MATHEMATICS-I

Curvature, Evolutes & Envelopes
Curve Tracing

I YEAR B.Tech

By

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UNIT-III

CURVATURE

The shape of a plane Curve C is characterized by the degree of Bentness or Curvedness.

RADIUS OF CURVATURE

The reciprocal of the curvature of a curve is called the radius of curvature of curve.

FORMULAE FOR THE EVALUATION OF RADIUS OF CURVATURE

In this we have three types of problems

- Problems to find Radius of Curvature in Cartesian Co-ordinates
- Problems to find Radius of Curvature in Polar Co-ordinates
- Problems to find Radius of Curvature in Parametric Form.

In Cartesian Co-ordinates

Let us consider $y = f(x)$ be the given curve, then radius of curvature is given by

$$(\rho)_{at\ P} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)}$$

If the given equation of the curve is given as $x = g(y)$, then the radius of curvature is given by

$$(\rho)_{at\ P} = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2x}{dy^2}\right)}$$

In Parametric Form

(i.e. x in terms of other variable and y in terms of other variable say $t, \theta\ etc$)

If the equation of the curve is given in parametric form $x = x(t), y = y(t)$ then

$$(\rho)_{at\ P} = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right)}$$

In Polar co-ordinates

If the equation of the curve is given in polar form i.e. $r = f(\theta)$ then

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$$

where, $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$.

Radius of Curvature at origin (Newton's Theorem)

Suppose a curve is passing through the origin and x-axis or y-axis is tangent to the curve at the origin. Then the following results will be useful to find ρ at (0,0).

If x-axis is tangent at $(0,0)$, then

$$(\rho)_{at\ (0,0)} = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2}{2y} = \lim_{x\to 0} \frac{x^2}{2y}$$

If y-axis is tangent at $(0,0)$, then

$$(\rho)_{at\ (0,0)} = \lim_{\substack{y\to 0 \\ x\to 0}} \frac{y^2}{2x} = \lim_{y\to 0} \frac{y^2}{2x}$$

Note: If the given curve $y = f(x)$ passes through (0,0) and neither x-axis nor y-axis is tangent to the curve, then

Using Maclaurin's series expansion of $f(x)$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ \dots$$

$$= 0 + px + qx^2 + \ \ldots$$

then, $p = f'(0), q = \frac{f''(0)}{2}$

By this we can calculate ρ at $(0,0)$ using the formula for finding Radius of Curvature in Cartesian Co-ordinates.

Problems on Radius of Curvature

1) Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve $x^3 + y^3 = 3axy$.

Sol: Clearly, the given equation of curve belongs to Cartesian coordinates.

We know that, the radius of curvature (ρ) at any point on the curve $y = f(x)$ is given by

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Now, consider $x^3 + y^3 = 3axy$

Differentiate w.r.t x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

$$\Longrightarrow\ x^2 + y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay = 0$$

$$\Longrightarrow\ (y^2 - ax)\frac{dy}{dx} = (ay - x^2)$$

$$\Longrightarrow\ \boxed{\frac{dy}{dx} = \frac{(ay-x^2)}{(y^2-ax)}} \qquad \ldots \qquad (1)$$

Here, $\boxed{\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1}$

Again, differentiating (1), w.r.t x, we get

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\frac{dy}{dx}-2x\right)-(ay-x^2)\left(2y\frac{dy}{dx}-a\right)}{(y^2-ax)^2}$$

$$\Longrightarrow\ \boxed{\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{32}{3a}}$$

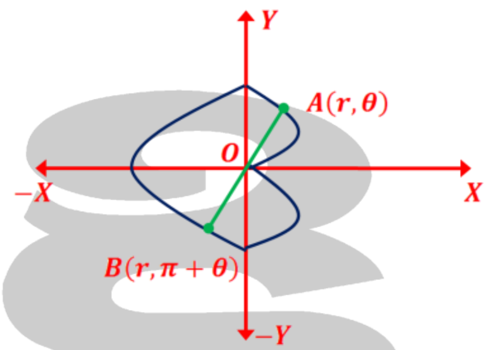
Now, the Radius of curvature at $\left(\frac{3a}{2},\frac{3a}{2}\right)$ is given by $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$$\Rightarrow (\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{[1+(-1)^2]^{\frac{3}{2}}}{\left(-\frac{32}{3a}\right)} = -\frac{2^{\frac{3}{2}}}{32}.3a = -\frac{2\sqrt{2}}{32}3a$$

$$\Rightarrow (\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{3\sqrt{2}}{16}a$$ (numerically… since radius cannot be negative)

