 Change of variable |
| Unit-V<br>Differential equations of first order and their applications | 5.1 Overview of differential equations<br>5.2 Exact and non exact differential equations<br>5.3 Linear differential equations<br>5.4 Bernoulli D.E<br>5.5 Newton's Law of cooling<br>5.6 Law of Natural growth and decay<br>5.7 Orthogonal trajectories and applications |
| Unit-VI<br>Higher order Linear D.E and their applications | 6.1 Linear D.E of second and higher order with constant coefficients<br>6.2 R.H.S term of the form *exp(ax)*<br>6.3 R.H.S term of the form *sin ax and cos ax*<br>6.4 R.H.S term of the form exp(ax) v(x)<br>6.5 R.H.S term of the form exp(ax) v(x)<br>6.6 Method of variation of parameters<br>6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion |
| Unit-VII<br>Laplace Transformations | 7.1 LT of standard functions<br>7.2 Inverse LT –first shifting property<br>7.3 Transformations of derivatives and integrals<br>7.4 Unit step function, Second shifting theorem<br>7.5 Convolution theorem-periodic function<br>7.6 Differentiation and integration of transforms<br>7.7 Application of laplace transforms to ODE |
| Unit-VIII<br>Vector Calculus | 8.1 Gradient, Divergence, curl<br>8.2 Laplacian and second order operators<br>8.3 Line, surface , volume integrals<br>8.4 Green's Theorem and applications<br>8.5 Gauss Divergence Theorem and applications<br>8.6 Stoke's Theorem and applications |

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UNIT-7

## LAPLACE TRANSFORMS

- **Laplace Transforms of standard functions**
- **Inverse LT- First shifting Property**
- **Transformations of derivatives and integrals**
- **Unit step function, second shifting theorem**
- **Convolution theorem – Periodic function**
- **Differentiation and Integration of transforms**
- **Application of Laplace Transforms to ODE**

# LAPLACE TRANSFORMATION

## INTRODUCTION

Laplace Transformations were introduced by Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician known as a Newton of French.

Laplace Transformations is a powerful Technique; it replaces operations of calculus by operations of Algebra.

Suppose an Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation.

**Definition of Laplace Transformation:** Let $f(t)$ be a given function defined for all $t \geq 0$, then the Laplace Transformation of $f(t)$ is defined as $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{dt} = F(s)$

Here, $L$ is called Laplace Transform Operator. The function $f(t)$ is known as determining function, depends on $t$. The new function which is to be determined (i.e. F($s$) ) is called generating function, depends on $s$.

Here $F(s) = \overline{f}(s)$

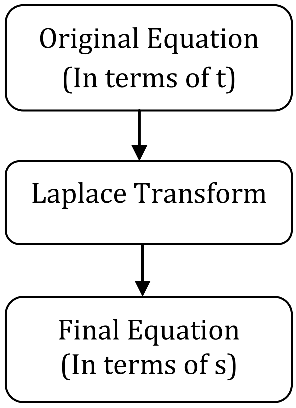
**NOTE:** Here Question will be in $t$ and Answer will be in $s$.

**Laplace Transformation** is useful since

- Particular Solution is obtained without first determining the general solution
- Non-Homogeneous Equations are solved without obtaining the complementary Integral
- Solutions of Mechanical (or) Electrical problems involving discontinuous force functions (R.H.S function $F(x)$ ) (or) Periodic functions other than *cos* and *sin* are obtained easily.
- The Laplace Transformation is a very powerful technique, that it replaces operations of calculus by operations of algebra. For e.g. With the application of L.T to an Initial value problem, consisting of an Ordinary( or Partial ) differential equation (O.D.E) together with Initial conditions is reduced to a problem of solving an algebraic equation ( with any given Initial conditions automatically taken care )

## APPLICATIONS

Laplace Transformation is very useful in obtaining solution of Linear D.E's, both Ordinary and Partial, Solution of system of simultaneous D.E's, Solutions of Integral equations, solutions of Linear Difference equations and in the evaluation of definite Integral.




Thus, Laplace Transformation transforms one class of complicated functions $f(t)$ to produce another class of simpler functions $F(s)$.

## ADVANTAGES

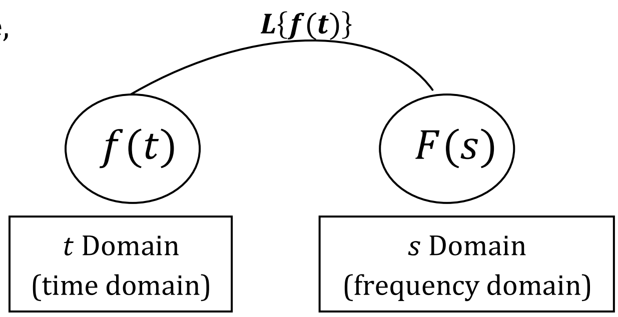
- With the application of Linear Transformation, Particular solution of D.E is obtained directly without the necessity of first determining general solution and then obtaining the particular solution (by substitution of Initial Conditions).
- L.T solves non-homogeneous D.E without the necessity of first solving the corresponding homogeneous D.E.
- L.T is applicable not only to continuous functions but also to piece-wise continuous functions, complicated periodic functions, step functions, Impulse functions.
- L.T of various functions are readily available.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt = F(s)$$

The symbol '$L$' denotes the L.T operator, when it operated on a function $f(t)$, it transforms into a function $F(s)$ of complex variable $s$. We say the operator transforms the function $f(t)$ in the $'t'$ domain (usually called time domain) into the function $F(s)$ in the $'s'$ domain (usually called complex frequency domain or simply the frequency domain)

Because the Upper limit in the Integral is Infinite, the domain of Integration is Infinite. Thus the Integral is an example of an Improper Integral.

$$\int_0^\infty e^{-st} f(t)\ dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt$$




The Laplace Transformation of $f(t)$ is said to exist if the Integral $\int_0^\infty e^{-st} f(t)\ dt$ Converges for some values of $s$, Otherwise it does not exist.

Definition: A function $f(t)$ is said to be piece wise Continuous in any Interval $[a, b]$ , if it is defined on that Interval and is such that the Interval can be broken up into a finite number of sub-Intervals in each of which $f(t)$ is Continuous.

