MATHEMATICS-I

SEQUENCES & SERIES

I YEAR B.Tech

By

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SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

Name of the Unit	Name of the Topic
Unit-I Sequences and Series	1.1 Basic definition of sequences and series 1.2 Convergence and divergence. 1.3 Ratio test 1.4 Comparison test 1.5 Integral test 1.6 Cauchy's root test 1.7 Raabe's test 1.8 Absolute and conditional convergence
Unit-II Functions of single variable	2.1 Rolle's theorem 2.2 Lagrange's Mean value theorem 2.3 Cauchy's Mean value theorem 2.4 Generalized mean value theorems 2.5 Functions of several variables 2.6 Functional dependence, Jacobian 2.7 Maxima and minima of function of two variables
Unit-III Application of single variables	3.1 Radius , centre and Circle of curvature 3.2 Evolutes and Envelopes 3.3 Curve Tracing-Cartesian Co-ordinates 3.4 Curve Tracing-Polar Co-ordinates 3.5 Curve Tracing-Parametric Curves
Unit-IV Integration and its applications	4.1 Riemann Sum 4.3 Integral representation for lengths 4.4 Integral representation for Areas 4.5 Integral representation for Volumes 4.6 Surface areas in Cartesian and Polar co-ordinates 4.7 Multiple integrals-double and triple 4.8 Change of order of integration 4.9 Change of variable
Unit-V Differential equations of first order and their applications	5.1 Overview of differential equations 5.2 Exact and non exact differential equations 5.3 Linear differential equations 5.4 Bernoulli D.E 5.5 Newton's Law of cooling 5.6 Law of Natural growth and decay 5.7 Orthogonal trajectories and applications
Unit-VI Higher order Linear D.E and their applications	6.1 Linear D.E of second and higher order with constant coefficients 6.2 R.H.S term of the form *exp(ax)* 6.3 R.H.S term of the form *sin ax and cos ax* 6.4 R.H.S term of the form exp(ax) v(x) 6.5 R.H.S term of the form exp(ax) v(x) 6.6 Method of variation of parameters 6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion
Unit-VII Laplace Transformations	7.1 LT of standard functions 7.2 Inverse LT –first shifting property 7.3 Transformations of derivatives and integrals 7.4 Unit step function, Second shifting theorem 7.5 Convolution theorem-periodic function 7.6 Differentiation and integration of transforms 7.7 Application of laplace transforms to ODE
Unit-VIII Vector Calculus	8.1 Gradient, Divergence, curl 8.2 Laplacian and second order operators 8.3 Line, surface , volume integrals 8.4 Green's Theorem and applications 8.5 Gauss Divergence Theorem and applications 8.6 Stoke's Theorem and applications

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UNIT-1

Sequences and Series

Number System

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ \dots\}$
- Whole Numbers : $\mathbb{Z} = \ \{0, 1, 2, 3, \dots\}$
  $= \{n/n = 0 \ or \ n \in \mathbb{N}\}$
- Integers : $I = \mathbb{Z} = \{\dots, -2, -1, 0, 1\ 2, \dots\}$
  $= \{n/n = 0 \ or \ n \in \mathbb{N} \ or \ -n \in \mathbb{N}\}$
- Rational Numbers : $\mathbb{Q} = \left\{\frac{p}{q}/p \in \mathbb{Z} \ and \ q \in \mathbb{N}\right\}$
- Irrational Numbers : $\mathbb{Q}^c = \mathbb{R} - \mathbb{Q}$
  i.e., Numbers which are having Infinite non-recurring decimal points
- Real Numbers : $\mathbb{R} = \mathbb{Q} \ \cup \ \mathbb{Q}^c$
- Complex Numbers : $\mathbb{C} = \{a + ib/a, b \in \mathbb{R}\}$

Sequences

Definition: Sequence is a function whose domain is set of all Natural numbers

Ex: $S_n = \left\{\frac{n}{n+1} / \ n \in \mathbb{Z}^+\right\}$, then it is a Sequence.

since $S_1 = \frac{1}{2}, S_2 = \frac{2}{3}, S_3 = \frac{3}{4}, \dots$, where $\{1, 2, 3, \dots\}$ is domain of above function.

Notation of a Sequence: If S_n is a sequence then it is denoted by $\{S_n\} \ or \ (S_n) \ or \ \langle S_n \rangle$.