2) If ρ_1, ρ_2 be the radius of curvature at the extremities of an chord of the cardioid $r = a(1-\cos\theta)$ which passes through the pole, show that $\rho_1{}^2 + \rho_2{}^2 = \frac{16a^2}{9}$

Sol: Let us consider the equation of the cardioids to be
$r = a(1-\cos\theta)$



Let us consider A and B to be the extremities of the chord, whose coordinates are given by $A(r,\theta)$ and $B(r,\pi+\theta)$

Let ρ_1, ρ_2 be the radius of curvature at the point A and B respectively.

Let us find ρ_1 :

We know that the radius of curvature for the curve $r = f(\theta)$ is given by

$$\boldsymbol{\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}}$$

Now, consider $r = a(1-\cos\theta)$

$\Rightarrow r_1 = a\sin\theta$ and $r_2 = a\cos\theta$

Hence, $\boldsymbol{\rho_1} = \frac{\left((a(1-\cos\theta))^2+(a\sin\theta)^2\right)^{\frac{3}{2}}}{(a(1-\cos\theta))^2+2(a\sin\theta)^2-a(1-\cos\theta)\,a\cos\theta}$

$$\Rightarrow \boldsymbol{\rho_1} = \frac{\left(a^2[(1-\cos\theta)^2+\sin^2\theta]\right)^{\frac{3}{2}}}{(a^2)\left[((1-\cos\theta))^2+2(\sin\theta)^2-(1-\cos\theta)\cos\theta\right]}$$

$$= \frac{a^3(2-2\cos\theta)^{\frac{3}{2}}}{a^2[1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}{(3-3\cos\theta)} = \frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}{3(1-\cos\theta)}$$

$$\Rightarrow \boxed{\boldsymbol{\rho_1} = \frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{1}{2}}}{3}}$$

Similarly, $\boldsymbol{\rho_2} = \frac{a\, 2^{\frac{3}{2}}(1-\cos(\pi+\theta))^{\frac{1}{2}}}{3} = \frac{a\, 2^{\frac{3}{2}}(1+\cos\theta)^{\frac{1}{2}}}{3}$

$$\Rightarrow \boxed{\boldsymbol{\rho_2} = \frac{a\, 2^{\frac{3}{2}}(1+\cos\theta)^{\frac{1}{2}}}{3}}$$

Now, consider L.H.S:

i.e. $\rho_1{}^2 + \rho_2{}^2 = \left(\frac{a\, 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{1}{2}}}{3}\right)^2 + \left(\frac{a\, 2^{\frac{3}{2}}(1+\cos\theta)^{\frac{1}{2}}}{3}\right)^2$

$$= \frac{8a^2}{9}([1-\cos\theta] + [1+\cos\theta])$$

$$= \frac{8a^2}{9}\,(2) = \frac{16a^2}{9}$$

$$\therefore \boxed{\boldsymbol{\rho_1}^2 + \boldsymbol{\rho_2}^2 = \frac{\mathbf{16a^2}}{\mathbf{9}}}$$

Hence the result .

3) Show that the radius of curvature at each point of the curve $x = a\Big(\cos t +$ logtan$t2$,

$y = a\sin t$ is inversely proportional to the length of the normal intercepted between the point on the curve and the x −axis.

Sol: Clearly, the equation of the curve is in parametric form

Let us change the problem of solving radius of curvature in parametric form to problem of solving radius of curvature in Cartesian form.

$$i.e. \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Here, given $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$

$$\Longrightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \,.\sec^2\frac{t}{2} \,.\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2}\,\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \,.\frac{1}{\cos^2\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\,\sin\frac{t}{2}\,\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right) = a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$\Longrightarrow \boxed{\frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right)}$$

Similarly, $\boxed{\frac{dy}{dt} = a\;\cos t}$

Now, $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \tan t$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \tan t}$$