In Mathematics, a transform is usually a device that converts one type of problem into another type.

The main application of D.E using Laplace Transformation and Inverse Laplace Transformation is that, By solving D.E directly by using Variation of Parameters, etc methods, we first find the general solution and then we substitute the Initial or Boundary values. In some cases it will be more critical to find General solution.

By suing Laplace and Inverse Laplace Transformation, we will not going to find General solution and in the middle we substitute the Boundary conditions, so the problem may becomes simple.

Note: Some Problems will be solved more easier in Laplace than by doing using Methods (variation of Parameter etc) and vice-versa.

## PROPERTIES OF LAPLACE TRANSFORMATION

- Laplace Transformation of constant $K$ is $\frac{K}{s}$

*Sol: We know that*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Longrightarrow L\{k\} = \int_0^\infty e^{-st}\, k\, dt$$

$$\Longrightarrow L\{k\} = k \int_0^\infty e^{-st}\, dt$$

$$\Longrightarrow L\{k\} = k \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} = \frac{k}{s}$$

- Definition of Gama Function

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt \,, n \geq 0$$

*(OR)*

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dt \,, n \geq 0 \,.$$

Note: i) $\Gamma(n+1) = n\Gamma(n) = n!$

ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

- The Laplace Transformation of $e^{at}$ is $\frac{1}{s-a}$

*Sol: We know that*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Longrightarrow L\{f(t)\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{(-s+a)t}\, dt$$

$$= \frac{1}{(-s+a)} \left[e^{-(s-a)}\right]_{t=0}^{\infty} = \frac{1}{s-a}$$

## ▶ The Laplace Transformation of $t^n$ , where $n$ is a non-negative Real number.

*Sol:* We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$\Rightarrow L\{f(t)\} = \int_0^\infty e^{-st} t^n\, dt$$

$$Put\ st = x \ \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{1}{s}\, dx$$

$$As\ t \to 0\ to\ \infty \quad \Rightarrow x \to 0\ to\ \infty$$

$$\Rightarrow L\{t^n\} = \int_{x=0}^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{1}{s} dx$$

$$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dt\,, n \geq 0\right]$$

$$\Rightarrow L\{t^n\} = \frac{n!}{s^{n+1}} \qquad [\because \Gamma(n+1) = n!]$$

## Problems

1) Find the Laplace Transformation of $t^{\frac{1}{2}}$

*Sol:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\left(t^{\frac{1}{2}}\right) = \frac{\left(\frac{1}{2}\right)!}{s^{n+1}}$$

$$= \frac{1}{s^{\frac{1}{2}+1}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{s^{\frac{3}{2}}} \frac{\sqrt{\pi}}{2} \qquad [\because n\Gamma(n) = n!\,]$$

2) Find the Laplace Transformation of $t^{-\frac{1}{2}}$

*Sol:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\left(t^{-\frac{1}{2}}\right) = \frac{\left(-\frac{1}{2}\right)!}{s^{-\frac{1}{2}+1}}$$

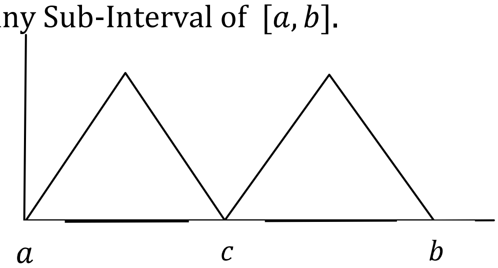
$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{\frac{1}{2}}} \qquad [\because \Gamma(n+1) = n!\,]$$

$$\Rightarrow L\left(t^{-\frac{1}{2}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

## SECTIONALLY CONTINUOUS CURVES (Or) PIECE-WISE CONTINUOUS

A function $f(x)$ is said to be Sectionally Continuous (or) Piece-wise Continuous in any Interval $[a, b]$, if it is continuous and has finite Left and Right limits in any Sub-Interval of $[a, b]$.

In the above case Laplace Transformation holds good.



## FUNCTIONS OF EXPONENTIAL ORDER

A function $f(x)$ is said to be exponential of order $'a'$ as $x \to \infty$ if $\lim_{x\to\infty} e^{-ax} f(x)$ is a finite value.

***Example:* Verify $x^n$ is an exponential order (or) not?**

***Sol:*** $lim_{x\to\infty} e^{-ax} f(x) = lim_{x\to\infty} e^{-ax}\ x^n = lim_{x\to\infty} \frac{x^n}{e^{ax}} = \frac{\infty}{\infty}$

$$= lim_{x\to\infty} \frac{n!}{a^n e^{ax}} = \frac{n!}{\infty} = 0, \text{ a finite value.}$$

$\therefore$ $x^n$ is an exponential order.

## Sufficient conditions for the Existence of Laplace Transformation

The Laplace Transformation of $f(t)$ exists i.e. The Improper Integral $\int_0^\infty e^{-st} f(t)\, dt$ of $L\{f(t)\}$ Converges (finite value) when the following conditions are satisfied.

1) $f(t)$ is a piece-wise continuous
2) $f(t)$ is an exponential of order $'a'$.

## PROPERTIES OF LAPLACE TRANSFORMATION

### LINEAR PROPERTY

*Statement:* If $L\{f(t)\} = F(s), L\{g(t)\} = G(s)$, then $L\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$

*Proof:* Given that $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$

***L.H.S:*** $L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}$

$= c_1 F(s) + c_2 G(s)$ ***=R.H.S***

### FIRST SHIFTING PROPERTY (or) FIRST TRANSLATION PROPERTY

*Statement:* If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$

*Proof:* We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt$$

*(Here we have taken exponential quantity as negative, but not positive, because as* $t \to \infty \Rightarrow e^t \to \infty \Rightarrow e^{-t} \to 0$ *)*

*Put* $s - a = p,\ p > 0$

$$\Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-pt} f(t) dt$$

$$= F(p)$$

$$= F(s-a)$$

*Hence, If* $L\{f(t)\} = F(s)$ *then* $L\{e^{at} f(t)\} = F(s-a)$

- whenever we want to evaluate $\boldsymbol{L\{e^{at} f(t)\}}$, first evaluate $\boldsymbol{L\{f(t)\}}$ which is equal to $\boldsymbol{F(s)}$ and then evaluate $\boldsymbol{L\{e^{at} f(t)\}}$, which will be obtained simply, by substituting $\boldsymbol{s - a}$ in place of $\boldsymbol{a}$ in $\boldsymbol{F(s)}$.