In the Sequence $\{S_n\} : S_1, S_2, S_3, \ \dots$ are called terms of the sequence $\{S_n\}$

Range of the Sequence: The set of all terms of the Sequence is called as Range of the Sequence.

Here $\{S_1, S_2, S_3, \ \dots\}$ is the Range of the Sequence $\{S_n\}$.

Note: The Basic difference between Range and co-domain is that

Co-domain means it includes all the elements in the image set, where as

Range means, it contains elements which have mapping from domain set elements.

Here if we consider the adjacent fig. :

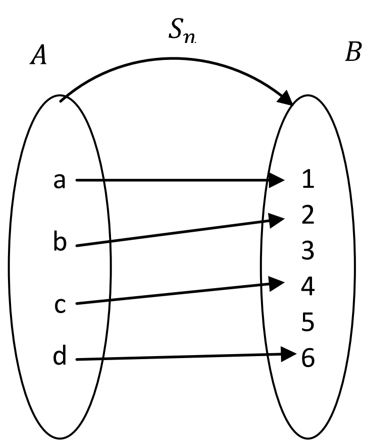
let us consider S_n to be any function from $A \ to \ B$

i.e $\ S_n: A \to B$ then,

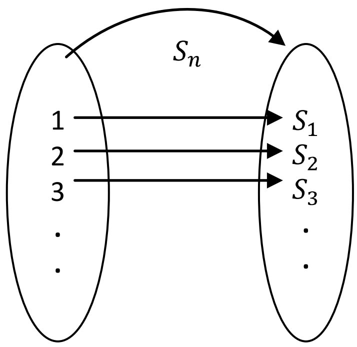
Domain: $\{a, b, c, d\}$

Co-domain: $\{1, 2, 3, 4, 5, 6\}$

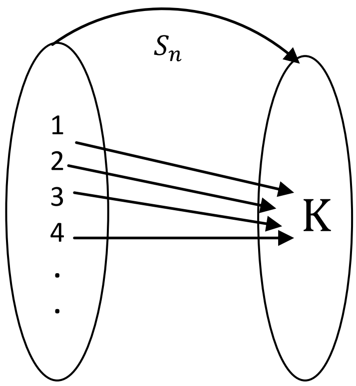
Range: $\{1, 2, 4, 6\}$



Sequence



Constant Sequence: A Sequence $\{S_n\}$ is said to be constant Sequence if $S_n = k \ \forall \ n \in \mathbb{Z}^+$



Boundedness of a Sequence

Bounded above Sequence: A sequence $\{S_n\}$ is said to be Bounded above if the Range of the Sequence is Bounded above.

(OR)

$\exists$ a real number k such that $S_n \leq k \ \forall \ n \in \mathbb{Z}^+$

In this case k is called as Upper Bound of Sequence $\{S_n\}$.

Least Upper Bound (or) Supremum of the Sequence

If u is a upper bound of the Sequence and any number less than u is not upper bound then u is called as Supremum of the Sequence $\{S_n\}$.

Note: Every decreasing Sequence is Bounded above.

Ex: 1) $\{S_n\} = \{-n \ / \ n \in \mathbb{Z}^+\}$, then it is Bounded above.

2) $\{S_n\} = \left\{\frac{1}{n} \ / \ n \in \mathbb{Z}^+\right\}$ is Bounded above. Since 1 is Supremum of it.

Bounded Below Sequence: A Sequence $\{S_n\}$ is said to be Bounded below if the Range of the Sequence is Bounded below.

(OR)

$\exists$ a real number k such that $S_n \geq k \ \forall \ n \in \mathbb{Z}^+$

In this case k is called as Lower Bound of Sequence $\{S_n\}$.

Greatest Lower Bound (or) Infimum of the Sequence

If v is a upper bound of the Sequence and any number greater than v is not Lower bound then v is called as Infimum of the Sequence $\{S_n\}$.

Note: Every increasing Sequence is Bounded below.

Ex: 1) $\{S_n\} = \{n \ / \ n \in \mathbb{Z}^+\}$, then it is Bounded below.

2) $\{S_n\} = \left\{\frac{1}{n} \ / \ n \in \mathbb{Z}^+\right\}$ is Bounded above. Since 0 is Infimum of it.

Bounded Sequence

A Sequence $\{S_n\}$ is said to be Bounded if it is Bounded above and Bounded Below.