Again, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t\,.\frac{dt}{dx}$

$$= \sec^2 t\,.\frac{1}{\left(\frac{dx}{dt}\right)} = \sec^2 t\,.\frac{1}{a\left(\frac{\cos^2 t}{\sin t}\right)}$$

$$= \frac{\sec^4 t\sin t}{a}$$

$$\Longrightarrow \boxed{\frac{d^2y}{dx^2} = \frac{\sec^4 t\sin t}{a}}$$

Hence, the radius of curvature for the equation of the curve $y = f(x)$ at any point is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\Longrightarrow \rho = \frac{[1+(\tan t)^2]^{\frac{3}{2}}}{\frac{\sec^4 t\sin t}{a}}$$

$$= \frac{a\left(\sec^2 t\right)^{\frac{3}{2}}}{\sec^4 t \sin t} = a\frac{\sec^3 t}{\sec^4 t \sin t}$$

$$= a\frac{\cos t}{\sin t} = a\cot t$$

$\Longrightarrow \boxed{\boldsymbol{\rho = a\cot t}} \quad \ldots \quad (1)$

Also, we know that the length of the normal is given by $L = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\Longrightarrow L = (a\sin t)\sqrt{1+\tan^2 t}$

$$= a\sin t \sec t$$

$$= a\tan t = \frac{a}{\cot t}$$

$\Longrightarrow \boxed{\boldsymbol{L = \frac{a}{\cot t}}} \quad \ldots \quad (2)$

Hence, from (1) & (2) we can say that, the radius of curvature at each point of the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ is inversely proportional to the length of the normal intercepted between the point on the curve and the x −axis.

Hence the result.

CENTRE OF CURVATURE

Definition : The Centre of Curvature at a point 'P' of a curvature is the point "C" which lies on the Positive direction of the normal at 'P' and is at a distance $'\rho'$ (in magnitude) from it.

CIRCLE OF CURVATURE

Definition: The circle of curvature at a point 'P' of a curve is the circle whose centre is at the centre of Curvature 'C' and whose radius is $'\rho'$ in magnitude

Problems on Centre of Curvature and circle of curvature

1) Find the centre of curvature at the point $\left(\frac{a}{4},\frac{a}{4}\right)$ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$. Find also the equation of the circle of curvature at that point.

Sol: We know that, if (X,Y) are the coordinates of the centre of curvature at any point $P(x,y)$ on the curve $y=f(x)$, then

$$\boxed{(X,Y)=\left(x-\frac{y_1(1+y_1^2)}{y_2}\ ,y+\frac{(1+y_1^2)}{y_2}\right)}$$

Now, given $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiate w.r.t x , we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Longrightarrow \boxed{\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}} \qquad \ldots \qquad (1)$$

Now, $\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1$

Again, Differentiate (1) w.r.t x , we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$\Longrightarrow \frac{d^2y}{dx^2}=-\left(\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}\right)$$

$$\Longrightarrow \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{4}{a}$$

Hence, the coordinates of the centre of curvature are

$$(X,Y)=\left(x-\frac{y_1(1+y_1^2)}{y_2}\ ,y+\frac{(1+y_1^2)}{y_2}\right)$$

$$=\left(\frac{a}{4}-\frac{(-1)\left(1+(-1)^2\right)}{\left({}^4/_a\right)}\ ,\frac{a}{4}+\frac{\left(1+(-1)^2\right)}{\left({}^4/_a\right)}\right)=\left(\frac{3a}{4},\frac{3a}{4}\right)$$

Hence, $\boxed{(\boldsymbol{X},\boldsymbol{Y})=\left(\frac{\boldsymbol{3a}}{\boldsymbol{4}},\frac{\boldsymbol{3a}}{\boldsymbol{4}}\right)}$

Now, radius of curvature at the point $\left(\frac{a}{4},\frac{a}{4}\right)$ is given by

$$\rho_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$=\frac{[1+(-1)^2]^{\frac{3}{2}}}{\frac{4}{a}}=\frac{a}{\sqrt{2}}$$

$\Rightarrow \boxed{\boldsymbol{\rho}_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{\boldsymbol{a}}{\sqrt{2}}}$

Hence, the equation of circle of curvature at the given point $\left(\frac{a}{4},\frac{a}{4}\right)$ is given by

$$(x-X)^2+(y-Y)^2=\rho^2$$

$\Rightarrow \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$

Hence the result.