## Problem

❖ Find the Laplace Transformation of

$f(t) = t^{\frac{7}{2}} . e^{3t}$

Sol: To find $L\left\{t^{\frac{7}{2}} . e^{3t}\right\}$, first we shall evaluate

$L\left\{t^{\frac{7}{2}}\right\}$

Now, $L\left\{t^{\frac{7}{2}}\right\} = \frac{(7/2)!}{s^{(7/2)+1}} \left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$

$\Longrightarrow L\left\{t^{\frac{7}{2}}\right\} = \frac{\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)}{s^{\frac{9}{2}}}$ $[\because n! = n\,\Gamma(\mathrm{n})]$

$= \frac{\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{s^{\frac{9}{2}}}$ $[\because \ \Gamma(n+1) = n\Gamma(n)]$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{9}{2}}} = \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{\frac{9}{2}}}$

$= \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}{s^{\frac{9}{2}}}$ $\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

$\Longrightarrow L\left\{t^{\frac{7}{2}}\right\} = \frac{105\sqrt{\pi}}{16\, s^{\frac{9}{2}}} = F(s)$

$\therefore L\left\{t^{\frac{7}{2}} . e^{3t}\right\} = \frac{105\sqrt{\pi}}{16\,(s-a)^{\frac{9}{2}}}$

## CHANGE OF SCALE PROPERTY

Statement: If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

*Proof:* We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = F(s)$

$$\Longrightarrow L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,\mathrm{dt}$$

Put $at = x \ \Longrightarrow t = \frac{x}{a}$

$$\Longrightarrow dt = \frac{1}{a}\,dx$$

$$\Longrightarrow L\{f(at)\} = \int_0^\infty e^{-\frac{s}{a}x} f(x)\ \frac{1}{a}\ dx$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}x} f(x)\ \ dx$$

Put $\frac{s}{a} = p$ then, $L\{f(at)\} = \frac{1}{a}\int_0^\infty e^{-px} f(x)\ \ dx$

$$= \frac{1}{a}\ F(p) = \ \frac{1}{a}\ F\left(\frac{s}{a}\right)$$

## Problems

**1)** Find the Laplace Transformation of $\cosh at$ $\quad i.e. L\{\cosh at\}$

*Sol:* We know that $\cosh at = \frac{e^{at}+e^{-at}}{2}$

and $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = F(s)$

Now, $L\{\cosh at\} = \frac{1}{2}[L\{e^{at} + e^{-at}\}]$

$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$\therefore L[\cosh at] = \frac{s}{s^2 - a^2}$

**2)** Find the Laplace Transformation of $\sinh at$ $\quad i.e. L\{\sinh at\}$

*Sol:* We know that $\sinh at = \frac{e^{at}-e^{-at}}{2}$

and $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = F(s)$

Now, $L\{\sinh at\} = \frac{1}{2}[L\{e^{at} - e^{-at}\}]$

$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$\therefore L[\sinh at] = \frac{a}{s^2 - a^2}$

**3)** Find $L\{\sin at\}$ *and* $L\{\cos at\}$

*Sol:* We know that $e^{iat} = \cos at + i \sin at$

$\Rightarrow L\{e^{iat}\} = L\{\cos at + i \sin at\} \longrightarrow$ I

But $L\{e^{iat}\} = \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} \longrightarrow$ II

$\therefore$ From I &II, $L\{\cos at + i \sin at\} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}$

Equating the corresponding coefficients, we get

$L\{\sin at\} = \frac{a}{s^2+a^2}$, and $L\{\cos at\} = \frac{s}{s^2+a^2}$

## LAPLACE TRANSFORMATION OF DERIVATIVES

*Statement:* If $L\{f(t)\} = F(s)$, then $L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\vdots$

$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$

*Proof:* We know that

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = F(s)$

Let us consider $L\{f'(t)\} = \int_0^\infty e^{-st} \frac{df}{dt}\,\mathrm{dt}$

$= \left[e^{-st}\int \frac{df}{dt}\,dt - \int\left\{\frac{d}{dt}(e^{-st})\int \frac{df}{dt}\,dt\right\}dt\right]_{t=0}^{\infty}$

$= [e^{-st} f(t)]_{t=0}^{\infty} - [\int_{t=0}^{\infty}(-se^{-st} f(t)\,dt]$

$= -f(0) + s\,L\{f(t)\}$

$= -f(0) + s\,F(s)$

$\therefore L\{f'(t)\} = sF(s) - f(0)$

In view of this, $L\{g'(t)\} = s\,G(s) - g(0)$

Now, put $g(t) = f'(t)$

$\Rightarrow L\{g(t)\} = L\{f''(t)\} = G(s)$

Since, $L\{g'(t)\} = s\,G(s) - g(0)$

$\therefore L\{f''(t)\} = s\,L\{g(t)\} - g(0)$

$\Rightarrow L\{f''(t)\} = s\,L\{f'(t)\} - f'(0)$

$\Rightarrow L\{f''(t)\} = s\big(sF(s) - f(0)\big) - f'(0)$

$\Rightarrow L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

Generalizing this, we get finally

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ \dots . - f^{n-1}(0)$$

***Problem:*** *Find the $L\{t^2\}$ by using derivatives method.*

Sol: we know that $L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

Given that $f(t) = t^2 \Rightarrow f(0) = 0$

$\Rightarrow f'(t) = 2t \Rightarrow f'(0) = 0$

$\therefore L\{f''(t)\} = s^2 F(s)$

$\Rightarrow L(2) = s^2 F(s)$

$\Rightarrow \frac{2}{s} = s^2 F(s)$

$\Rightarrow F(s) = \frac{2}{s^3}$

## DIVISION BY 't' METHOD (or) Laplace Integrals

*Statement:* If $L\{f(t)\} = F(s)$ , then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

*Proof:* Let us consider $\boldsymbol{F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt}$

Now, Integrate on both sides w.r.t $'s'$ by taking the Limits from $s$ *to* $\infty$ then

$$\int_s^\infty F(s)\,ds = \int_{s=s}^\infty \left(\int_{t=0}^\infty e^{-st} f(t)\,dt\right)\,ds$$

