MATHEMATICS-I

MEAN VALUE THEOREMS FUNCTIONS OF SINGLE & SEVERAL VARIABLES

I YEAR B.TECH

By

Y. Prabhaker Reddy
Asst. Professor of Mathematics
Guru Nanak Engineering College
Ibrahimpatnam, Hyderabad.

SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

Name of the Unit	Name of the Topic
Unit-I Sequences and Series	• Basic definition of sequences and series • Convergence and divergence. • Ratio test • Comparison test • Integral test • Cauchy's root test • Raabe's test • Absolute and conditional convergence
Unit-II Functions of single variable	• Rolle's theorem • Lagrange's Mean value theorem • Cauchy's Mean value theorem • Generalized mean value theorems • Functions of several variables • Functional dependence, Jacobian • Maxima and minima of function of two variables
Unit-III Application of single variables	• Radius , centre and Circle of curvature • Evolutes and Envelopes • Curve Tracing-Cartesian Co-ordinates • Curve Tracing-Polar Co-ordinates • Curve Tracing-Parametric Curves
Unit-IV Integration and its applications	• Riemann Sum • Integral representation for lengths • Integral representation for Areas • Integral representation for Volumes • Surface areas in Cartesian and Polar co-ordinates • Multiple integrals-double and triple • Change of order of integration • Change of variable
Unit-V Differential equations of first order and their applications	• Overview of differential equations • Exact and non exact differential equations • Linear differential equations • Bernoulli D.E • Newton's Law of cooling • Law of Natural growth and decay • Orthogonal trajectories and applications
Unit-VI Higher order Linear D.E and their applications	• Linear D.E of second and higher order with constant coefficients • R.H.S term of the form *exp(ax)* • R.H.S term of the form *sin ax and cos ax* • R.H.S term of the form exp(ax) v(x) • R.H.S term of the form exp(ax) v(x) • Method of variation of parameters • Applications on bending of beams, Electrical circuits and simple harmonic motion
Unit-VII Laplace Transformations	• LT of standard functions • Inverse LT –first shifting property • Transformations of derivatives and integrals • Unit step function, Second shifting theorem • Convolution theorem-periodic function • Differentiation and integration of transforms • Application of laplace transforms to ODE
Unit-VIII Vector Calculus	• Gradient, Divergence, curl • Laplacian and second order operators • Line, surface , volume integrals • Green's Theorem and applications • Gauss Divergence Theorem and applications • Stoke's Theorem and applications

CONTENTS

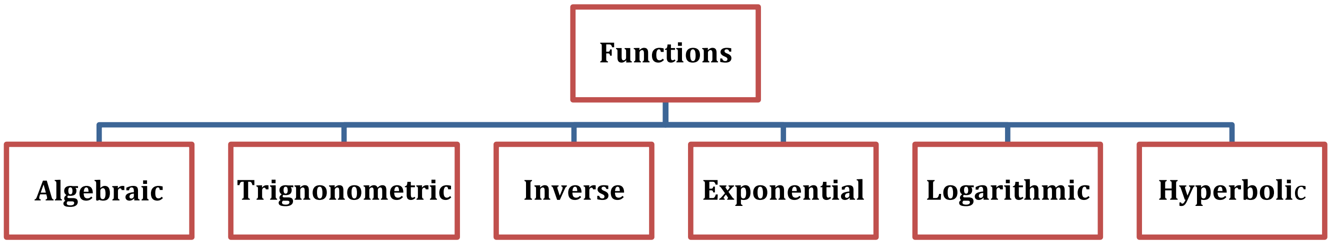
UNIT-2

Functions of Single & Several Variables

- **Rolle's Theorem(without Proof)**
- **Lagrange's Mean Value Theorem(without Proof)**
- **Cauchy's Mean Value Theorem(without Proof)**
- **Generalized Mean Value Theorem (without Proof)**
- **Functions of Several Variables-Functional dependence**
- **Jacobian**
- **Maxima and Minima of functions of two variables**

Introduction

Real valued function: Any function $\boldsymbol{f}: \boldsymbol{S} \longrightarrow \mathbb{R}$ $(\boldsymbol{S} \subseteq \mathbb{R})$ is called a Real valued function.