EVOLUTE

Corresponding to each point on a curve we can find the curvature of the curve at that point. Drawing the normal at these points, we can find Centre of Curvature corresponding to each of these points. Since the curvature varies from point to point, centres of curvature also differ. The totality of all such centres of curvature of a given curve will define another curve and this curve is called the evolute of the curve.

The Locus of centres of curvature of a given curve is called the evolute of that curve.

The locus of the centre of curvature C of a variable point P on a curve is called the evolute of the curve. The curve itself is called involute of the evolute.

Here, for different points on the curve, we get different centre of curvatures. The locus of all these centres of curvature is called as Evolute.

The external curve which satisfies all these centres of curvature is called as Evolute. Here Evolute is nothing but an curve equation.

To find Evolute, the following models exist.

1) If an equation of the curve is given and If we are asked to show / prove L.H.S=R.H.S, Then do as follows.

   First find Centre of Curvature $C(X, Y)$, where $X = x - \frac{y_1[1+(y_1)^2]}{y_2}$

   $$Y = y + \frac{[1+(y_1)^2]}{y_2}$$

   And then consider L.H.S: In that directly substitute X in place of x and Y in place of y. Similarly for R.H.S. and then show that L.H.S=R.H.S

2) If a curve is given and if we are asked to find the evolute of the given curve, then do as follows:

   First find Centre of curvature $C(X, Y)$ and then re-write as

   x in terms of X and y in terms of Y. and then substitute in the given curve, which gives us the required evolute.

3) If a curve is given, which is in parametric form, then first find Centre of curvature, which will be in terms of parameter. then using these values of X and Y eliminate the parameter, which gives us evolute.

Problem

1) Find the coordinates of centre of curvature at any point of the parabola $y^2 = 4ax$ and also show its evolute is given by $27ay^2 = 4(x-2a)^2$

Sol: Given curve is $y^2 = 4ax$

We know that, if (X, Y) are the coordinates of the centre of curvature at any point $P(x, y)$ on the curve $y = f(x)$, then

$$\boxed{(X, Y) = \left(x - \frac{y_1(1+y_1^2)}{y_2}, y + \frac{(1+y_1^2)}{y_2}\right)}$$

Now, $y^2 = 4ax \Rightarrow 2y\frac{dy}{dx} = 4a$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2a}{y}}$

Also, $\frac{d^2y}{dx^2} = -\frac{2a}{y^2} \cdot \frac{dy}{dx}$

$= -\frac{2a}{y^2} \cdot \frac{2a}{y} = -\frac{4a^2}{y^3}$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}}$

$$\therefore (X, Y) = \left(x - \frac{\left(\frac{2a}{y}\right)\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}, y + \frac{\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}\right)$$

Consider, $\boldsymbol{X} = \boldsymbol{x} - \frac{\left(\frac{2a}{y}\right)\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}$

$= x + \frac{y^2}{2a}\left(1 + \frac{a}{x}\right)$

$= x + \frac{4ax}{2a}\left(\frac{x+a}{x}\right)$

$= x + 2(x+a)$

$= 3x + 2a$

$\Longrightarrow \boxed{\boldsymbol{X = 3x + 2a}}$

Consider, $\boldsymbol{Y = y +} \dfrac{\left(\boldsymbol{1+\left(\frac{2a}{y}\right)^2}\right)}{\boldsymbol{-\frac{4a^2}{y^3}}}$

$$= y - \left(1 + \left(\frac{2a}{y}\right)^2\right)\frac{y^3}{4a^2}$$

$$= y - \left(1 + \frac{4a^2}{y^2}\right)\frac{y^3}{4a^2}$$

$$= y - \left(\frac{y^2+4a^2}{y^2}\right)\frac{y^3}{4a^2}$$

$$= y - (x + a)\frac{y}{a} = -\frac{xy}{a}$$

$\Longrightarrow \boxed{\boldsymbol{Y = -\frac{xy}{a}}}$

Now, required to prove is $27aY^2 = 4(X - 2a)^3$

L.H.S $\Longrightarrow 27aY^2 = 27a\left(-\frac{xy}{a}\right)^2 = 108x^3$

R.H.S $\Longrightarrow 4(X - 2a)^3 = 4(3x + 2a - 2a)^3 = 4(3x)^3 = 108x^3$

Hence, $\mathbf{L.H.S} = \mathbf{R.H.S}$

Hence the Result.