Since $'s'$ and $'t'$ are Independent variables, by Interchanging the order of Integration,

$$\Rightarrow \int_s^\infty F(s)\,ds = \int_{t=0}^\infty \left(\int_{s=s}^\infty e^{-st}\,ds\right)\,f(t)\,dt$$

$$= \int_{t=0}^\infty \left[\frac{e^{-st}}{-t}\right]_{s=s}^\infty f(t)\,dt$$

$$= \int_{t=0}^\infty e^{-st}\left(\frac{f(t)}{t}\right)\,dt$$

$$\Rightarrow \int_s^\infty F(s)\,ds = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

## Problems

- Find the $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol: Here $f(t) = e^{-at} - e^{-bt}$

$\Rightarrow F(s) = L\{f(t)\} = L\{e^{-at} - e^{-bt}\}$

$= \frac{1}{s+a} - \frac{1}{s+b}$

We know that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

$\Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right]\,ds$v

$= [\log|s+a| - \log|s+b|]_s^\infty$

$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_{s=s}^\infty$

$= \log 1 - \log\left(\frac{s+b}{s+b}\right) = log\left(\frac{s+a}{s+b}\right)$

- If $\{f(t)\} = F(s)$ , then $L\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\ F(s), s > 0$ .

***Sol:*** Let us consider $g(t) = \int_0^t f(t)\ dt$ and $g(0) = 0$

$$\Rightarrow g'(t) = f(t)$$

Now, $L\{g'(t)\} = s\ L\{g(t)\} - g(0)$

$$L\{\ f(t)\} = s\ L\left\{\int_0^t f(t)\ dt\right\} - 0$$

$$\Rightarrow F(s) = s\ L\left\{\int_0^t f(t)\ dt\right\}$$

$$\Rightarrow\ L\left\{\int_0^t f(t)\ dt\right\} = \frac{F(s)}{s}$$

Generalization of above one:

$$L\left\{\underbrace{\int_0^t \int_0^t \int_0^t \dots\ \int_0^t f(t)}_{n\text{-times}}\ dt\right\} = \frac{F(s)}{s^n}$$

## Multiplication of $'t'$

*Statement: If* $L\{f(t)\} = F(s)$, *then* $L\{t.f(t)\} = -\frac{d}{ds}[F(s)]$

*Proof:* We know that $F(s) = \int_0^\infty e^{-st}\ f(t)dt$

$$\Rightarrow \frac{d}{ds}\ F(s) = \frac{d}{ds}\ \left[\int_0^\infty e^{-st}\ f(t)dt\right]$$

$$= \int_0^\infty\ f(t)\ \frac{\partial}{\partial s}(e^{-st})dt$$

$$= \int_0^\infty f(t)\ e^{-st}\ (-t)\ dt$$

$$= -\int_0^\infty e^{-st}\ t\ f(t)dt$$

$$= -L\{t\ f(t)\}$$

$$\therefore \boldsymbol{L\{t.f(t)\} = -\frac{dF}{ds}}$$

Generalization: $L\{t^n.f(t)\} = -\frac{d^n}{ds^n}[F(s)] = -\ \frac{d^nF}{ds^n}$

## Problem

- Find $L\{t\ \sin at\}$

*Sol:* we know that $L\{t.f(t)\} = -\frac{dF}{ds}$

*Here* $f(t) = \sin at$

$$\Rightarrow F(s) = \left[\frac{a}{s^2+a^2}\right]$$

$$\therefore\ L\{t\ \sin at\} = -\frac{\boldsymbol{d}}{\boldsymbol{ds}}\left\{\frac{\boldsymbol{a}}{\boldsymbol{s^2+a^2}}\right\} = \frac{\boldsymbol{2as}}{\boldsymbol{s^2+a^2}}$$

## INVERSE LAPLACE TRANSFORMATION

Definition: If $\{f(t)\} = F(s)$ , then $f(t)$ is known as Inverse Laplace Transformation of $F(s)$ and it is denoted by $L^{-1}[F(s)] = f(t)$ , where $L^{-1}$ is known as Inverse Laplace Transform operator and is such that $L\,L^{-1} = L^{-1}\,L = 1$.

## Inverse Elementary Transformations of Some Elementary Functions

- $L^{-1}\left(\frac{1}{s}\right) = 1$
- $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
- $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
- $L^{-1}\left(\frac{1}{(s-a)^n}\right) = e^{at}\frac{t^{n-1}}{(n-1)!}$
- $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
- $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
- $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$
- $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
- $L^{-1}\left(\frac{s}{(s-a)^2+b^2}\right) = \frac{1}{b}\,e^{at}\sin bt$
- $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

## Problems based on Partial Fractions

A fraction of the form $\frac{a_0\,s^m + a_1\,s^{m-1} + \ldots + a_m}{b_0\,s^n + b_1\,s^{n-1} + \cdots + b_n}$ in which both powers $m$ and $n$ are positive numbers is called rational algebraic function.

When the degree of the Numerator is Lower than the degree of Denominator, then the fraction is called as Proper Fraction.

To Resolve Proper Fractions into Partial Fractions, we first factorize the denominator into real factors. These will be either Linear (or) Quadratic and some factors may be repeated.

From the definitions of Algebra, a Proper fraction can be resolved into sum of Partial fractions.

| S.No | Factor of the Denominator | Corresponding Partial Fractions |
|---|---|---|
| **1.** | Non-Repeated Linear Factor<br>Ex: $\frac{1}{s-a}$, [ $(s-a)$occurs only one time] | $\frac{A}{s-a}$ |
| **2.** | Repeated Linear Factor, repeated 'r' times<br>Ex: $(s-a)^r$ | $\frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_r}{(s-a)^r}$ |
| **3.** | Non-repeated Quadratic Expression<br>Ex: $s^2 + as + b$ | $\frac{As+B}{s^2+as+b}$ , Here atleast one of $A\ or\ B \neq 0$ |
| **4.** | Repeated Quadratic Expression, repeated 'r' times<br>Ex: $(s^2 + as + b)^r$ | $\frac{A_1 s + B_1}{s^2 + as + b} + \frac{A_2 s + B_2}{(s^2 + as + b)^2} + \ldots \frac{A_r s + B_r}{(s^2 + as + b)^r}$ |