Limit of a function: Let $f: S \longrightarrow \mathbb{R}$ is a real valued function and $a \in S$. Then, a real number $l \in \mathbb{R}$ is said to be limit of f at $x = a$ if for each $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$ $\forall$ $x \in S$. It is denoted by $\lim_{x \longrightarrow a} f(x) = l$.

Continuity: Let $f: S \longrightarrow \mathbb{R}$ is a real valued function and $a \in S$. Then, f is said to be continuous at $x = a$ if for each $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ $\forall$ $x \in S$. It is denoted by $\lim_{x \longrightarrow a} f(x) = f(a)$.

Note: 1) The function $y = \sin x$ is continuous every where

2) Every $\cos x$ function is continuous every where.

3) Every $\tan x$ function is not continuous, but in $\left(0, \frac{\pi}{2}\right)$ the function $\tan x$ is continuous.

4) Every polynomial is continuous every where.

5) Every exponential function is continuous every where.

6) Every log function is continuous every where.

Differentiability: Let $y = f(x)$ be a function, then f is said to be differentiable or derivable at a point $x = a$, if $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

MEAN VALUE THEOREMS

Let $y = f(x)$ be a given function.

ROLLE'S THEOREM

Let $f: [a, b] \longrightarrow \mathbb{R}$ such that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable (or) derivable in (a, b)

(iii) $f(a) = f(b)$

then $\exists$ atleast one point c in (a, b) such that $f'(c) = 0$

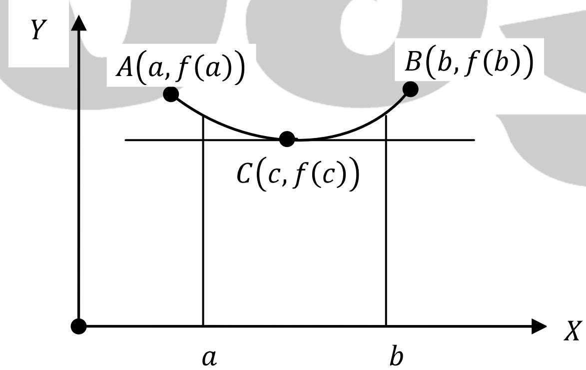
Geometrical Interpretation of Rolle's Theorem:

From the diagram, it can be observed that

(i) there is no gap for the curve $y = f(x)$ from $A(a, f(a))$ to $B(b, f(b))$. Therefore, the function is continuous

(ii) There exists unique tangent for every intermediate point between A and B

(ii) Also the ordinates of a and b are same, then by Rolle's theorem, there exists atleast one point $C(c, f(c))$ in between $A$and B such that tangent at C is parallel to X −axis.



LAGRANGE'S MEAN VALUE THEOREM

Let $f: [a, b] \longrightarrow \mathbb{R}$ such that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable (or) derivable in (a, b)

then $\exists$ atleast one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

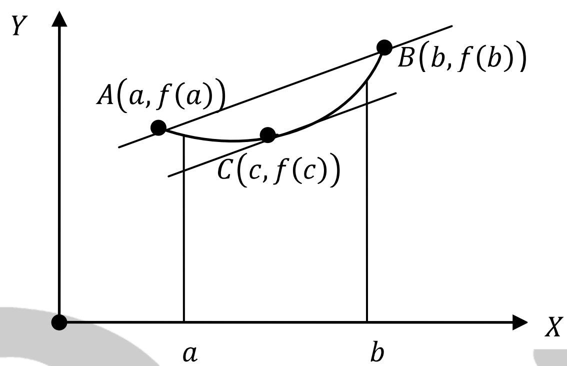
Geometrical Interpretation of Rolle's Theorem:

From the diagram, it can be observed that

(i) there is no gap for the curve $y = f(x)$ from $A\big(a, f(a)\big)$ to $B\big(b, f(b)\big)$. Therefore, the function is continuous

(ii) There exists unique tangent for every intermediate point between A and B

Then by Lagrange's mean value theorem, there exists atleast one point $C\big(c, f(c)\big)$ in between $A$and B such that tangent at C is parallel to a straight line joining the points A and B