ENVELOPE

A curve which touches each member of a given family of curves is called envelope of that family.

<u>Procedure to find envelope for the given family of curves:</u>

Case 1: Envelope of one parameter

Let us consider $y = f(x)$ to be the given family of curves.

Step 1: Differentiate w.r.t to the parameter partially, and find the value of the parameter

Step 2: By Substituting the value of parameter in the given family of curves, we get required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter, then envelope is given by $discriminant = 0$

Case 2: Envelope of two parameter

Let us consider $y = f(x)$ to be the given family of curves, and a relation connecting these two parameters

Step 1: Obtain one parameter in terms of other parameter from the given relation

Step 2: Substitute in the given equation of curve, so that the problem of two parameter converts to problem of one parameter.

Step 3: Use one parameter technique to obtain envelope for the given family of curve

Problem

1) Find the envelope of the family of straight line $y = mx + \sqrt{a^2m^2 + b^2}$, m is the parameter.

Sol: Given equation of family of curves is $y = mx + \sqrt{a^2m^2 + b^2}$

$\Longrightarrow (y - mx) = \sqrt{a^2m^2 + b^2}$

$\Longrightarrow (y - mx)^2 = (a^2m^2 + b^2)$

$\Longrightarrow y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$

$\Longrightarrow \boxed{\boldsymbol{m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0}}$

Step 1: Differentiate partially w.r.t the parameter $(i.e. m)$

$\Longrightarrow 2m(x^2 - a^2) - 2xy = 0$

$\Longrightarrow \boxed{\boldsymbol{m = \frac{xy}{(x^2 - a^2)}}}$

Step 2: Substitute the value of m in the given family of curves

$\Rightarrow m^2(x^2-a^2)-2mxy+(y^2-b^2)=0$

$\Rightarrow \left(\frac{xy}{(x^2-a^2)}\right)^2(x^2-a^2)-2\frac{xy}{(x^2-a^2)}\,xy+(y^2-b^2)=0$

$\Rightarrow \frac{x^2y^2}{x^2-a^2}-\frac{2x^2y^2}{x^2-a^2}+(y^2-b^2)=0$

$\Rightarrow -\frac{x^2y^2}{x^2-a^2}+(y^2-b^2)=0$

$\Rightarrow \frac{x^2y^2}{x^2-a^2}=(y^2-b^2)$

$\Rightarrow x^2y^2=(x^2-a^2)(y^2-b^2)$

$\Rightarrow x^2y^2=x^2y^2-x^2b^2-a^2y^2+a^2b^2$

$\Rightarrow x^2b^2+a^2y^2=a^2b^2$

$\Rightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$

∴ The envelope of the given family of straight lines is an ellipse.

2) Find the envelope of family of straight line $\frac{x}{a}+\frac{y}{b}=1$, where a, b are two parameters which are connected by the relation $a+b=c$.

Sol: Given equation of family of straight lines is $\boxed{\frac{x}{a}+\frac{y}{b}=1}$ ... (1)

Also given, $a+b=c$

$$\Rightarrow \boxed{b=c-a} \quad \ldots(2)$$

Substituting (2) in (1), we get $\frac{x}{a}+\frac{y}{c-a}=1$

Step 1: Differentiate w.r.t a partially, we get $-\frac{x}{a^2}+\frac{y}{(c-a)^2}=0$

$\Rightarrow \frac{x}{a^2}=\frac{y}{(c-a)^2}$

$\Rightarrow \frac{(c-a)^2}{a^2}=\frac{y}{x}$

$\Rightarrow \left(\frac{c-a}{a}\right)^2=\frac{y}{x}$

$\Rightarrow \frac{c-a}{a}=\sqrt{\frac{y}{x}}$

$\Rightarrow \frac{c}{a}-1=\sqrt{\frac{y}{x}}$

$$\Longrightarrow \frac{c}{a} = 1 + \sqrt{\frac{y}{x}}$$

$$\Longrightarrow \frac{c}{a} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$$

$$\Longrightarrow \boxed{\boldsymbol{a} = \frac{\boldsymbol{c\sqrt{x}}}{\boldsymbol{\sqrt{x}+\sqrt{y}}}}$$

