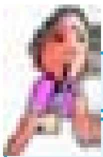
CHAPTER 4

Pair of Linear Equations in Two Variables

4.1 Introduction

One day Siri went to a book shop with her father and bought 3 notebooks and 2 pens. Her father paid ₹80 for them. Her friend Laxmi liked the notebooks and pens so she bought 4 notebooks and 3 pens of the same kind for ₹110 and again her classmates Rubina liked the pens and Joseph liked the notebooks. They asked Siri the cost of one pen and one notebook. But, Siri did not know the cost of one notebook and one pen separately. How can they find the costs of these items?

In this example, the cost of a notebook and a pen are not known. These are unknown quantities. We come across many such situations in our day-to-day life.



Think - Discuss

Two situations are given below:

(i) The cost of 1kg potatoes and 2kg tomatoes was ₹30 on a certain day. After two days, the cost of 2kg potatoes and 4kg tomatoes was found to be ₹66.

(ii) The coach of a cricket team of M.K.Nagar High School buys 3 bats and 6 balls for ₹3900. Later he buys one more bat and 2 balls for ₹1300.

Identify the unknowns in each situation. We observe that there are two unknowns in each case.

4.1.1 How Do We Find Unknown Quantities?

In the introduction, Siri bought 3 notebooks and 2 pens for ₹80. How can we find the cost of a notebook or the cost of a pen?

Rubina and Joseph tried to guess. Rubina said that price of each notebook could be ₹25. Then three notebooks would cost ₹75, the two pens would cost ₹5 and each pen could be for ₹2.50.

Joseph felt that ₹2.50 for one pen was too little. It should be at least ₹16. Then the price of each notebook would also be ₹16.

We can see that there can be many possible values for the price of a notebook and of a pen so that the total cost is ₹80. So, how do we find cost price at which Siri and Laxmi bought them? By only using Siri's situation, we cannot find the costs. We have to use Laxmi's situation also.

4.1.2 Using Both Equations Together

Laxmi also bought the same types of notebooks and pens as Siri. She paid ₹110 for 4 notebooks and 3 pens.

So, we have two situations which can be represented as follows:

(i) Cost of 3 notebooks + 2 pens = ₹80.

(ii) Cost of 4 notebooks + 3 pens = ₹110.

Does this help us find the cost of a pen and a notebook?

Consider the prices mentioned by Rubina. If the price of one notebook is ₹25 and the price of one pen is ₹2.50 then,

The cost of 4 notebooks would be : $4 \times 25 =$ ₹100

And the cost for 3 pens would be : $3 \times 2.50 =$ ₹7.50

If Rubina is right then Laxmi should have paid ₹ 100 +₹ 7.50 = ₹ 107.50 but she paid ₹110.

Now, consider the prices mentioned by Joseph. Then,

The cost of 4 notebooks, if one is for ₹16, would be : $4 \times 16 =$ ₹64

And the cost for 3 pens, if one is for ₹16, would be : $3 \times 16 =$ ₹ 48

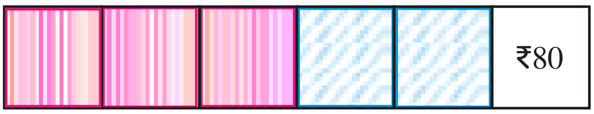
If Joseph is right then Laxmi should have paid ₹64 + ₹48 = ₹112 but this is more than the price she paid.

So what do we do? How to find the exact cost of the notebook and the pen?

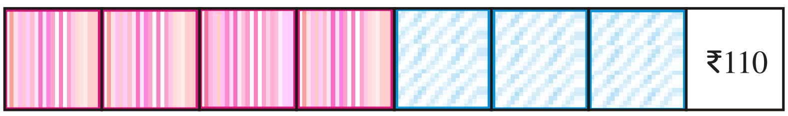
If we have only one equation but two unknowns (variables), we can find many solutions. So, when we have two variables, we need at least two independent equations to get a unique solution. One way to find the values of unknown quantities is by using the Model method. In this method, rectangles or portions of rectangles are often used to represent the unknowns. Let us look at the first situation using the model method:

Step-1 : Represent notebooks by  and pens by .