## SHIFTING PROPERTY OF INVERSE LAPLACE TRANSFORMATION

We know that $L\{e^{at} f(t)\} = F(s-a)$

$$\Rightarrow L^{-1}[F(s-a)] = e^{at} f(t)$$

## FORMULAS

- If $L^{-1}[F(s)] = f(t)$ then, $L^{-1}[F(s-a)] = e^{at} f(t)$
- If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then, $L^{-1}[s\, F(s)] = \frac{d}{dt}[f(t)]$

  In general, $L^{-1}[s^n F(s)] = \frac{d^n}{dt^n}[f(t)]$ , provided $f(0) = f'(0) = f''(0) = \ldots = f^{n-1} = 0$
- If $L^{-1}[F(s)] = f(t)$ then, $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\, dt$
- If $L^{-1}[F(s)] = f(t)$ then, $t.f(t) = L^{-1}\left[-\frac{d}{ds}\{F(s)\}\right]$
- If $L^{-1}[F(s)] = f(t)$ then, $\frac{f(t)}{t} = L^{-1}\left[\int_0^\infty F(s)\, ds\right]$

## CONVOLUTION THEOREM

(A Differential Equation can be converted into Inverse Laplace Transformation)

(In this the denominator should contain atleast two terms)

Convolution is used to find Inverse Laplace transforms in solving Differential Equations and Integral Equations.

**Statement:** *Suppose two Laplace Transformations $F(s)$ and $G(s)$ are given. Let $f(t)$ and $g(t)$ are their Inverse Laplace Transformations respectively i.e.* $L^{-1}[F(s)] = f(t)$

$$L^{-1}[G(s)] = g(t)$$

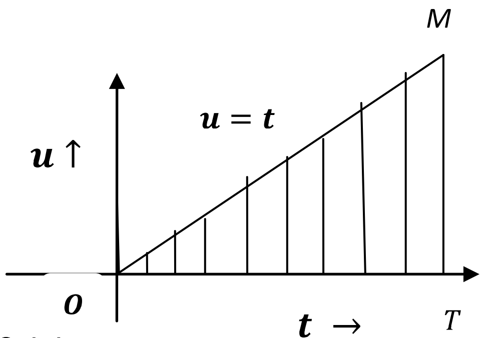
*Then,* $L^{-1}[F(s).G(s)] = \int_0^t f(u)\, g(t-u) du = F * G$

*Where $F * G$ is called Convolution. (Or) Falting of $F$ & $G$.*

*Proof:* Let $\phi(t) = \int_0^t f(u)\; g(t-u)\, du$

Now $L\{\phi(t)\} = \int_0^\infty e^{-st} \left[\int_{u=0}^t f(u) g(t-u)\, du\right] dt$

$$\Rightarrow L\{\phi(t)\} = \int_0^\infty \left[\int_{u=0}^t e^{-st} f(u) g(t-u)\, du\right] dt$$



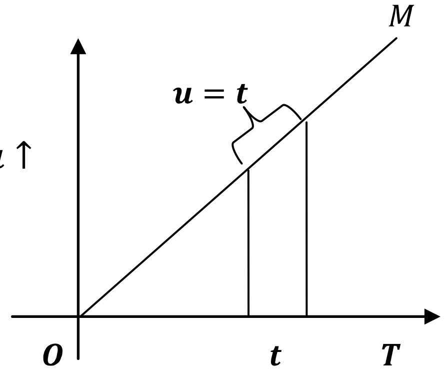

The above Integration is within the region lying below the line, and above $OT$.

(Here equation of $OM$ is $u = t$)

Let $'t'$ is taken on $OT$ line and $u$ is taken on $OM$ line, with $'O'$ as Origin.

The axes are perpendicular to each other.

If the order of Integration is changed, the strip will be taken parallel to $OT$. So that the limits of $t$ are from $u \ to \ \infty$ and $u$ is taken as $0 \ to \ \infty$.




$$L\{\phi(t)\} = \int_{u=0}^{\infty} \left[\int_{t=u}^{t} e^{-st} f(u) g(t-u)\, du\right] dt$$

$$= \int_{u=0}^{\infty} \left[\int_{t=u}^{t} e^{-s(t-u)} e^{-su} f(u) g(t-u)\, du\right] dt$$

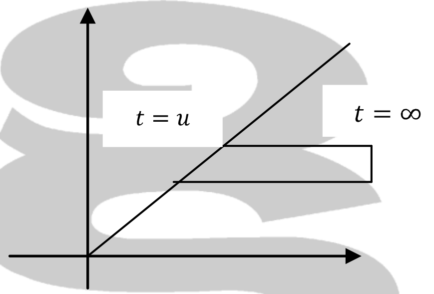
$$= \int_{u=0}^{\infty} (e^{-su} f(u) du) \left[\int_{t=u}^{t} e^{-s(t-u)} g(t-u)\, dt\right]$$

Put $t-u = v$, then Lower Limit: $t = u \implies v = 0$

Upper Limit: $t = \infty \implies v = \infty$

Then, Consider $\int_{t=u}^{\infty} e^{-s(t-u)} g(t-u)\ dt \ = \int_{v=0}^{\infty} e^{-sv}\, g(v)\, dv$

$$= L\{g(v)\} = G(v)$$




$$L\{\phi(t)\} = \left[\int_{u=0}^{\infty} e^{-su} f(u) du\right] \{G(v)\}$$

$$= F(u) G(v)$$

Again, $\phi(t) = L^{-1}[F(u).G(v)]$

$$= L^{-1}[F(u).G(t-u)]$$

## Problem

- Apply Convolution Theorem to evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