Now, substitute the value of a in $b = c - a$

$$\Longrightarrow b = c - \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{c\sqrt{x}+c\sqrt{y}-c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$\Longrightarrow \boxed{\boldsymbol{b} = \frac{\boldsymbol{c\sqrt{y}}}{\boldsymbol{\sqrt{x}+\sqrt{y}}}}$$

Step 2: Substitute the values of a & b in the given family of curves $\frac{x}{a}+\frac{y}{b} = 1$, we get

$$\frac{x}{\left(\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}\right)} + \frac{y}{\left(\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)} = 1$$

$$\Longrightarrow \frac{x(\sqrt{x}+\sqrt{y})}{c\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{c\sqrt{y}} = 1$$

$$\Longrightarrow \frac{x(\sqrt{x}+\sqrt{y})}{\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{\sqrt{y}} = c$$

$$\Longrightarrow \left(\sqrt{x}+\sqrt{y}\right)\left(\frac{x}{\sqrt{x}}+\frac{y}{\sqrt{y}}\right) = c$$

$$\Longrightarrow \left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right) = c$$

$$\Longrightarrow \left(\sqrt{x}+\sqrt{y}\right)^2 = c$$

$$\Longrightarrow \left(\sqrt{x}+\sqrt{y}\right) = \sqrt{c}$$ is the required envelope

CURVE TRACING

Drawing a rough sketch of the curve is called as Curve Tracing.

Aim: To find the appropriate shape of a curve for the given equation.

METHOD OF TRACING CURVES

Cartesian Co-ordinates: In order to obtain general shape of the curve from the given equations, we have to examine the following properties.

1) SYMMETRY

a) If the equation contains even powers of y only, the curve is symmetrical about $x - axis$.

Example: $y^2 = 4ax$, $y^4 = x$, $xy^2 = 4(2-x)$

b) If the equation contains even powers of x only, the curve is symmetrical about $y - axis$.

Example: $x^2 = 4ay, x^4 = y$, $y = x^{100} + x^2 + 7$

c) If all the powers of x and y in the given equation are even, the curve is symmetrical about the both the axes. i.e. about the origin.

Example: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x^2 + y^2 = a^2, x^2y^2 = a^2(y^2 - x^2)$

d) If the equations of the curve is not changed by interchanging x and y, then the curve is symmetrical about the line $y = x$.

Example: $xy = c^2, x^3 + y^3 = 3axy$.

e) If the equation of a curve remains unchanged when both x and y are interchanged by $-x$ and $-y$ respectively, then the curve is symmetrical in opposite-quadrants.

Examples: $xy = c^2, x^3 + y^3 = 3ax, y = x^3$.

2) ORIGIN

If the equation of a curve is satisfied by $x = 0, y = 0$ then the curve passes through the origin.

Example: $y^2 = 4ax, x^3 + y^3 = 3axy$.

3) INTERSECTION WITH CO-ORDINATE AXIS

Put $x = 0$ in the given equation to get points of intersection with $y - axis$.

Put $y = 0$ in the given equation to get points of intersection with $x - axis$.

4) REGION

If possible write the given equation in the form of $y = f(x)$. Give values of x to make y imaginary. Let y be imaginary for the values of x lying between $x = a$ and $x = b$. Then, no part of the curve lies between $x = a$ and $x = b$. Similarly, the curve does not lies between those values of y for which x is imaginary.

5) TANGENTS

a) If the curve passes through the origin, the tangents at the origin are given by equating the lowest degree terms to zero.

Example: For the curve $x^3 + y^3 = 3axy$, the tangents are given by equating the lowest degree terms to zero. i.e. $3axy = 0 \Longrightarrow x = 0, y = 0$ are tangents at origin.

b) If the curve is not passing through the origin, the tangents at any point are given by finding $\frac{dy}{dx}$ at that point and this indicates the direction of the tangent at that point.

Note: If there are two tangents at the origin, the origin is a double point.

A) If the two tangents are real and coincident, the origin is a **cusp**.

B) If the two tangents are real and different, the origin is a **node**.

C) If the two tangents are Imaginary, the origin is a conjugate point or Isolated point.