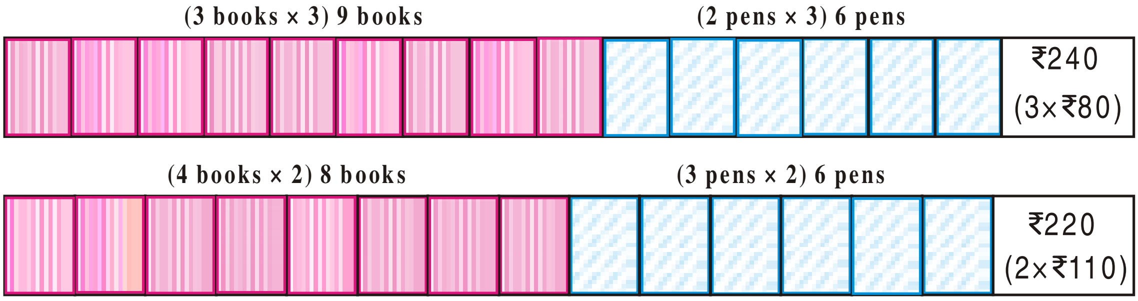
Siri bought 3 books and 2 pens for ₹80.



Laxmi bought 4 books and 3 pens for ₹110.



Step-2 : Increase (or decrease) the quantities in proportion to make one of the quantities equal in both situations. Here, we make the number of pens equal.



In Step 2, we observe a simple proportional reasoning.

Since Siri bought 3 books and 2 pens for ₹80, so for 9 books and 6 pens:

$3 \times 3 = 9$ books and $3 \times 2 = 6$ pens, the cost will be $3 \times 80 =$ ₹240 (1)

Similarly, Laxmi bought 4 books and 3 pens for ₹110, so:

$2 \times 4 = 8$ books and $2 \times 3 = 6$ pens will cost $2 \times 110 =$ ₹220 (2)

After comparing (1) and (2), we can easily observe that 1 extra book costs

₹240 - ₹ 220 = ₹20. So one book is of ₹20.

Siri bought 3 books and 2 pens for ₹80. Since each book costs ₹20, 3 books cost ₹ 60. So the cost of 2 pens become ₹ 80 - ₹ 60 = ₹ 20.

So, cost of each pen is ₹20 ÷ 2 = ₹10.

Let us try these costs in Laxmi's situation. 4 books will cost ₹ 80 and three pens will cost ₹ 30 for a total of ₹ 110, which is true.

From the above discussion and calculation, it is clear that to get exactly one solution (unique solution) we need at least two independent linear equations in the same two variables.

In general, an equation of the form $ax + by + c = 0$ where a, b, c are real numbers and where at least one of a or b is not zero, is called a linear equation in two variables x and y. [We often write this condition as $a^2 + b^2 \neq 0$].

Try This

Mark the correct option in the following questions:

1. Which of the following equations is not a linear equation?

a) $5 + 4x = y + 3$ b) $x + 2y = y - x$

c) $3 - x = y^2 + 4$ d) $x + y = 0$

2. Which of the following is a linear equation in one variable?

   a) $2x + 1 = y - 3$ b) $2t - 1 = 2t + 5$

   c) $2x - 1 = x^2$ d) $x^2 - x + 1 = 0$

3. Which of the following numbers is a solution for the equation $2(x + 3) = 18$?

   a) 5 b) 6 c) 13 d) 21

4. The value of x which satisfies the equation $2x - (4 - x) = 5 - x$ is

   a) 4.5 b) 3 c) 2.25 d) 0.5

5. The equation $x - 4y = 5$ has

   a) no solution b) unique solution

   c) two solutions d) infinitely many solutions

4.2 Solutions of Pairs of Linear Equations in Two Variables

In the introductory example of notebooks and pens, how many equations did we have? We had two equations or a pair of linear equations in two variables. What do we mean by the solution for a pair of linear equations?

The pair of values of the variables x and y which together satisfy each one of the equations is called the solution for a pair of linear equations.