Ex: 1) $\{S_n\} = \{(-1)^n \ / \ n \in \mathbb{Z}^+\}$, then -1 is Infimum and 1 is Supremum of the Sequence $\{S_n\}$

2) $\{S_n\} = \left\{\frac{1}{n} \ / \ n \in \mathbb{Z}^+\right\}$ is Bounded above. Since 0 is Infimum and 1 is Supremum.

Un Bounded Sequence

A Sequence which is not Bounded is called as Un Bounded Sequence.

Ex: 1) $\{S_n\} = \{-n \ / \ n \in \mathbb{Z}^+\}$, then it is Bounded above , but not Bounded below.

2) $\{S_n\} = \{n \ / \ n \in \mathbb{Z}^+\}$, then it is Bounded below , but not Bounded above.

3) $\{S_n\} = \{(-1)^n \, n \ / \ n \in \mathbb{Z}^+\}$ is neither Bounded above nor Bounded below.

Limit of a Sequence

If $\{S_n\}$ is a Sequence, then a Real number " l " is said to be limit of Sequence $\{S_n\}$ if

$\forall \, \varepsilon > 0, \exists \, m \in \mathbb{Z}^+ \ni |S_n - l| < \varepsilon \ \forall \, n \geq m$.

i.e. For larger values of n if S_n reaches to " l " (i.e. $S_n \cong l$), then we say that " l " is limit of S_n .

Above relation can be written as $\lim_{n\to\infty} S_n = l$

Ex: If $\{S_n\} = \left\{\frac{1}{n} \ / \ n \in \mathbb{Z}^+\right\}$ then prove that ' 0 ' is limit of the Sequence $\{S_n\}$.

Sol: Given that $\{S_n\} = \left\{\frac{1}{n} \ / \ n \in \mathbb{Z}^+\right\}$

$\forall \, \varepsilon > 0$, consider $|S_n - 0| < \varepsilon \ \forall \, n \geq m$

$\Rightarrow \left|\frac{1}{n}\right| < \varepsilon \ \forall \, n \geq m$

$\Rightarrow \frac{1}{n} < \varepsilon$

$\Rightarrow n > \frac{1}{\varepsilon}$

In particular let $n = m$, so that $m > \frac{1}{\varepsilon}$

$\Rightarrow \ \forall \, \varepsilon > 0$ it is possible to find a positive Integer m such that $m > \frac{1}{\varepsilon}$

Hence " 0 " is limit of Sequence $\{S_n\}$

Note: 1) For a Sequence, limit may or may not exist.

2) If limit exist then it is Unique.

Convergence Sequence

A Sequence $\{S_n\}$ is said to be converges to " l " if " l " is limit of Sequence $\{S_n\}$.

- Limit and Convergence point is same.

- If a Sequence $\{S_n\}$ is converges to " l " and $\{t_n\}$ is converges to " l' " then
  i) $\{S_n t_n\} \xrightarrow{converges} ll'$
  ii) $\{S_n \pm t_n\} \xrightarrow{converges} l \pm l'$
  iii) $\left\{\frac{S_n}{t_n}\right\} \xrightarrow{converges} \frac{l}{l'}$
- If a Sequence $\{S_n\}$ is converges to " l " and $\{t_n\}$ is not converges to " l' " then
  i) $\{S_n t_n\} \xrightarrow{converges} ll'$ may or may not converges

     Ex: $\{S_n\} = \left\{\frac{1}{n}\right\}, \{t_n\} = n$

  ii) $\{S_n \pm t_n\} \xrightarrow{converges} l \pm l'$ is not converges always
  iii) $\left\{\frac{S_n}{t_n}\right\} \xrightarrow{converges} \frac{l}{l'}$, $l' \neq 0$ may or may not converges

     Ex: $\{S_n\} = \{1\}, \{t_n\} = \{n\}$

Divergence of a Sequence

A Sequence which is not Converges is called as a Divergence Sequence.

Divergence of a Sequence is separated into 3 types.

1) Diverges to $+\infty$
2) Diverges to $-\infty$
3) Oscillating Sequence

Diverges to $+\infty$ **:** A Sequence $\{S_n\}$ is said to be diverges to $+\infty$ if $Lt\ S_n = +\infty$

Ex: $\{S_n\} = \{n \,/\, n \in \mathbb{Z}^+\}$

Diverges to $-\infty$: A Sequence $\{S_n\}$ is said to be diverges to $-\infty$ if $Lt\ S_n = -\infty$

Ex: $\{S_n\} = \{-n \,/\, n \in \mathbb{Z}^+\}$

Oscillating Sequence: Oscillating Sequence is sub-divided into two types

1) Oscillating Finite Sequence: A Sequence which is Bounded but not converges is called as Oscillating Finite Sequence.