6) EXTENSION OF THE CURVE TO INFINITY

Give values to x for which the value of y is Infinity and also give values to y for which x is Infinity. These values indicate the direction in which the curve extends to Infinity.

7) ASYMPTOTES

An Asymptote is a straight line which cuts a curve in two points, at an infinite distance from the origin and yet is not itself wholly at Infinity.

To find Asymptotes

a) Asymptotes parallel to axis

$\boldsymbol{x-axis}$: Asymptotes parallel to x-axis are obtained by equating the coefficient of the highest power of x to zero.

$\boldsymbol{y-axis}$: Asymptotes parallel to y-axis are obtained by equating the coefficient of the highest power of y to zero.

b) Oblique Asymptotes: (Asymptotes which are not parallel to axis)

Let $y = mx + c$ be an asymptote. Put $y = mx + c$ in the given equation of the curve. Equate the coefficients of highest powers of x to zero and solve for m and c.

Problems on Curve Tracing - Cartesian Coordinates

1) Trace the curve $x^3 + y^3 = 3axy$

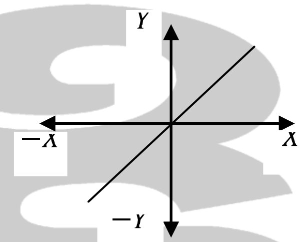
Sol: In order to trace a curve, we need to check the following properties

- ✓ Symmetry
- ✓ Origin
- ✓ Intersection with coordinate axis
- ✓ Region
- ✓ Tangents
- ✓ Intersection of curve to ∞
- ✓ Asymptotes

Let us consider given equation of curve to be $\boldsymbol{F(x,y) = x^3 + y^3 = 3axy}$

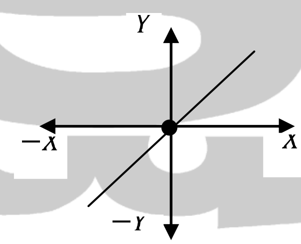
<u>Symmetry:</u> If we interchange x and y, the equation of curve is not changing.

Hence, the curve is symmetric about the line $y = x$



<u>Origin:</u> If we substitute $x = 0$ and $y = 0$, the equation of curve is satisfied.

Hence, we can say that the curve passes through the origin. $i.e.(0,0)$



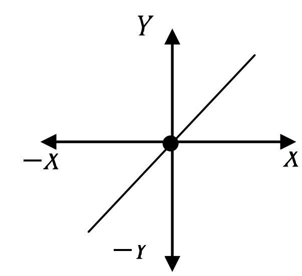
<u>Intersection with coordinate axis:</u>

Put $y = 0 \Longrightarrow x = 0$

$\therefore$ The curve meets the X −axis only at the Origin

Again $x = 0 \Longrightarrow y = 0$

$\therefore$ The curve meets the Y −axis only at the Origin



<u>Region:</u> Consider $x^3 + y^3 = 3axy$

- x is positive $\Longrightarrow$ y is also positive

  $\therefore$ The curve lies in 1st Quadrant
- x is negative $\Longrightarrow$ y is positive

  $\therefore$ The curve lies in 2nd Quadrant
- x is negative & y is negative

Then the equation of curve is not satisfied

$\therefore$ The curve does not lie in 3rd Quadrant

- x is positive $\Longrightarrow y$ is negative

$\therefore$ The curve lies in 4th Quadrant

Tangents: The given equation of curve $\boldsymbol{F(x,y) = x^3 + y^3 - 3axy = 0}$ passes through Origin.

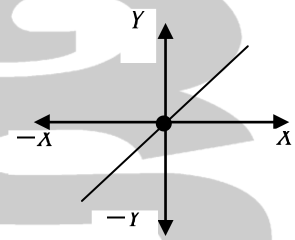
Hence, the tangents at origin are given by equating lowest degree terms to zero.

i.e., $\boldsymbol{3axy = 0} \Longrightarrow \boldsymbol{xy = 0}$

$\Longrightarrow \boldsymbol{x = 0, y = 0}$

$\therefore$ There exists two tangents namely, $\boldsymbol{x = 0, y = 0}$

Here, the two tangents at origin are real and distinct. Hence, we get a node



Now, let us check, at what points the curve meets the line $\boldsymbol{y = x}$

$\Longrightarrow x^3 + x^3 - 3ax^2 = 0$

$\Longrightarrow x^2(2x - 3a) = 0$

$\therefore x = 0, x = \frac{3a}{2}$

$\Longrightarrow y = 0, y = \frac{3a}{2}$

Hence, the curve meets at $\boldsymbol{(0,0)}$, $\boldsymbol{\left(\frac{3a}{2}, \frac{3a}{2}\right)}$

Extension of the curve to Infinity:

As $y \longrightarrow \infty \Longrightarrow \boldsymbol{x} \longrightarrow -\boldsymbol{\infty}$

$\therefore$ The curve extends to infinity in the second quadrant.