4.2.1 Graphical Method of Finding Solution of a Pair of Linear Equations

What will be the number of solutions for a pair of linear equations in two variables? Is the number of solutions infinite or unique or none?

In an earlier section, we used the model method for solving the pair of linear equations. Now we will use graphs to solve the equations.

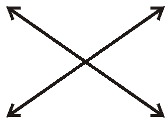
Let: $a_1x + b_1y + c_1 = 0$, $(a_1^2 + b_1^2 \neq 0)$ and $a_2x + b_2y + c_2 = 0$; $(a_2^2 + b_2^2 \neq 0)$ form a pair of linear equation in two variables.

The graph of a linear equation in two variables is a straight line. Ordered pairs of real numbers (x, y) representing points on the line are solutions of the equation and ordered pairs of real numbers (x, y) that do not represent points on the line are not solutions.

When we have a pair of equations, they represent lines in the same plane. So, if we have two lines in the same plane, what can be the possible relations between them? What is the significance of this relation?

When two lines are drawn in the same plane, only one of the following three situations is possible:

i) The two lines may intersect at one point.



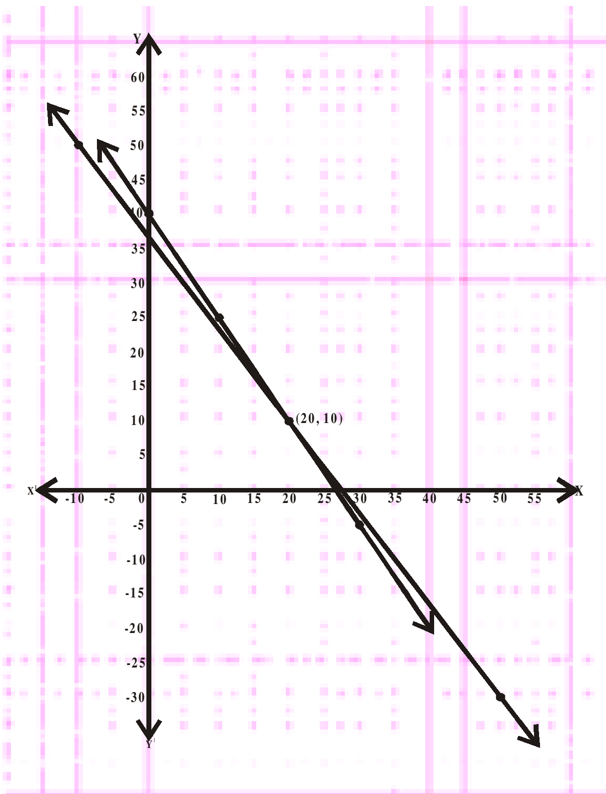
ii) The two lines may not intersect i.e., they are parallel.

iii) The two lines may be coincident.
(actually both are same)

Let us write the equations in the first example in terms of x and y where x is the cost of a notebook and y is the cost of a pen. Then, the equations are $3x + 2y = 80$ and $4x + 3y = 110$.

For the equation $3x + 2y = 80$		
x	$y = \frac{80-3x}{2}$	(x, y)
0	$y = \frac{80-3(0)}{2} = 40$	(0, 40)
10	$y = \frac{80-3(10)}{2} = 25$	(10, 25)
20	$y = \frac{80-3(20)}{2} = 10$	(20, 10)
30	$y = \frac{80-3(30)}{2} = -5$	(30, -5)

For the equation $4x + 3y = 110$		
x	$y = \frac{110-4x}{3}$	(x, y)
-10	$y = \frac{110-4(-10)}{3} = 50$	(-10, 50)
20	$y = \frac{110-4(20)}{3} = 10$	(20, 10)
50	$y = \frac{110-4(50)}{3} = -30$	(50, -30)



After plotting the above points in the Cartesian plane, we observe that the two straight lines are intersecting at the point (20, 10).

Substituting the values of x and y in equation we get $3(20) + 2(10) = 80$ and $4(20) + 3(10) = 110$.

Thus, as determined by the graphical method, the cost of each book is ₹20 and of each pen is ₹10. Recall that we got the same solution using the model method.