   Ex: $\{S_n\} = \{(-1)^n \,/\, n \in \mathbb{Z}^+\}$

2) Oscillating Infinite Sequence: A Sequence which is neither Bounded nor diverges to $+\infty$ or $-\infty$ is called as Oscillating Infinite Sequence

   Ex: $\{S_n\} = \{(-1)^n\, n \,/\, n \in \mathbb{Z}^+\}$

- If a Sequence $\{S_n\}$ is diverges to " l " and $\{t_n\}$ is diverges to " l' " then
  i) $\{S_n t_n\}$ may or may not diverges
  ii) $\{S_n \pm t_n\}$ is always diverges
  iii) $\left\{\frac{S_n}{t_n}\right\}$ may or may not diverges

Cauchy's Sequence

A Sequence $\{S_n\}$ is said to be Cauchy's Sequence if $\forall\ \epsilon > 0\ \exists\ m \in \mathbb{Z}^{+} \ni \left|S_p - S_q\right| < \epsilon\ \forall\ p, q\ \geq m$.

i.e. For larger values of $p\ and\ q$ if S_p & S_q closed together, then Sequence $\{S_n\}$ is called as Cauchy's Sequence.

Note: 1) If a Sequence $\{S_n\}$ is Converges then it is Cauchy's.

2) Every Cauchy's Sequence is Bounded.

3) If a Sequence is Cauchy's then it is Converges.

Cauchy's General Principle of Convergence

A Sequence $\{S_n\}$ is Cauchy's iff it is Converges.

(Or)

A Sequence $\{S_n\}$ is Cauchy's iff $\forall\ \epsilon > 0\ \exists\ m \in \mathbb{Z}^{+} \ni \left|S_{n+p} - S_n\right| < \epsilon\ \forall\ p > 0, n\ \geq m$

Real-Life Application:

- If we consider a Simple Pendulum, in order to count the Oscillations, when it moves To and Fro, these Sequences are used.
- Let us consider an cinema theatre having 30 seats on the first row, 32 seats on the second row, 34 seats on the third row, and so on and has totally 40 rows of seats. How many seats are in the theatre?
  To solve such type of problems, we need to learn sequences and series.
  Here, we need to know how many seats are in the cinema theatre, which means we are counting things and finding a total. In other words, we need to add up all the seats on each row. Since we are adding things up, this can be looked at as a series.

INFINITE SERIES

Definition: The sum of terms of a Sequence is called as an Infinite Series.

I.e. An expression of the form $u_1 + u_2 + u_3 + \ ... + u_n + \ ...$ is called as an Infinite Series and it is denoted by $\sum_{n=1}^{\infty} u_n$

In this case u_n is called as n^{th} term of the series

Here $u_1, u_2, u_3, \ ..., u_n \ ...$ are terms of the Sequence. (previously , we have taken $S_1, S_2, S_3, \ ...$)

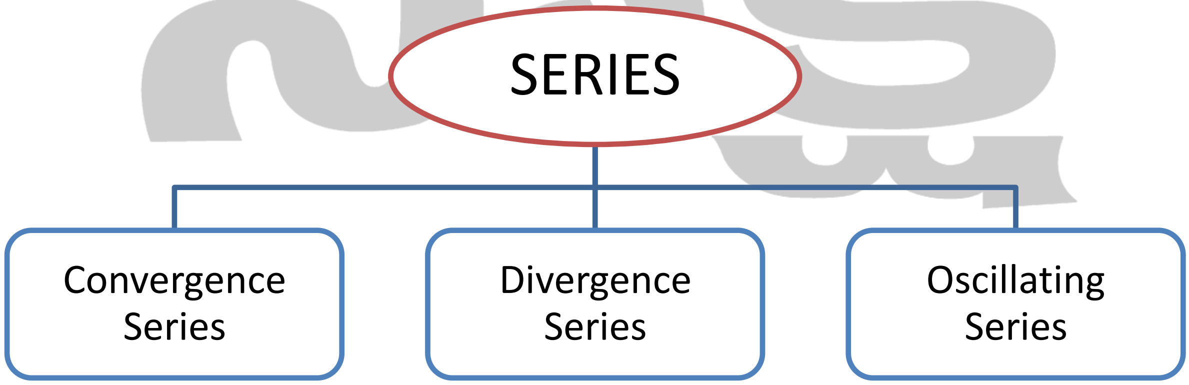
n^{th} Partial Sum of the Series

The sum of beginning n terms of the series is called as n^{th} Partial Sum of the Series

i.e. $S_n = u_1 + u_2 + u_3 + \ ... + u_n$ is called as n^{th} Partial Sum of $\sum u_n$