Also, as $x \longrightarrow \infty \Longrightarrow \boldsymbol{y} \longrightarrow -\boldsymbol{\infty}$

The curve extends to infinity in the fourth quadrant.

Asymptotes:

Asymptotes parallel to axis:

No Asymptote is parallel to $X - axis$

No Asymptote is parallel to $Y - axis$

Asymptotes not parallel to axis (Oblique Asymptotes)

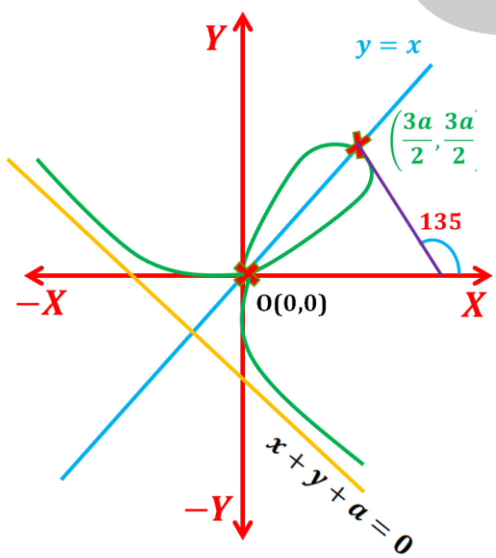
Let us consider $y = mx + c$ to be the required asymptote.

Now, substitute in the given equation of the curve and solving for m and c, we get

$$m = -1, c = -a$$

Hence, the required asymptote is $x + y + a = 0$

Hence the shape of the curve is as follows



PROCEDURE FOR TRACING CURVES IN POLAR CO-ORDINATES

1) SYMMETRY

i) If the equation does not alter by changing θ to $-\theta$, the curve is symmetrical about the initial line or X-axis.

Example: $r = a(1 + \cos\theta)$

ii) If the equation does not alter by changing r to $-r$, the curve is symmetrical about the pole.

Example: $r^2 = a^2 \cos 2\theta$

iii) If the equation of the curve remains unaltered when θ is changed to $\pi - \theta$ or by changing θ to $-\theta, r$ to $-r$, the curve is symmetric about the line $\theta = \frac{\pi}{2}$ or Y-axis.

Example: $r = a \sin 3\theta$

iv) If the equation of the curve remains unaltered when θ is changed to $\frac{\pi}{2} - \theta$, then the curve is symmetrical about the line $\theta = \frac{\pi}{4}$ or $y = x$.

Example: $r = a \sin 2\theta$

v) If the equation of the curve does not alter by changing θ to $\frac{3\pi}{2} - \theta$, then the curve is symmetrical about the line $\theta = \frac{3\pi}{4}$ or $y = -x$.

Example: $r = a \sin 2\theta$.

2) Discussion for r and θ

Give certain values to θ and find the corresponding values of r and then plot the points. Sometimes it is inconvenient to find the corresponding values of r for certain values of θ. In such cases, a particular region for θ may be considered and find out whether r increases or decreases in that region.

For example: It is inconvenient to find the value of r for $\theta = 32^o$ but it is equal to know whether r increases or decreases in the region from 0 to 45^0 in which $\theta = 32^o$ value is also included.

3) Region

No part of the curve exists for those values of θ which make corresponding value of r imaginary.

4) Tangents

Find $tan\, \emptyset = r\, \frac{d\theta}{dr}$, where $\emptyset$ is the angle between the radius vector OP and the tangent at $P\ (r, \theta)$. It will indicate the direction of the tangents at any point $P\ (r, \theta)$.

5) Asymptotes

Find the value of θ which makes r infinity. The curve has an asymptote in that direction.

* * *