*Sol:* Given $L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$

Let us choose one quantity as $F(s)$ and other quantity as $G(s)$

Now, Let $F(s) = \frac{1}{s^2+a^2}$, $G(s) = \frac{s}{s^2+a^2}$

Now $f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

$$\implies f(u) = \frac{1}{a}\sin au$$

Again $g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$\therefore$ By Convolution Theorem, $L^{-1}[F(s).G(s)] = \int_0^u f(u)\, g(t-u) du$

$$\implies L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_{u=0}^{t} \frac{1}{a}\sin au \cos a(t-u)\ du$$

$$= \frac{1}{2a} \int_{u=0}^{t} 2 \sin au \cos a(t-u) \; du$$

$$= \frac{1}{2a} \int_{u=0}^{t} [\{\sin(au + at - au\} + \sin\{au - at + au\}] \; du$$

$$= \frac{1}{2a} \int_{u=0}^{t} \sin at \; du + \frac{1}{2a} \int_{u=0}^{t} \sin(2au - at) \; du$$

$$= \frac{1}{2a} \sin at \; [u]_0^t + \frac{1}{2a} \left[\frac{-\cos(2au-at)}{2a}\right]_{u=0}^{t}$$

$$= \frac{1}{2a} \sin at \; t + \frac{(-1)}{4a^2} \{-\cos at + \cos at\}$$

$$= \frac{1}{2a} \; t \sin at$$

## APPLICATIONS OF D.E's BY USING LAPLACE AND INVERSE LAPLACE TRANSFORMATIONS

Laplace Transform Method of solving Differential Equations yields particular solutions without necessity of first finding General solution and elimination of arbitrary constants.

Suppose the given D.Eq is of the form $a \frac{d^2y}{dt^2} + b \frac{dy}{dt} + y = f(t)$ ⟶ I

is a Linear D.Eq of order 2 with constants a, b.

Case 1: Suppose in Equation I, we assume a,b are constants and the boundary conditions are $y(0) = y'(0) = 0$.

We Know that $L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0)$

and $L[f''(t)] = s^2 F(s) - sf(0) - f'(0)$

(Or) $L[y^2(t)] = s^2 \bar{y}(s) - s\, y(0) - y'(0)$ Here $\bar{y}(s) = L[y(t)]$ and $y^2(t)$ is Second derivative

and $L[y'(t)] = s\bar{y}(s) - y(0)$

Procedure: Apply Laplace Transformation to equation ( I )

i.e. $a\, L\{y''\} + b\, L\{y'\} + L\{y\} = L\{f(t)\}$

$\Rightarrow a\{s^2 \bar{y}(s) - s\, y(0) - y'(0)\} + b\, \{s\bar{y}(s) - y(0)\} + \bar{y}(s) = F(s)$

$\Rightarrow as^2 \bar{y}(s) + bs\, \bar{y}(s) + \bar{y}(s) = F(s)$

($\because$ *we have taken Initial conditions as* $y(0) = y'(0) = 0$ )

$\Rightarrow (as^2 + bs + 1)\, \bar{y}(s) = F(s)$

$$\Rightarrow \bar{y}(s) = \frac{F(s)}{(as^2 + bs + 1)}$$

Now, apply Inverse Laplace Transformation i.e. $L^{-1}\{\bar{y}(s)\} = L^{-1}\left\{\frac{F(s)}{(as^2+bs+1)}\right\}$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{F(s)}{(as^2+bs\ +1)}\right\}$$

By solving this, we get the required answer.

Case 2: If a, b are not constants (i.e. D.E with Variable Co-efficient)

Let the D.E is of the form $t^2\ \frac{d^2y}{dt^2} + t\ \frac{dy}{dt} + y = f(t) \longrightarrow$ II

Here $a, b$ are some functions of $'t'$, with Initial conditions $y(0) = y'(0) = 0$

We know that $L\{t^m\ f^n(t)\} = (-1)^m\ \frac{d^m}{ds^m}\ [L\{f^n(t)\}]$

Now, $L\{t^2y''\} = (-1)^2\ \frac{d^2}{ds^2}\ [L\{y''(t)\}]$

$$= \frac{d^2}{ds^2}\ [s^2\bar{y}(s) - s\ y(0) - y'(0)]$$

And, $L\{t\ y'\} = (-1)\frac{d}{ds}L\{y'(t)\} = -\frac{d}{ds}\ \{s\bar{y}(s) - y(0)\}$

Apply Laplace Transformation on both sides to ( II ), we get

$$\Rightarrow L\{t^2y''\} + L\{t\ y'\} + L\{y\} = L\{f(t)\}$$

$$\Rightarrow \frac{d^2}{ds^2}\ [s^2\bar{y}(s) - s\ y(0) - y'(0)] - \frac{d}{ds}\ \{s\bar{y}(s) - y(0)\} + \bar{y}(s) = F(s)$$

Substituting the boundary conditions in equation II and get the values of $\bar{y} \longrightarrow$ III

Required solution is obtained by taking Inverse Laplace Transformation for equation III.

## Problem

- Solve by the method of Transformations, the equation $y''' + 2y'' - y' - 2y = 0$ and $y(0) = y'(0) = 1$ and $y''(0) = -6$.

Sol: Given $y''' + 2y'' - y' - 2y = 0 \longrightarrow$ I

Apply Laplace Transformation on both sides, we get

$$\Rightarrow L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\}$$

$$\Rightarrow \{s^3\bar{y} - s^2\bar{y}(0) - s\ y'(0) - y''(0)\} + 2\{s^2\bar{y}(s) - s\ y(0) - y'(0)\} - \{s\bar{y}(s) - y(0)\} - 2\bar{y} = 0 \longrightarrow \text{II}$$

Now substitute boundary conditions Immediately before solving in equation II, we get

$$\Rightarrow \{s^3\bar{y} + 6\} + 2\{s^2\bar{y}\} - \{s\bar{y}\} - 2\bar{y} = 0$$

$$\Rightarrow \bar{y}(s^3 + 2s^2 - s - 2) = -6$$

$$\Rightarrow \bar{y} = \frac{-6}{s^3 + 2s^2 - s - 2}$$

$$\Rightarrow \bar{y} = \frac{-6}{(s-1)(s+1)(s-2)}$$

$$= \frac{-1}{s-1} + \frac{3}{s+1} - \frac{2}{s+2} \qquad \text{(By resolving into partial fractions)}$$

$$\Rightarrow L^{-1}\{\bar{y}(s)\} = -L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{s+1}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$\Rightarrow y(t) = -e^{t} + 3e^{-t} - 2e^{-2t}$ is the required solution.