Note: If Sequence $\{S_n\}$ is Converges to " l " , then we say that its corresponding Series $\sum u_n$ is also converges to " l " .

Series



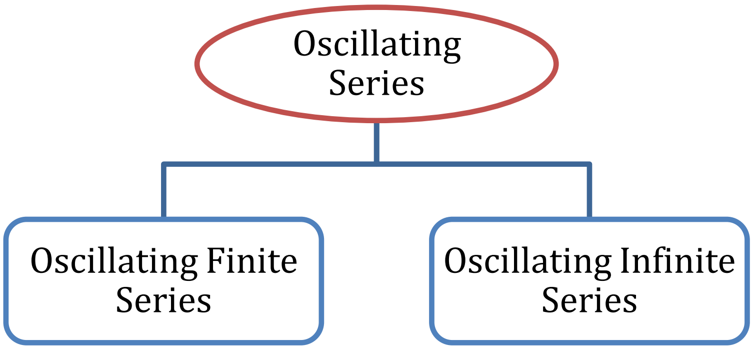
Series are separated into 3-types:

1) Convergence Series
2) Divergence Series
3) Oscillating Series

Convergence Series: If a Series is Converges to " l " i.e lim S_n $as \ n \to \infty$ is , then we say that $\sum u_n$ is convergence series.

Divergence Series: If lim $S_n = +\infty \ (or) - \infty$, then $\sum u_n$ is said to be divergence Series.

Oscillating Series



Oscillating series is separated into two types

1) **Oscillating Finite Series:** If $Lt\ S_n$ exists but not Unique then the corresponding Series $\sum u_n$ is Called as Oscillating Finite Series

   Ex: $\{(-1)^n \ \ / n \in \mathbb{Z}^+\}$

2) **Oscillating Infinite Series:** If an Infinite Series $\sum u_n$ neither diverges to $+\infty$ nor $-\infty$ then the series is called as Oscillating Infinite Series

   Ex: $\{(-1)^n n \ / n \in \mathbb{Z}^+\}$

Nature of the Series

Nature of the Series $\Longrightarrow$ we have to see whether the given Series is Converges (or) diverges

To check the nature of the series, we have following tests.

1) Geometric Series Test
2) Auxillary Series Test
3) Limit of Comparison Test
4) Cauchy's n^{th} root Test
5) De Alembert's Ratio Test
6) Raabe's Test
7) Demorgan's and Bertrand's Test
8) Logarithmic Test
9) Cauchy's Integral Test
10) Alternating Series & Leibnitz's Test
11) Absolutely and Conditionally Convergence.

Geometric Series Test

Statement: A Series of the form $\sum_{n=0}^{\infty} r^n$

$(constant)^n$

1) Converges to $\frac{1}{1-r}$ if $0 \leq r < 1$ and
2) Diverges if $r \geq 1$.

Example: The series $\sum \left(\frac{1}{2}\right)^n$ is converges [since, $o < \frac{1}{2} < 1$]

Auxillary Series Test

Statement: A Series of the form $\sum_{n=1}^{\infty} \frac{1}{n^P}$

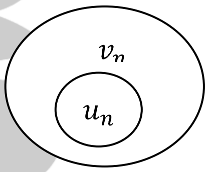
$$\frac{1}{n^{Constant}}$$

1) Converges if $P > 1$ and
2) Diverges if $P \leq 1$

Example: The series $\sum \left(\frac{1}{n}\right)^{\frac{1}{2}}$ is diverges [since, $\frac{1}{2} < 1$]

Comparison Test

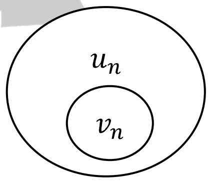
Comparison Test I: If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \leq k v_n \ \forall \ k \in \mathbb{R}^+$ and if $\sum v_n$ converges then $\sum u_n$ is also converges.