CAUCHY'S MEAN VALUE THEOREM

Let $f: [a, b] \longrightarrow \mathbb{R}, g: [a, b] \longrightarrow \mathbb{R}$ are such that

(i) f, g are continuous in $[a, b]$

(ii) f, g are differentiable (or) derivable in (a, b)

(iii) $g'(x) \neq 0 \;\; \forall \; x \in (a, b)$

then $\exists$ atleast one point $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-g(a)}{g(b)-g(a)}$

Generalised Mean Value Theorems

Taylor's Theorem: If $f: [a, b] \longrightarrow \mathbb{R}$ such that

(i) $f, f, f'', \dots, f^{(n-1)}$ are continuous on $[a, b]$

(ii) $f, f, f'', \dots, f^{(n-1)}$ are derivable or differentiable on (a, b)

(iii) $p \in \mathbb{Z}^+$, then $\exists$ $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f(a) + \frac{(b-a)^2}{2!} f(a) + \frac{(b-a)^3}{3!} f(a) + \dots + R_n$$

where, R_n is called as Schlomilch – Roche's form of remainder and is given by

$$R_n = \frac{(b-a)^n(b-c)^{n-p}}{(n-1)!\,p}\, f^{(n)}(c)$$

- **Lagrange's form of Remainder:**

  Substituting $\boldsymbol{p = n}$ in R_n we get Lagrange's form of Remainder

  i.e. $R_n = \frac{(b-a)^n}{(n)!}\, f^{(n)}(c)$

- **Cauchy's form of Remainder:**

  Substituting $\boldsymbol{p = 1}$ in R_n we get Cauchy's form of Remainder

  i.e. $R_n = \frac{(b-a)^n(b-c)^{n-1}}{(n-1)!}\, f^{(n)}(c)$

Maclaurin's Theorem: If $f: [0, x] \longrightarrow \mathbb{R}$ such that

(i) $f, f, f'', \dots, f^{(n-1)}$ are continuous on $[0, x]$

(ii) $f, f, f'', \dots, f^{(n-1)}$ are derivable or differentiable on $(0, x)$

(iii) $p \in \mathbb{Z}^+$, then $\exists\, \theta \in (0, x)$ such that

$$f(x) = f(0) + \frac{x}{1!} f(0) + \frac{x^2}{2!} f(0) + \frac{x^3}{3!} f(0) + \dots + R_n$$

where, R_n is called as Schlomilch – Roche's form of remainder and is given by

$$R_n = \frac{x^n(1-\theta)^{n-p}}{(n-1)!\,p}\, f^{(n)}(\theta x)$$

- **Lagrange's form of Remainder:**

  Substituting $p = n$ in R_n we get Lagrange's form of Remainder

  i.e. $R_n = \frac{x^n}{(n)!}\, f^{(n)}(\theta x)$

- **Cauchy's form of Remainder:**

  Substituting $p = 1$ in R_n we get Cauchy's form of Remainder

  i.e. $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}\, f^{(n)}(\theta x)$

Jacobian

Let $u = u(x,y), v = v(x,y)$ are two functions , then the Jacobian of u and v w.r.t x and y is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$ and is defined as

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \; or \; \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Properties:

- $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$
- If u, v are functions of r, s and r, s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

Functional Dependence:

Two functions $u(x,y), v(x,y)$ are said to be functional dependent on one another if the Jacobian of (u,v) $w.r.t$ (x,y) is zero.

If they are functionally dependent on one another, then it is possible to find the relation between these two functions.

MAXIMA AND MINIMA

Maxima and Minima for the function of one Variable:

Let us consider a function $y = f(x)$

To find the Maxima and Minima, the following procedure must be followed:

Step 1: First find the first derivative and equate to zero. i.e. $\frac{dy}{dx} = 0$

Step 2: Since $y = f(x)$ is a polynomial $\Longrightarrow \frac{dy}{dx} = 0$ is a polynomial equation. By solving this equation we get roots.

Step 3: Find second derivative i.e. $\frac{d^2y}{dx^2}$

Step 4: Now substitute the obtained roots in $\frac{d^2y}{dx^2}$

Step 5: Depending on the Nature of $\frac{d^2y}{dx^2}$ at that point we will solve further. The following cases will be there.