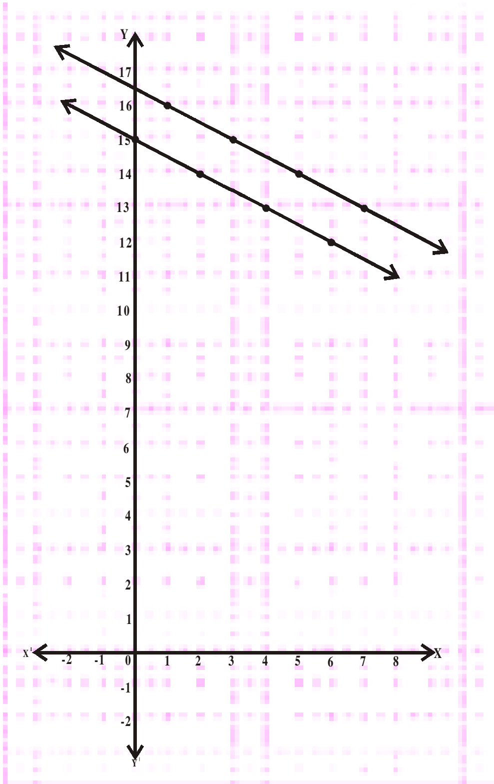
Since (20, 10) is the only common point, there is only one solution for this pair of linear equations in two variables. Such equations are known as consistent pairs of linear equations. They will always have only a unique solution.

Now, let us look at the first example from the think and discuss section. We want to find the cost of 1kg of potatoes and the cost of 1 kg of tomatoes each. Let the cost of 1kg potatoes be ₹x and cost of 1kg of tomato be ₹y. Then, the equations will become $1x+2y=30$ and $2x+4y=66$.

For the equation $x + 2y = 30$		
x	$y = \frac{30-x}{2}$	(x, y)
0	$y = \frac{30-0}{2} = 15$	(0, 15)
2	$y = \frac{30-2}{2} = 14$	(2, 14)
4	$y = \frac{30-4}{2} = 13$	(4, 13)
6	$y = \frac{30-6}{2} = 12$	(6, 12)

For the equation $2x + 4y = 66$		
x	$y = \frac{66-2x}{4}$	(x, y)
1	$y = \frac{66-2(1)}{4} = 16$	(1, 16)
3	$y = \frac{66-2(3)}{4} = 15$	(3, 15)
5	$y = \frac{66-2(5)}{4} = 14$	(5, 14)
7	$y = \frac{66-2(7)}{4} = 13$	(7, 13)

Here, we observe that the situation is represented graphically by two parallel lines. Since the lines do not intersect, the equations have no common solution. This means that the cost of the potato and tomato was different on different days. We see this in real life also. We cannot expect the same cost price of vegetables every day; it keeps changing. Also, the change is independent.



Such pairs of linear equations which have no solution are known as inconsistent pairs of linear equations.

In the second example from the think and discuss section, let the cost of each bat be ₹ x and each ball be ₹ y. Then we can write the equations as $3x + 6y = 3900$ and $x + 2y = 1300$.

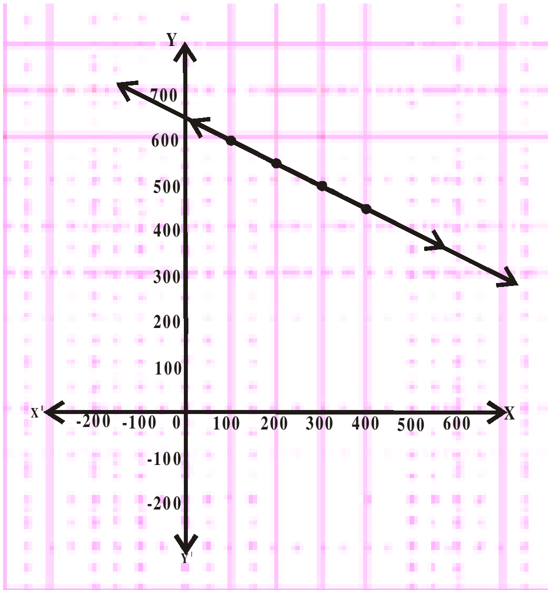
For the equation $3x + 6y = 3900$		
x	$y = \frac{3900-3x}{6}$	(x, y)
100	$y = \frac{3900-3(100)}{6} = 600$	(100, 600)