Here $u_n \leq k v_n$.

Here, if big portion v_n converges $\Longrightarrow$ small portion u_n also converges .

Comparison Test II: If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \geq k v_n \ \forall \ k \in \mathbb{R}^+$ and if $\sum v_n$ diverges then $\sum u_n$ is also diverges.



Here $u_n \geq k v_n$.

Here, if small portion v_n diverges $\Longrightarrow$ big portion u_n also diverges .

Limit of Comparison Test

Statement: If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $\left(\frac{u_n}{v_n}\right) = l$, where $l \neq 0$ then $\sum u_n$ and $\sum v_n$ are converges (or) diverges together.

Note: Here consider given series as $\sum u_n$ and we have to select $\sum v_n$ from the given series $\sum u_n$ by taking maximum term as common in both Numerator and Denominator.

i.e $\sum v_n$ is a part of $\sum u_n$.

Note: In series, we commonly use two formulas. They are

1) $Lt(\sqrt[n]{n}) = 1$
2) $Lt\left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$

Problem on Limit of comparison Test

❖ **Test for the convergence of $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$**

Sol: Let us consider $\sum u_n = \sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$\Rightarrow u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Now, take maximum term in the Numerator and Denominator as common

$\Rightarrow u_n = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$

Let us choose v_n in such a way that $Lt\left(\frac{u_n}{v_n}\right) \neq 0$

Now, let $v_n = \frac{1}{\sqrt{n}}$ so that $Lt\left(\frac{u_n}{v_n}\right) = Lt \frac{1}{\left[1+\sqrt{1+\frac{1}{n}}\right]} = \frac{1}{2} \neq 0 \quad \left[\because \text{As } n \to \infty \Rightarrow \frac{1}{n} \to 0\right]$

Now, consider $\sum v_n = \sum \frac{1}{\sqrt{n}}$, which is divergent by $p - test$

Hence, by limit of comparison test, $\sum u_n$ is also diverges.

Cauchy's n^{th} Root Test (or) n^{th} Root Test (or) Root Test

Statement: If $\sum u_n$ is a series of positive terms such that $Lt\,(u_n)^{\frac{1}{n}} = l$ then

(i) $\sum u_n$ is converges when $l < 1$

(ii) $\sum u_n$ is diverges when $l > 1$

(iii) Test fails to decide the nature of the series when $l = 1$

Note: While solving problems, Method recognition is that , if the terms of the series involves power terms, then use this method.

Problem on Cauchy's n^{th} Root Test

❖ **Test for the convergence of the series $\sum \left(1+\frac{1}{n}\right)^{-n^2}$**

Sol: Let us consider given series to be $\sum u_n = \sum \left(1+\frac{1}{n}\right)^{-n^2}$

$\Rightarrow u_n = \left(1+\frac{1}{n}\right)^{-n^2}$

Now, $(u_n)^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{-n}$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$$

$$= e^{-1} \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{-bx} = e^{ab}\right]$$

$$= \frac{1}{e} < 1$$

$\therefore$ By Cauchy's n^{th} root test, given series is convergent.

2) Test for the convergence of the series $\sum \left(\frac{nx}{1+n}\right)^n$

Sol: Let us consider given series to be $\sum u_n = \sum \left(\frac{nx}{1+n}\right)^n$

$$\Rightarrow u_n = \left(\frac{nx}{1+n}\right)^n$$

Now, $(u_n)^{\frac{1}{n}} = \frac{nx}{1+n}$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{nx}{1+n}$$

$$= x \lim_{\frac{1}{n}\to 0} \frac{1}{1+\frac{1}{n}} = x$$

∴ By Cauchy's n^{th} root test, given series is convergent if $x < 1$

and divergent if $x > 1$ and test fails if $x = 1$

Now, if $x = 1$, then $\boldsymbol{u_n} = \left(\frac{n}{1+n}\right)^n$

$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0$, which is diverges.