- Solve the D.E $ty'' + 2y' + ty = \cos t$ , $y(0) = 1, y'(0) = 1$

*Sol:* Taking Laplace Transform on both sides, we get

$$L\{ty''\} + 2L\{y'\} + 2L\{ty\} = L\{Cos\ t\}$$

$$\Rightarrow (-1)\frac{d}{ds}[s^2\,Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + (-1)\frac{d}{ds}[Y(s)] = \frac{s}{s^2+1}$$

Here, given Initial/Boundary conditions are $y(0) = 1, y'(0) = 1$

$$\Rightarrow (-1)\frac{d}{ds}[s^2\,Y(s) - s.1 - 1] + 2[sY(s) - 1)] + (-1)\frac{d}{ds}[Y(s)] = \frac{s}{s^2+1}$$

$$\Rightarrow -s^2\frac{d}{ds}Y(s) - 2sY(s) + 1 + 2sY(s) - 2 - \frac{d}{ds}Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow -\frac{dY}{ds}\{s^2+1\} = \frac{s}{s^2+1} + 1$$

$$\Rightarrow -\frac{dY}{ds} = \frac{s}{(s^2+1)^2} + \frac{1}{s^2+1}$$

Apply Inverse Laplace Transformation on both sides, we get

$$\Rightarrow -L^{-I}\left(\frac{dY}{ds}\right) = L^{-I}\left(\frac{s}{(s^2+1)^2}\right) + L^{-I}\left(\frac{1}{s^2+1}\right)$$

We know that $L\{t\,f(t)\} = -\frac{d}{ds}F(s)$

$$\Rightarrow t.y(t) = \frac{1}{2}t\sin t + \sin t$$

(Or) $y(t) = \frac{1}{2}\sin t + \frac{\sin t}{t}$

## Hints for solving problems in Inverse Laplace Transformation

- If it is possible to express denominator as product of factors then use partial fraction method (i.e. resolve into partial fractions and solve further)
- Sometimes it is not possible to express as product of partial fractions. In such case, express denominator quantity in the form of $[(s-a)^2 + b^2]$ $or$ $[(s-a)^2 - b^2]$ etc
- Note that, the problems in which the denominator is possible to express as product of partial fractions can be solved in other methods also.

## Some Formulas

- If $L\{f(t)\} = F(s)$ then $L\{e^{at}\, f(t)\} = F(s-a)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F(s-a)\} = e^{at} f(t)$

  $$\Rightarrow L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$$

- If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\ F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt$

  ***Generalization:*** If $L\{f(t)\} = F(s)$ then $L\left\{\underbrace{\int_0^t \int_0^t \dots \int_0^t f(t)}_{n\ times}\ dt\right\} = \frac{1}{s^n}\ F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \underbrace{\int_0^t \int_0^t \dots \int_0^t f(t)}_{n\ times}\ dt$

- If $L\{f(t)\} = F(s)$ then $L\{t\, f(t)\} = -\frac{d}{ds} F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t\, f(t)$

  ***Generalization***: If $L\{f(t)\} = F(s)$ then $L\{t^n\, f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{d^n}{ds^n} F(s)\right\} = (-1)^n\, t^n\, f(t)$

- If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_{s=s}^{\infty} F(s)\ ds$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{\int_{s=s}^{\infty} F(s)\ ds\} = \frac{f(t)}{t}$

- If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = [s\, F(s) - f(0)]$

  Now, If $L^{-1}\{F(s)\} = f(t)$, and $f(0) = 0$, then $L^{-1}[s\, F(s)] = f'(t)$

  ***Generalization:*** If $L\{f(t)\} = F(s)$ then $L\{f''(t)\} = [s^2\, F(s) - s\, f(0) - f'(0)]$

  Now, If $L^{-1}\{F(s)\} = f(t)$, and $f(0) = f'(0) = 0$, then $L^{-1}[s^2\, F(s)] = f''(t)$

  ***Similarly,*** If $L\{f(t)\} = F(s)$ then $L\{f^n(t)\} = [s^n\, F(s) - s^{n-1}\, f(0) - \ \dots - f^{n-1}(0)]$

  Now, If $L^{-1}\{F(s)\} = f(t)$, and $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$, then $L^{-1}[s^n\, F(s)] = f^n(t)$

- If $L\{f(t)\} = F(s)$ and $g(t) = f(t-a)\, u(t-a) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases}$ Then,

  $L\{g(t)\} = L\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$

  Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{e^{-as} F(s)\} = g(t) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases}$

## Problems

❖ Find $L^{-1}\left\{\frac{1}{s(s+2)}\right\}$

Sol: Here, if we observe, the denominator is in the product of factors form. So we can use partial fractions method. Or we can use the following method.

We know that, If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\ F(s)$

Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt \longrightarrow \text{I}$

Let us consider $F(s) = \frac{1}{s+2}$

$$\Rightarrow L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow f(t) = e^{-2t}$$

$\therefore$ Substituting in I, we get $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt$

$$\Rightarrow L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t e^{-2t}\ dt$$

$$= \left(\frac{1-e^{-2t}}{2}\right)$$

❖ Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

**Sol:** Here if we observe, it is possible to express denominator as product of partial fractions. So we can use partial fractions method also. But the method will be lengthy.

So, we go for another method, by which we can solve the problem easily.

Now, We know that If $L\{f(t)\} = F(s)$ then $L\{t\ f(t)\} = -\frac{d}{ds}F(s)$

Now, If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\ f(t) \longrightarrow \text{I}$

Let us consider $F(s) = \frac{1}{s^2+a^2}$

> If Numerator is $'s'$ and denominator is $(\quad)^2$ term, then always use $t\,f(t)$ model.

$$\Rightarrow L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$\Rightarrow \boxed{f(t) = \frac{1}{a}\sin at}$$

$\therefore$ From I, we have $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\ f(t)$

$$\Rightarrow -tf(t) = L^{-1}\left\{\frac{d}{ds}(s^2+a^2)^{-1}\right\}$$

$$= L^{-1}\{-2s(s^2+a^2)^{-2}\}$$

$$\Rightarrow -tf(t) = -2\ L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow tf(t) = 2\ L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow \frac{t}{2}f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow \frac{t}{2}\frac{1}{a}\sin at = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} \text{ (Or)}$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

- If Numerator is $'s'$ and denominator is $(\quad)^2$ term, then always use $t\,f(t)$ model.
- If Numerator is 1, and denominator is $(\quad)^2$ term, then always use $t\,f(t)$ model.
- If we are asked to find $L^{-1}$ of $\log(\quad)$, $\tan(\quad)$, $\cot(\quad)$ etc. or any unknown quantity, where we don't have any direct formula, in such cases always use $t\,f(t)$ model.