For the equation $x + 2y = 1300$		
x	$y = \frac{1300-x}{2}$	(x, y)
100	$y = \frac{1300-100}{2} = 600$	(100, 600)

200	$y = \dfrac{3900-3(200)}{6}$	= 550	(200, 550)
300	$y = \dfrac{3900-3(300)}{6}$	= 500	(300, 500)
400	$y = \dfrac{3900-3(400)}{6}$	= 450	(400, 450)

200	$y = \dfrac{1300-200}{2}$	= 550	(200, 550)
300	$y = \dfrac{1300-300}{2}$	= 500	(300, 500)
400	$y = \dfrac{1300-400}{2}$	= 450	(400, 450)

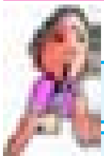
We see that the equations are geometrically shown by a pair of coincident lines. If the solutions of the equations are given by the common points, then what are the common points in this case?



From the graph, we observe that every point on the line is a common solution to both the equations. So, they have infinitely many solutions as both the equations are equivalent. Such pairs of equations are called **dependent** pair of linear equations in two variables.

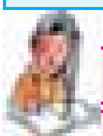
TRY THIS

In the example given above, can you find the cost of each bat and ball?



THINK - DISCUSS

Is a dependent pair of linear equations always consistent. Why or why not?



DO THIS

1. Solve the following systems of equations :

   i) $x - 2y = 0$, $3x + 4y = 20$

   ii) $x + y = 2$, $2x + 2y = 4$

   iii) $2x - y = 4$, $4x - 2y = 6$

2. Two rails on a railway track are represented by the equations. $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation graphically.

4.2.3 Relation Between Coefficients and Nature of System of Equations

Let a_1, b_1, c_1 and a_2, b_2, c_2 denote the coefficients of a given pair of linear equations in two variables. Then, let us write and compare the values of $\dfrac{a_1}{a_2}$, $\dfrac{b_1}{b_2}$ and $\dfrac{c_1}{c_2}$ in the above examples.

Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Comparison of ratios	Graphical representation	Algebraic interpretation
1. $3x+2y-80=0$ $4x+3y-110=0$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{-80}{-110}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Unique solution
2. $1x+2y-30=0$ $2x+4y-66=0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-30}{-66}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution
3. $3x+6y=3900$ $x+2y=1300$	$\frac{3}{1}$	$\frac{6}{2}$	$\frac{3900}{1300}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines (Dependent lines)	Infinite number of solutions

Let us look few examples.

Example-1. Check whether the given pair of equations repressent intersecting, parallel or coincident lines. Find the solution if the equations are consistent.

$2x + y - 5 = 0$

$3x - 2y - 4 = 0$

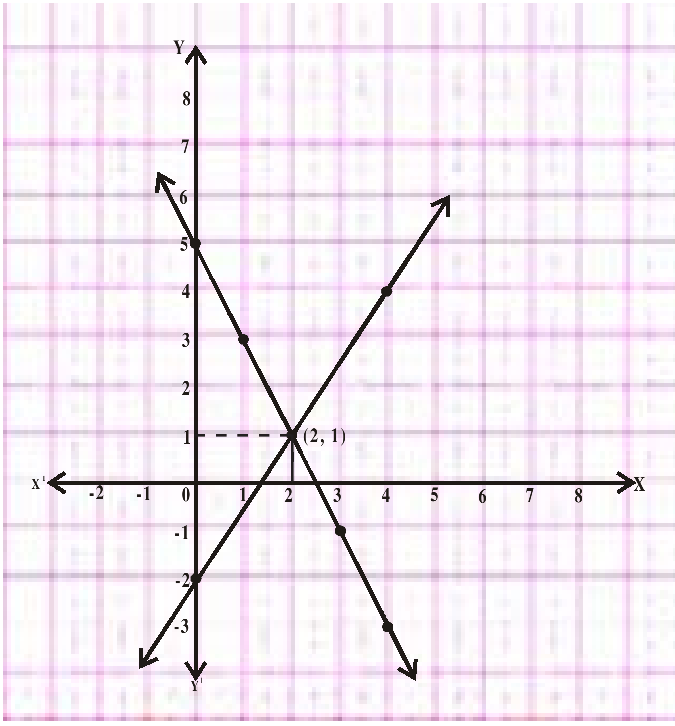
Solution : $\frac{a_1}{a_2} = \frac{2}{3}$ $\qquad \frac{b_1}{b_2} = \frac{1}{-2}$ $\qquad \frac{c_1}{c_2} = \frac{-5}{-4}$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, therefore they are intersecting lines and hence, consistent pair of linear equation.