Hence, the given series is converges if $x < 1$ and diverges if $x \geq 1$

D'Alemberts Ratio Test (or) Ratio Test

Statement: If $\sum u_n$ is a series of positive terms such that $Lt \frac{u_{n+1}}{u_n} = l$ then

(i) $\sum u_n$ is converges when $l < 1$

(ii) $\sum u_n$ is diverges when $l > 1$

(iii) Test fails to decide the nature of the series when $l = 1$

(OR)

Statement: If $\sum u_n$ is a series of positive terms such that $Lt \frac{u_n}{u_{n+1}} = l$ then

(i) $\sum u_n$ is converges when $l > 1$

(ii) $\sum u_n$ is diverges when $l < 1$

(iii) Test fails to decide the nature of the series when $l = 1$

Remember this formula, because we use the same conditions in the next series tests also.

Note: Above method is applicable when the terms of series Involves power terms (in n) (or) factors terms (or) Factorials.

Problem on Ratio Test

❖ **Test the convergence of the series $\sum \frac{x^{2n}}{(n+1)\sqrt{n}}$**

Sol: Let us consider given series to be $\sum u_n = \sum \frac{x^{2n}}{(n+1)\sqrt{n}}$

$$\Longrightarrow u_n = \frac{x^{2n}}{(n+1)\sqrt{n}}$$

$$\Longrightarrow u_{n+1} = \frac{x^{2n+2}}{(n+2)\sqrt{n+1}}$$

Now, $\frac{u_n}{u_{n+1}} = \frac{\frac{x^{2n}}{(n+1)\sqrt{n}}}{\frac{x^{2n+2}}{(n+2)\sqrt{n+1}}} = \frac{1}{x^2}\frac{(n+2)}{(n+1)}\frac{\sqrt{n+1}}{\sqrt{n}}$

$$\Longrightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{x^2}\frac{(n+2)}{(n+1)}\frac{\sqrt{n+1}}{\sqrt{n}}$$

$$= \frac{1}{x^2}\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{1+\frac{1}{n}}$$

$$= \frac{1}{x^2}$$

∴ By ratio test, given series is convergent if $\frac{1}{x^2} > 1$ (or) $x^2 < 1$

and divergent if $\frac{1}{x^2} < 1$ (or) $x^2 > 1$ and test fails if $x^2 = 1$

Now, if $x^2 = 1$, then $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Let us choose $v_n = \frac{1}{n^{3/2}}$ so that $Lt\left(\frac{u_n}{v_n}\right) \neq 0$

$\Longrightarrow \sum v_n = \sum \frac{1}{n^{3/2}}$, which is converges by Auxillary series test

Hence, by limit of comparison test $\sum u_n$ is also converges

∴ The given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$

❖ **Test the convergence of the series $\frac{2}{1} + \frac{2.5}{1.5} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \cdots \infty$**

Sol: Let us first find the n^{th} term of the given series

Consider Numerator: $2, 5, 8, 11, \ldots$, which is in Arthemetic Progression, which Initial value $a = 2$ and with common difference $d = 3$

Hence, the n^{th} of the Numerator is $a + (n-1)d = 2 + (n-1)3 = 3n - 1$

Similarly, the n^{th} of the Denominator is $4n - 3$

Therefore, the given series will be $\sum u_n = \sum \frac{2.5.8.11\ \dots\ (3n-1)}{1.5.9.13\ \dots(4n-3)}$

$$\Rightarrow u_n = \frac{2.5.8.11\ \dots\ (3n-1)}{1.5.9.13\ \dots(4n-3)}$$

$$\Rightarrow u_{n+1} = \frac{2.5.8.11\ \dots\ (3n-1)(3n+2)}{1.5.9.13\ \dots(4n-3)(4n+1)}$$

Now, $\frac{u_n}{u_{n+1}} = \frac{2.5.8.11\ \dots\ (3n-1)}{1.5.9.13\ \dots(4n-3)} \times \frac{1.5.9.13\ \dots(4n-3)(4n+1)}{2.5.8.11\ \dots\ (3n-1)(3n+2)} = \frac{4n+1}{3n+2}$

$$\Rightarrow \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{4n+1}{3n+2}$$

$$= \lim_{n \to \infty} \frac{4+\frac{1}{n}}{3+\frac{2}{n}} = \frac{4}{3} > 1$$

∴ By ratio test, given series is convergent.

Raabe's Test

Statement: If $\sum u_n$ is a series of positive terms such that

$$Lt\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = l \text{ then}$$

(i) $\sum u_n$ is converges when $l > 1$

(ii) $\sum u_n$ is diverges when $l < 1$

(iii) Test fails to decide the nature of the series when $l = 1$

$Lt\ n\left(\underbrace{\frac{u_n}{u_{n+1}}} - 1\right) = l$

Here $\frac{u_n}{u_{n+1}}$ term is there. so all the conditions of Ratio test will be carried here also.