## PERIODIC FUNCTIONS OF LAPLACE TRANSFORMATIONS

Periodic Function: A Function $f(t)$ is said to be periodic function of the period $T\ (>0)$ if

$$f(t) = f(t+T) = f(t+2T) = \ \dots = f(t+nT)$$

Example: $\sin t$, $\cos t$ are the periodic functions of period $2\pi$.

Theorem: The Laplace Transformation of Piece-wise periodic function $f(t)$ with period $'p'$ is

$$L\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st}\ f(t)\ dt$$

Proof: Let $f(t)$ be the given function then

$$L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)dt$$

$$= \int_0^P e^{-st}\ f(t)dt + \int_P^{2P} e^{-st}\ f(t)dt + \int_{2P}^{3P} e^{-st}\ f(t)dt + \ \dots$$

Put $t = u + p$ in the second Integral

$t = u + 2p$ in the third Integral

⋮

$t = u + np$ in the $n^{th}$ Integral etc

$$\therefore L\{f(t)\} = \int_0^p e^{-st}\ f(t)dt + \int_{u=0}^{p} e^{-s(u+p)}\ f(u+p)du + \ \dots + \int_0^p e^{-s[u+(n-1)p]}\ f(u+(n-1)p)du + \ \dots$$

Since by substituting $t = u \implies dt = du$ in 1st Integral and also by the definition of Periodic function $f(t) = f(t+T) = f(t+2T) = \ \dots = f(t+nT)$

$$\implies L\{f(t)\} = \int_0^p e^{-su}\ f(u)du + e^{-sp}\int_{u=0}^{p} e^{-su}\ f(u)du + \ e^{-2sp}\int_0^p e^{-su}\ f(u)du + \ \dots$$

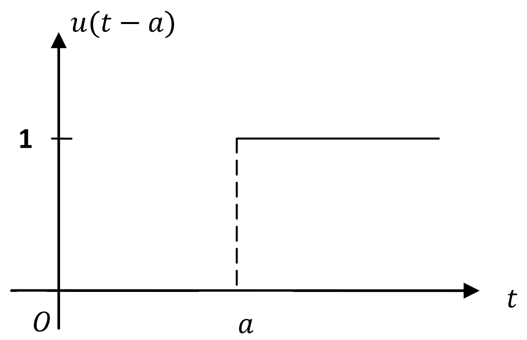
$$\implies L\{f(t)\} = [1 + e^{-sp} + e^{-2sp} + \ \dots]\int_0^p e^{-su}\ f(u)du$$

This is a Geometric Progression $S_\infty = \frac{a}{1-r}$

$$\implies \boxed{L\{f(t)\} = \left(\frac{1}{1-e^{-sp}}\right)\left[\int_0^p e^{-su}\ f(u)du\right]}$$

## Unit Step Function (Heaviside's Unit Function)

The Unit Step Function $u(t-a)$ or $H(t-a)$ is defined as $H(t-a) = \begin{cases} 0\ , & for\ t < a \\ 1\ , & for\ t \geq a \end{cases}$, where $'a'$ is positive number always.




## Second Shifting Property (or) Second Translation Property

*Statement:* If $L\{f(t)\} = F(s)$ and the shifted function $g(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases}$ Then, $L\{g(t)\} = L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

Proof: We know that $L\{g(t)\} = \int_0^\infty e^{-st}\, g(t)\, dt$

$$= \int_0^a e^{-st}\, g(t)\, dt + \int_a^\infty e^{-st}\, g(t)\, dt$$

$$= 0 + \int_a^\infty e^{-st}\, f(t-a)\, dt$$

Put $t - a = u \implies dt = du$ then

$$\implies L\{g(t)\} = \int_0^\infty e^{-s(u+a)}\, f(u) du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\, du$$

$$\implies L\{g(t)\} = e^{-sa}F(s)$$

*Note:* The Laplace Transform of Unit Step Function (put $f(t) = 1$) is $L\{H(t-a)\} = \frac{e^{-as}}{s}$

## Unit Impulse Function (or Diract delta Function)

Suppose a large force (like Earthquake, collision of two bodies) acts on a system, produces large effect when applied for a very short interval of time. To deal with such situations, we introduce a function called unit impulse function, which is a discontinuous function.

If a large force acts for a short time, the product of the force and time is called impulse. To deal with similar type of problems in applied mechanics, the unit impulse is to be introduced.

## Laplace Transform of Unit Step Function

$$L\{H(t-a)\} = \int_0^\infty e^{-st}\, H(t-a)\, dt$$

$$\Rightarrow\ L\{H(t-a)\} = \int_0^a e^{-st}\, H(t-a)\, dt + \int_a^\infty e^{-st}\, H(t-a)\, dt$$

$$= \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st}.1\, dt$$

$$= \frac{e^{-as}}{s}$$

❖ Find Laplace Transformation of $J_0(x)$

Sol: We know that $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ \ldots$

$$\Rightarrow L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2.4^2}\frac{4!}{s^5} + \ \ldots$$

$$\Rightarrow L\{J_0(x)\} = \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\frac{3}{4}\frac{1}{s^4} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^6} + \ \ldots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{\frac{-1}{2}} = \frac{1}{\sqrt{s^2+1}}$$

❖ Find Laplace Transformation of $J_1(x)$

Sol: We know that $J_1(x) = -J_0^{I}(x)$

$$= -\frac{2x}{2^2} + 4\frac{x^3}{2^2 4^2} - 6\frac{x^5}{2^2 4^2 6^2} + \ \ldots$$

$$\Rightarrow J_1(x) = -\frac{x}{2} + \frac{x^3}{2^2.4} - \ \ldots$$

$$\Rightarrow L^{-I}[J_1(x)] = -L^{-I}\left[\frac{x}{2}\right] + \frac{1}{2^2.4}L^{-I}[x^3] - \ \ldots$$

$$= -\frac{1}{2}\frac{1}{s^2} + \frac{1}{2^2.4}\frac{3!}{s^4} - \ \ldots$$

* * *