For the equation $2x + y = 5$		
x	$y = 5 - 2x$	(x, y)
0	$y = 5 - 2(0) = 5$	(0, 5)
1	$y = 5 - 2(1) = 3$	(1, 3)
2	$y = 5 - 2(2) = 1$	(2, 1)
3	$y = 5 - 2(3) = -1$	(3, -1)
4	$y = 5 - 2(4) = -3$	(4, -3)

For the equation $3x - 2y = 4$		
x	$y = \frac{4-3x}{-2}$	(x, y)
0	$y = \frac{4-3(0)}{-2} = -2$	(0, -2)
2	$y = \frac{4-3(2)}{-2} = 1$	(2, 1)
4	$y = \frac{4-3(4)}{-2} = 4$	(4, 4)

The unique solution of this pair of equations is (2,1).



Example-2. Check whether the following pair of equations is consistent.

$3x + 4y = 2$ and $6x + 8y = 4$. Verify by a graphical representation.

Solution : $3x + 4y - 2 = 0$

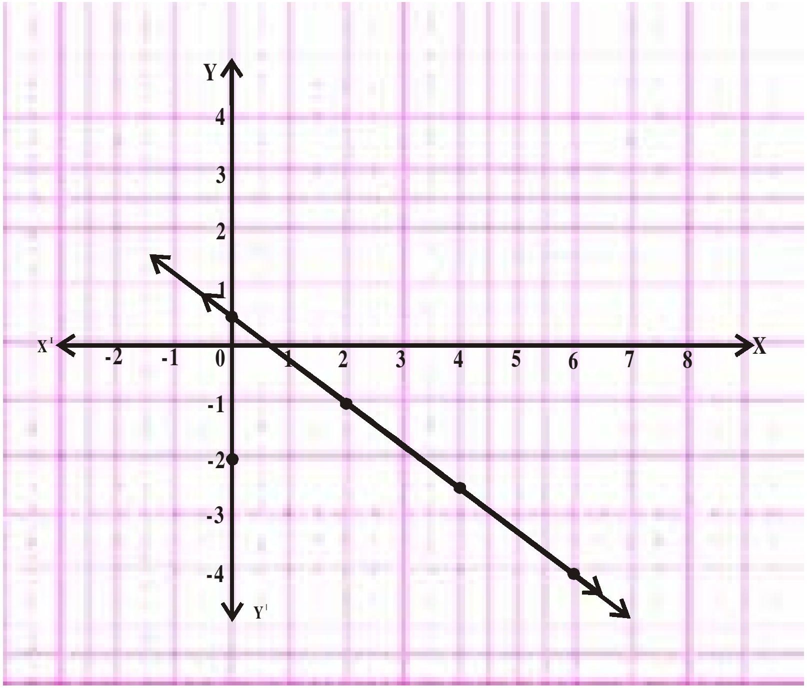
$6x+8y - 4 = 0$

$\frac{a_1}{a_2} = \frac{3}{6} = \frac{1}{2}$ $\frac{b_1}{b_2} = \frac{4}{8} = \frac{1}{2}$ $\frac{c_1}{c_2} = \frac{-2}{-4} = \frac{1}{2}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, therefore, they are coincident lines. So, the pair of linear equations is dependent and have infinitely many solutions.

