MATHEMATICS-I

DIFFERENTIAL EQUATIONS-II

I YEAR B.TECH

By

Y. Prabhaker Reddy
Asst. Professor of Mathematics
Guru Nanak Engineering College
Ibrahimpatnam, Hyderabad.

SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

Name of the Unit	Name of the Topic
Unit-I Sequences and Series	1.1 Basic definition of sequences and series 1.2 Convergence and divergence. 1.3 Ratio test 1.4 Comparison test 1.5 Integral test 1.6 Cauchy's root test 1.7 Raabe's test 1.8 Absolute and conditional convergence
Unit-II Functions of single variable	2.1 Rolle's theorem 2.2 Lagrange's Mean value theorem 2.3 Cauchy's Mean value theorem 2.4 Generalized mean value theorems 2.5 Functions of several variables 2.6 Functional dependence, Jacobian 2.7 Maxima and minima of function of two variables
Unit-III Application of single variables	3.1 Radius , centre and Circle of curvature 3.2 Evolutes and Envelopes 3.3 Curve Tracing-Cartesian Co-ordinates 3.4 Curve Tracing-Polar Co-ordinates 3.5 Curve Tracing-Parametric Curves
Unit-IV Integration and its applications	4.1 Riemann Sum 4.3 Integral representation for lengths 4.4 Integral representation for Areas 4.5 Integral representation for Volumes 4.6 Surface areas in Cartesian and Polar co-ordinates 4.7 Multiple integrals-double and triple 4.8 Change of order of integration 4.9 Change of variable
Unit-V Differential equations of first order and their applications	5.1 Overview of differential equations 5.2 Exact and non exact differential equations 5.3 Linear differential equations 5.4 Bernoulli D.E 5.5 Newton's Law of cooling 5.6 Law of Natural growth and decay 5.7 Orthogonal trajectories and applications
Unit-VI Higher order Linear D.E and their applications	6.1 Linear D.E of second and higher order with constant coefficients 6.2 R.H.S term of the form *exp(ax)* 6.3 R.H.S term of the form *sin ax and cos ax* 6.4 R.H.S term of the form exp(ax) v(x) 6.5 R.H.S term of the form exp(ax) v(x) 6.6 Method of variation of parameters 6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion
Unit-VII Laplace Transformations	7.1 LT of standard functions 7.2 Inverse LT –first shifting property 7.3 Transformations of derivatives and integrals 7.4 Unit step function, Second shifting theorem 7.5 Convolution theorem-periodic function 7.6 Differentiation and integration of transforms 7.7 Application of laplace transforms to ODE
Unit-VIII Vector Calculus	8.1 Gradient, Divergence, curl 8.2 Laplacian and second order operators 8.3 Line, surface , volume integrals 8.4 Green's Theorem and applications 8.5 Gauss Divergence Theorem and applications 8.6 Stoke's Theorem and applications

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UNIT-6

Differential Equations-II

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

A D.E of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n = Q(x)$ is called as a Linear Differential Equation of order n with constant coefficients, where $P_1, P_2, \ldots, P_n$ are Real constants.

Let us denote $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, \frac{d^3}{dx^3} \equiv D^3$ etc, then above equation becomes

$(D^n + P_1 D^{n-1} + \ldots + P_{n-1} D + P_n) y = Q(x)$ which is in the form of $f(D)y = Q(x)$, where

$f(D) = (D^n + P_1 D^{n-1} + \ldots + P_{n-1} D + P_n)$.

The **General Solution** of the above equation is $y = C.F + P.I$ C.F= Complementary Function

(or) $\boldsymbol{y = y_c + y_p}$ P.I= Particular Function

Now, to find Complementary Function y_c, we have to find Auxillary Equation

Auxillary Equation: An equation of the form $f(m) = 0$ is called as an Auxillary Equation.

Since $f(m) = 0$ is a polynomial equation, by solving this we get roots. Depending upon these roots we will solve further.

Complimentary Function: The General Solution of $f(D)y = 0$ is called as Complimentary Function and it is denoted by y_c

Depending upon the Nature of roots of an Auxillary equation we can define y_c

Case I: If the Roots of the A.E are real and distinct, then proceed as follows

If α_1, α_2 are two roots which are real and distinct (different) then complementary function is given by $y_c = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$

Generalized condition: If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are real and distinct roots of an A.E then

$$y_c = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + c_3 e^{\alpha_3 x} + \ldots + c_n e^{\alpha_n x}$$