Case (i): If $\frac{d^2y}{dx^2} < 0$ at a point say $= a$, then f has maximum at $x = a$ and the maximum value is given by $[f(x)]_{x=a} = f(a)$

Case (ii): If $\frac{d^2y}{dx^2} > 0$ at a point say $= a$, then f has minimum at $x = a$ and the minimum value is given by $[f(x)]_{x=a} = f(a)$

Case (iii): If $\frac{d^2y}{dx^2} = 0$ at a point say $= a$, then f has neither minimum nor maximum. i.e. stationary.

Maxima and Minima for the function of Two Variable

Let us consider a function $z = f(x, y)$

To find the Maxima and Minima for the given function, the following procedure must be followed:

Step 1: First find the first derivatives and equate to zero. i.e. $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$

(Here, since we have two variables, we go for partial derivatives, but not ordinary derivatives)

Step 2: By solving $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$, we get the different values of x and y.

Write these values as set of ordered pairs. i.e. (x, y)

Step 3: Now, find second order partial derivatives.

i.e., $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\, \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$

Step 4: Let us consider $l = \frac{\partial^2 z}{\partial x^2}$, $m = \frac{\partial^2 z}{\partial x\, \partial y}$, $n = \frac{\partial^2 z}{\partial y^2}$

Step 5: Now, we have to see for what values of x & y, the given function is maximum/minimum/ does not have extreme values/ fails to have maximum or minimum.

- If at a point, say (a, b) : $\boldsymbol{ln - m^2} > 0$ *and* $l < 0$, then f has **maximum** at this point and the maximum value will be obtained by substituting (a, b) in the given function.
- If at a point, say (a, b) : $\boldsymbol{ln - m^2} > 0$ *and* $l > 0$, then f has **minimum** at this point and the minimum value will be obtained by substituting (a, b) in the given function.
- If at a point, say (a, b) : $\boldsymbol{ln - m^2} < 0$, then f has **neither maximum nor minimum** and such points are called as saddle points.
- If at a point, say (a, b) : $\boldsymbol{ln - m^2 = 0}$, then f fails to have maximum or minimum and case needs further investigation to decide maxima/minima. *i.e.* **No conclusion**

Problem

1) Examine the function for extreme values $\boldsymbol{f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4}$ $(\boldsymbol{x} > 0, y > 0)$

Sol: Given $f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

The first order partial derivatives of f are given by

$\frac{\partial f}{\partial x} = \boldsymbol{3x^2 + 3y^2 - 6x}$ and

$\frac{\partial f}{\partial y} = \boldsymbol{6xy - 6y}$

Now, equating first order partial derivatives to zero, we get

$\frac{\partial f}{\partial x} = \boldsymbol{0 \Rightarrow 3x^2 + 3y^2 - 6x = 0} \quad \ldots(\boldsymbol{1})$

$\frac{\partial f}{\partial y} = \boldsymbol{0 \Rightarrow 6xy - 6y = 0} \quad \ldots(\boldsymbol{2})$

Solving (1) & (2) we get

$(\boldsymbol{2}) \Rightarrow 6y(x-1) = 0$

$\Rightarrow y = 0, x = 1$

Substituting $\boldsymbol{y = 0}$ in $(1) \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$\Rightarrow \boldsymbol{x = 0, x = 2}$

Substituting $\boldsymbol{x = 1}$ in $(1) \Rightarrow 3y^2 - 3 = 0$

$\Rightarrow \boldsymbol{y = 1, -1}$

$\therefore$ All possible set of values are $(\boldsymbol{0,0}), (\boldsymbol{2,0}), (\boldsymbol{1,1}), (\boldsymbol{1,-1})$

Now, the second order partial derivatives are given by

$\boldsymbol{l} = \frac{\partial^2 f}{\partial x^2} = \boldsymbol{6x - 6}$

$\boldsymbol{m} = \frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(\boldsymbol{6xy - 6y}) = \boldsymbol{6y}$

$\boldsymbol{n} = \frac{\partial^2 f}{\partial y^2} = \boldsymbol{6x - 6}$

Now, $\boldsymbol{ln - m^2} = (6x-6)(6x-6) - (6y)^2 = \boldsymbol{(6x-6)^2 - 36y^2}$

At a point $(0,0) \Rightarrow \boldsymbol{ln - m^2} = 36 > \boldsymbol{0}$ & $\boldsymbol{l} = -6 < \boldsymbol{0}$

$\therefore f$ has maximum at $(0,0)$ and the maximum value will be obtained by substituting $(0,0)$ in the function

i.e. $[f(x,y)]_{(0,0)} = 4$ is the maximum value.