For the equation 3x + 4y = 2		
x	$y = \frac{2-3x}{4}$	(x, y)
0	$y = \frac{2-3(0)}{4} = \frac{1}{2}$	$(0, \frac{1}{2})$
2	$y = \frac{2-3(2)}{4}$ = -1	(2, -1)
4	$y = \frac{2-3(4)}{4}$ = -2.5	(4, -2.5)
6	$y = \frac{2-3(6)}{4}$ = -4	(6, -4)

For the equation 6x + 8y = 4		
x	$y = \frac{4-6x}{8}$	(x, y)
0	$y = \frac{4-6(0)}{8} = \frac{1}{2}$	$(0, \frac{1}{2})$
2	$y = \frac{4-6(2)}{8}$ = -1	(2, -1)
4	$y = \frac{4-6(4)}{8}$ = -2.5	(4, -2.5)
6	$y = \frac{4-6(6)}{8}$ = -4	(6, -4)



Example-3. Check whether the equations $2x-3y=5$ and $4x-6y=15$ are consistent. Also verify by graphical representation.

Solution : $4x-6y - 15 = 0$

$2x-3y - 5 = 0$

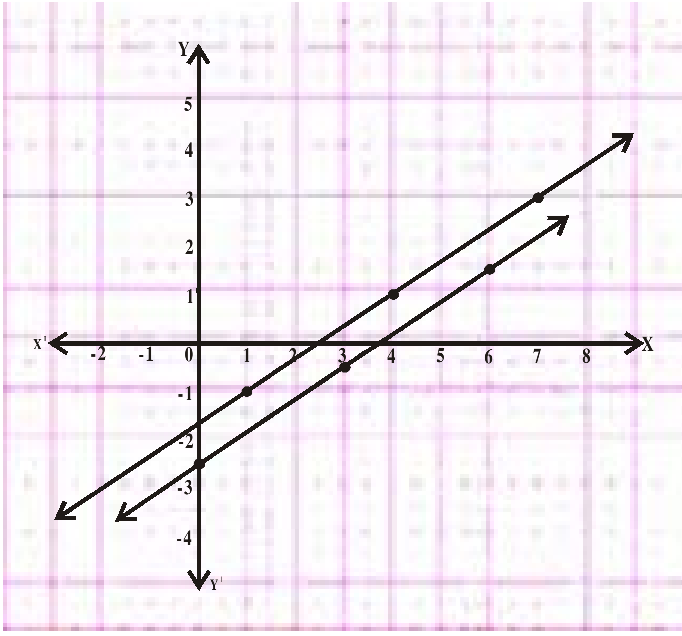
$\frac{a_1}{a_2} = \frac{4}{2} = \frac{2}{1}$ $\qquad \frac{b_1}{b_2} = \frac{-6}{-3} = \frac{2}{1}$

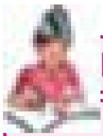
$\frac{c_1}{c_2} = \frac{-15}{-5} = \frac{3}{1}$ $\qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

So the equations are inconsistent. They have no solutions and its graph is of parallel lines.

For the equation $4x - 6y = 9$		
x	$y = \frac{15-4x}{-6}$	(x, y)
0	$y = \frac{15-0}{-6} = \frac{-5}{2}$	(0, -2.5)
3	$y = \frac{15-4(3)}{-6} = \frac{-1}{2}$	(3, -0.5)
6	$y = \frac{15-4(6)}{-6} = \frac{3}{2}$	(6, 1.5)

For the equation $2x - 3y = 5$		
x	$y = \frac{5-2x}{-3}$	(x, y)
1	$y = \frac{5-2(1)}{-3} = -1$	(1, -1)
3	$y = \frac{5-2(4)}{-3} = 1$	(4, 1)
6	$y = \frac{5-2(7)}{-3} = 3$	(7, 3)





Do this

Check each of the given systems of equations to see if it has a unique solution, infinitely many solutions or no solution. Solve them graphically.