Case II: If the roots of A.E are real and equal then proceed as follows

If $\alpha_1 = \alpha_2 = \alpha$ then $y_c = e^{\alpha x}(c_1 + c_2 x)$

Generalized condition: If $\alpha_1 = \alpha_2, = \alpha_3, = \ldots, = \alpha_n = \alpha$ then

$$y_c = e^{\alpha x}(c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1})$$

Case III: If roots of A.E are Complex conjugate i.e. $m = \alpha \pm i\beta$ then

$$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

(Or) $y_c = c_1 e^{\alpha x} \cos(\beta x + c_2)$

(Or) $y_c = c_1 e^{\alpha x} \sin(\beta x + c_2)$

Note: For repeated Complex roots say, $m = \alpha \pm i\beta,\ \alpha \pm i\beta$

$$y_c = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$$

Case IV: If roots of A.E are in the form of Surds i.e. $m = \alpha \pm \sqrt{\beta}$, where β is not a perfect square then,

$$y_c = e^{\alpha x}\left(c_1 \cosh\sqrt{\beta}x + c_2 \sin\sqrt{\beta}x\right)$$

$$\text{(Or)}\quad y_c = c_1 e^{\alpha x}\cos\left(\sqrt{\beta}x + c_2\right)$$

$$\text{(Or)}\quad y_c = c_1 e^{\alpha x}\sin\left(\sqrt{\beta}x + c_2\right)$$

Note: For repeated roots of surds say, $m = \alpha \pm \sqrt{\beta},\ \alpha \pm \sqrt{\beta}$

$$y_c = e^{\alpha x}\left[(c_1 + c_2 x)\cosh\sqrt{\beta}x + (c_3 + c_4 x)\sinh\sqrt{\beta}x\right]$$

Particular Integral

The evaluation of $\frac{1}{f(D)}Q(x)$ is called as Particular Integral and it is denoted by y_p

i.e. $y_p = \frac{1}{f(D)}Q(x)$

Note: The General Solution of $f(D)y = Q(x)$ is called as Particular Integral and it is denoted by y_p

Methods to find Particular Integral

Method 1: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = e^{ax}$, where a is a constant.

We know that $y_p = \frac{1}{f(D)}Q(x)$

$$= \frac{1}{f(D)}e^{ax}$$

$\therefore\ y_p = \frac{1}{f(a)}e^{ax}$ if $f(a) \neq 0$ — Directly substitute a in place of D

$= e^{ax}\frac{1}{f(D+a)}$ if $f(a) = 0$ — Taking e^{ax} outside the operator by replacing D with $D + a$

Depending upon the nature of $f(D + a)$ we can proceed further.

Note: while solving the problems of the type $\frac{1}{f(D)}Q(x)$, where Denominator =0, Rewrite the Denominator quantity as product of factors, and then keep aside the factor which troubles us. I.e the term which makes the denominator quantity zero, and then solve the remaining quantity. finally substitute $D + a$ in place of D.

Method 2: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = \sin ax\ (or) \cos ax$, a is constant

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} \sin ax \ \ (or) \ \frac{1}{f(D)} \cos ax$$

Let us consider $f(D) = \emptyset(D^2)$, then the above equation becomes

$$\therefore \ y_p = \frac{1}{\emptyset(D^2)} \sin ax \ \ (or) \ \frac{1}{\emptyset(D^2)} \cos ax$$

Now Substitute $D^2 = -a^2$ if $\emptyset(D^2) \neq 0$

If $\emptyset(D^2) = 0$ then i.e. $y_p = \frac{1}{D^2+a^2} \sin ax \ \ (or) \ \frac{1}{D^2+a^2} \cos ax$

Then $y_p = \frac{x}{2} \int \sin ax \ dx \ \ (or) \ \frac{x}{2} \int \cos ax \ dx$ respectively.

Method 3: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = x^k, k \in \mathbb{Z}^+$

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} x^k$$

Now taking Lowest degree term as common in $f(D)$, above relation becomes $y_p = \frac{1}{[1+\emptyset(D)]} x^k$

$$\Rightarrow y_p = [1 + \emptyset(D)]^{-1} x^k$$

Expanding this relation upto k^{th} derivative by using Binomial expansion and hence get y_p

Important Formulae:

1) $(1 - D)^{-1} = 1 + D + D^2 + \ ...$
2) $(1 + D)^{-1} = 1 - D + D^2 - \ ...$
3) $(1 - D)^{-2} = 1 + 2D + 3D^2 + \ ...$
4) $(1 + D)^{-2} = 1 - 2D + 3D^2 - \ ...$
5) $(1 - D)^{-3} = 1 + 3D + 6D^2 + \ ...$
6) $(1 + D)^{-3} = 1 - 3D + 6D^2 - \ ...$

Method 4: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = e^{ax} V$, where V is a function of x and a is constant

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} e^{ax} V$$

In such cases, first take e^{ax} term outside the operator, by substituting $D + a$ in place of D.

$$\Rightarrow y_p = e^{ax} \frac{1}{f(D+a)} V$$

Depending upon the nature of V we will solve further.