Also, at a point $(2,0) \Rightarrow \boldsymbol{ln - m^2} = 36 > \mathbf{0}$ & $\boldsymbol{l} = 6 > 0$

$\therefore f$ has minimum at $(2,0)$ and the minimum value is $[f(x,y)]_{(2,0)} = 0$.

Also, at a point $(1,1) \Rightarrow \boldsymbol{ln - m^2 < 0}$

$\therefore f$ has neither minimum nor maximum at this point.

Again, at a point $(1,-1) \Rightarrow \boldsymbol{ln - m^2 < 0}$

$\therefore f$ has neither minimum nor maximum at this point.

Lagrange's Method of Undetermined Multipliers

This method is useful to find the extreme values (i.e., maximum and minimum) for the given function, whenever some condition is given involving the variables.

To find the Maxima and Minima for the given function using Lagrange's Method , the following procedure must be followed:

Step 1: Let us consider given function to be $f(x,y,z)$ subject to the condition $\phi(x,y,z) = 0$

Step 2: Let us define a Lagrangean function $F = f + \lambda\,\phi$, where λ is called the Lagrange multiplier.

Step 3: Find first order partial derivatives and equate to zero

$$i.e.\ \frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda\,\frac{\partial \phi}{\partial x} = 0 \quad \dots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda\,\frac{\partial \phi}{\partial y} = 0 \quad \dots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda\,\frac{\partial \phi}{\partial z} = 0 \quad \dots (3)$$

Let the given condition be $\phi(x,y,z) = 0 \quad \dots (4)$

Step 4: Solve $(1), (2), (3)$ & (4) , eliminate λ to get the values of x, y, z

Step 5: The values so obtained will give the stationary point of $f(x,y,z)$

Step 6: The minimum/maximum value will be obtained by substituting the values of x, y, z in the given function.

Problem

1) Find the minimum value of $\boldsymbol{x^2 + y^2 + z^2}$ subject to the condition $\boldsymbol{xyz = a^3}$

Sol: Let us consider given function to be $f = x^2 + y^2 + z^2$ and $\phi = xyz - a^3$

Let us define Lagrangean function $F = f + \lambda\,\phi$, where λ is called the Lagrange multiplier.

$$\Rightarrow F = (x^2 + y^2 + z^2) + \lambda\,(xyz - a^3)$$

Now, $\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + \lambda yz = 0 \quad \Longrightarrow \frac{\lambda}{2} = -\frac{x}{yz} \quad \dots (1)$

$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + \lambda xz = 0 \quad \Longrightarrow \frac{\lambda}{2} = -\frac{y}{xz} \dots (2)$

$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + \lambda xy = 0 \quad \Longrightarrow \frac{\lambda}{2} = -\frac{z}{xy} \quad \dots (3)$

Solving $(1), (2)$ & $(3) \Longrightarrow \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xz}$

Now, consider $\frac{x}{yz} = \frac{y}{xz} \Longrightarrow x^2 = y^2 \quad \dots (4)$

Again, consider $\frac{y}{xz} = \frac{z}{xz} \Longrightarrow y^2 = z^2 \dots (5)$

Again solving (4) & $(5) \Longrightarrow x^2 = y^2 = z^2$

$\Longrightarrow x = y = z$

Given $\phi = xyz - a^3 = 0$

$At\ x = y = z \Longrightarrow x^3 = a^3$

$\Longrightarrow x = a$

Similarly, we get $y = a, z = a$

Hence, the minimum value of the function is given by $(f)_{(a,a,a)} = a^2 + a^2 + a^2 = 3a^2$

$\overline{\underline{* * *}}$