Note: The above test is applicable after the failure of Ratio test.

De-Morgan's and Bertrand's Test

Statement: If $\sum u_n$ is a series of positive terms such that $Lt\ \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] = l$ then

(i) $\sum u_n$ is converges when $l > 1$

(ii) $\sum u_n$ is diverges when $l < 1$

(iii) Test fails to decide the nature of the series when $l = 1$

Note: The above test is applicable after the failure of both Ratio test, and Raabe's Test.

Logarithmic Test

Statement: If $\sum u_n$ is a series of positive terms such that $Lt\, n \log \frac{u_n}{u_{n+1}} = l$ then

(i) $\sum u_n$ is converges when $l > 1$

(ii) $\sum u_n$ is diverges when $l < 1$

(iii) Test fails to decide the nature of the series when $l = 1$

$$\frac{u_n}{u_{n+1}} = e^{\blacksquare}$$

Note: The above test is applicable after the failure of both Ratio test, and when $\frac{u_n}{u_{n+1}}$ term contains/ involves e, (exponential term).

Cauchy's Integral Test

Statement: If $f(x)$ is a decreasing function of positive terms on $(1, \infty)$ then the series $\sum f(n)$ and improper Integral $\int_1^\infty f(x)\ dx$ converge (or) diverge together.

Note: 1) Here If the solution of the Improper Integral is a finite positive number, then it is converges. Otherwise, it is diverges.

2) This method is useful in finding the convergence point.

Alternating Series

Statement: An Infinite series whose terms are alternatively Positives and Negatives is called as an alternating series.

i.e. An expression of the form $u_1 - u_2 + u_3 - u_4\ \ldots + (-1)^{n-1} u_n +\ \ldots$ is called as an alternating series and it is denoted by $\sum (-1)^{n-1} u_n$.

Note: To check the convergence of the alternating series, we have Leibnitz test.

Leibnitz Test

Statement: An alternating series $\sum (-1)^{n-1} u_n$ where $u_n > 0$ is converges if

(i) $u_1 \geq u_2 \geq u_3 \geq\ \ldots$ and

(ii) $Lt\, u_n = 0$.

Ex: $\sum (-1)^{n-1} \frac{1}{n}$

Absolutely and Conditionally Convergence

Absolutely Convergence: An Infinite Series $\sum u_n$ (i.e. alternating Series $\sum (-1)^{n-1} u_n$) is said to be absolutely convergence if $\sum |u_n|$ is converges.

Ex: $\sum (-1)^{n-1} \frac{1}{n^2}$

Conditionally Convergence: If $\sum u_n$ (i.e. alternating Series $\sum(-1)^{n-1}u_n$) is converges and if $\sum|u_n|$ is diverges then $\sum u_n$ (i.e. alternating Series $\sum(-1)^{n-1}u_n$) is called as conditionally converges.

Ex: $\sum(-1)^{n-1}\frac{1}{n}$

Problem on Absolute Convergence/Conditional Convergence

- Test whether the series $\sum(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ is absolutely convergent or conditionally convergent.

Sol: Let us consider given alternating series to be $\sum(-1)^{n+1}u_n = \sum(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$

$\Rightarrow u_n = \left(\sqrt{n+1}-\sqrt{n}\right)$

Rationalizing the Numerator, we get $u_n = \left(\sqrt{n+1}-\sqrt{n}\right) \times \frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}$

$$\Rightarrow u_n = \frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

Now, take maximum term in the Numerator and Denominator as common

$\Rightarrow u_n = \frac{1}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+1\right]}$

Let us choose v_n in such a way that $Lt\left(\frac{u_n}{v_n}\right) \neq 0$

Now, let $v_n = \frac{1}{\sqrt{n}}$ so that $Lt\left(\frac{u_n}{v_n}\right) = Lt\frac{1}{\left[1+\sqrt{1+\frac{1}{n}}\right]} = \frac{1}{2} \neq 0$ $\left[\because \text{As } n \to \infty \Rightarrow \frac{1}{n} \to 0\right]$

Now, consider $\sum v_n = \sum\frac{1}{\sqrt{n}}$, which is divergent by $p-test$

Hence, by limit of comparison test, $\sum u_n$ is also diverges.

Hence, the given alternating series is conditionally convergent.

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