Method 5: Method to find P.I of $f(D)y = Q(x)$ where $Q(x) = x^k . v$, where $k \epsilon \mathbb{Z}^+$, v is any function of x (i.e. $v = sin\ ax\ (or) cos\ ax$)

We know that $y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{f(D)} x^k . v$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta \rightarrow R.P$, $\sin\theta \rightarrow I.P$

$\cos\theta = R.P(e^{i\theta})$

$\sin\theta = I.P(e^{i\theta})$

Case I: Let $k = 1$, then $y_p = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} v$

Case II: Let $k \neq 1$ and $v = \sin ax$

$$y_p = \frac{1}{f(D)} x^k \sin ax$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$y_p = \frac{1}{f(D)} x^k\ I.P(e^{iax})$

$= I.P \frac{1}{f(D)} x^k e^{iax}$

$= I.P\ e^{iax} \frac{1}{f(D+ia)} x^k$

By using previous methods we will solve further

Finally substitute $e^{iax} = \cos ax + i \sin ax$

Let $k \neq 1$ and $v = \sin ax$

$$y_p = \frac{1}{f(D)} x^k \cos ax$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$y_p = \frac{1}{f(D)} x^k\ R.P(e^{iax})$

$= R.P \frac{1}{f(D)} x^k e^{iax}$

$= R.P\ e^{iax} \frac{1}{f(D+ia)} x^k$

By using previous methods we will solve further

Finally substitute $e^{iax} = \cos ax + i \sin ax$

General Method

To find P.I of $f(D)y = Q(x)$ where $Q(x)$ is a function of x

We know that $y_p = \frac{1}{f(D)} Q(x)$

Let $f(D) = (D - \alpha)$ then $y_p = \frac{1}{(D-\alpha)} Q(x)$

$$= e^{\alpha x} \int e^{-\alpha x} Q(x)\, dx$$

Similarly, $f(D) = (D + \alpha)$ then $y_p = \frac{1}{(D+\alpha)} Q(x)$

$$= e^{-\alpha x} \int e^{\alpha x} Q(x)\, dx$$

Note: The above method is used for the problems of the following type

- $(D^2 - 3D + 2)y = \sin(e^{-x})$
- $(D^2 + a^2)y = \sec ax$
- $(D^2 + a^2)y = \tan ax$
- $(D^2 + a^2)y = \operatorname{cosec} ax$

Cauchy's Linear Equations (or) Homogeneous Linear Equations

A Differential Equation of the form $[x^n D_n + A_1 x^{n-1} D_{n-1} + \ \dots + A_{n-1} x D + A_n] y = Q(x)$ where $D \equiv \frac{d}{dx}$ is called as n^{th} order Cauchy's Linear Equation in terms of dependent variable y and independent variable x, where $A_1, A_2, A_3, \dots, A_n$ are Real constants and $D \equiv \frac{d}{dx}$.

Substitute $x = e^z \implies \log x = z$ and

$xD = \theta, x^2 D^2 = \theta(\theta - 1), x^3 D^3 = \theta(\theta - 1)(\theta - 2), \dots \quad \theta \equiv \frac{d}{dz}$

Then above relation becomes $(\theta)y = Q(z)$, which is a Linear D.E with constant coefficients. By using previous methods, we can find Complementary Function and Particular Integral of it, and hence by replacing z with $\log x$ we get the required General Solution of Cauchy's Linear Equation.

Legendre's Linear Equation

An D.E of the form $[(ax + b)^n D_n + A_1 (ax + b)^{n-1} D_{n-1} + \ \dots + A_{n-1}(ax + b)D + A_n] y = Q(x)$ is called as Legendre's Linear Equation of order , where $a, b, A_1, A_2, A_3, \dots, A_n$ are Real constants.

Now substituting, $(ax + b) = e^z \implies z = \log(ax + b)$

$(ax + b)D = a\theta, (ax + b)^2 D^2 = a^2\theta(\theta - 1), (ax + b)^3 D^3 = a^3\theta(\theta - 1)(\theta - 2), \dots \quad \theta \equiv \frac{d}{dz}$

Then, above relation becomes $f(\theta)y = Q(z)$ which is a Linear D.E with constant coefficients. By using previous methods we can find general solution of it and hence substituting $z = \log(ax + b)$ we get the general solution of Legendre's Linear Equation.

Method of Variation of Parameters

To find the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)$

Let us consider given D.E $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x) \longrightarrow$ **(I)**

Let the Complementary Function of above equation is $y_c = c_1 u + c_2 v$

Let the Particular Integral of it is given by $y_p = Au + Bv$, where

$$A = \int \frac{-vR}{uv' - vu'} dx \qquad B = \int \frac{uR}{uv' - vu'}$$

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