(i) $2x+3y = 1$, $3x-y = 7$ (ii) $x + 2y = 6$, $2x + 4y = 12$ (iii) $3x + 2y = 6$, $6x + 4y = 18$

Try this

1. For what value of 'p' the following pair of equations has a unique solution. $2x + py = -5$ and $3x + 3y = -6$
2. Find the value of 'k' for which the pair of equations $2x - ky + 3 = 0$, $4x + 6y - 5 = 0$ represent parallel lines.
3. For what value of 'k', the pair of equation $3x + 4y + 2 = 0$ and $9x + 12y + k = 0$ represent coincident lines.
4. For what positive values of 'p' the following pair of liner equations have infinitely many solutions?

   $px + 3y - (p - 3) = 0$

   $12x + py - p = 0$

Let us look at some more examples.

Example-4. In a garden there are some bees and flowers. If one bee sits on each flower then one bee will be left. If two bees sit on each flower, one flower will be left. Find the number of bees and number of flowers.

Solution : Let the number of bees = x and

the number of flowers = y

If one bee sits on each flower then one bee will be left. So, $x = y + 1$

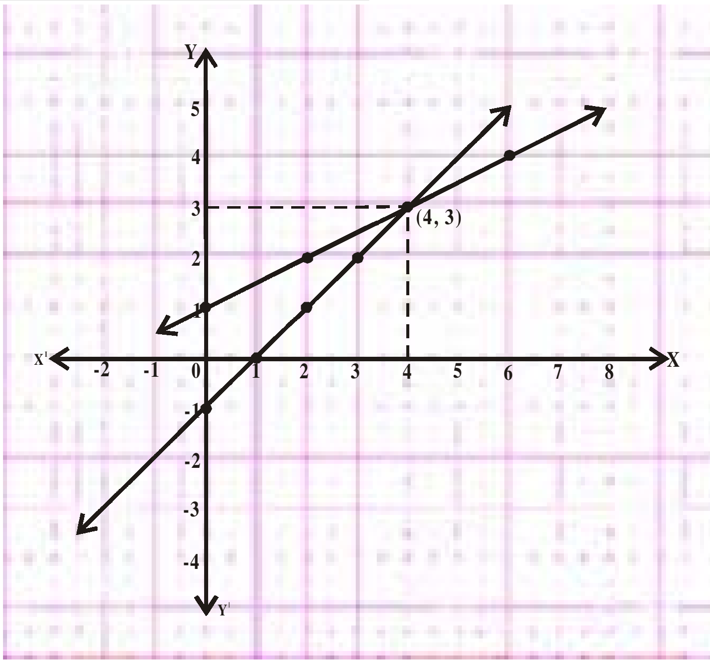
or $x - y - 1 = 0$ ... (1)

If two bees sit on each flower, one flower will be left. So, $x = 2(y - 1)$

or $x - 2y + 2 = 0$ ... (2)

For the equation $x - y - 1 = 0$		
x	$y = x - 1$	(x, y)
0	$y = 0 - 1 = -1$	(0, -1)
1	$y = 1 - 1 = 0$	(1, 0)
2	$y = 2 - 1 = 1$	(2, 1)
3	$y = 3 - 1 = 2$	(3, 2)
4	$y = 4 - 1 = 3$	(4, 3)

For the equation $x - 2y + 2 = 0$		
x	$y = \frac{x+2}{2}$	(x, y)
0	$y = \frac{0+2}{2} = 1$	(0, 1)
2	$y = \frac{2+2}{2} = 2$	(2, 2)
4	$y = \frac{4+2}{2} = 3$	(4, 3)
6	$y = \frac{6+2}{2} = 4$	(6, 4)



Therefore, there are 4 bees and 3 flowers.

Example-5. The perimeter of a rectangular plot is 32m. If the length is increased by 2m and the breadth is decreased by 1m, the area of the plot remains the same. Find the length and breadth of the plot.

Solution : Let length and breadth of the rectangular land be l and b respectively. Then,

area = lb and

Perimeter = $2(l + b) = 32\ m$.

$l + b = 16$ or $l + b - 16 = 0$ ... (1)

When length is increased by 2 *m*., then new length is $l + 2$. Also breadth is decreased by 1m so new breadth is $b - 1$.

Then, area= $(l + 2)(b - 1)$

Since there is no change in the area,

$(l + 2)(b - 1) = lb$

$lb - l + 2b - 2 = lb$ or $lb - lb = l - 2b + 2$

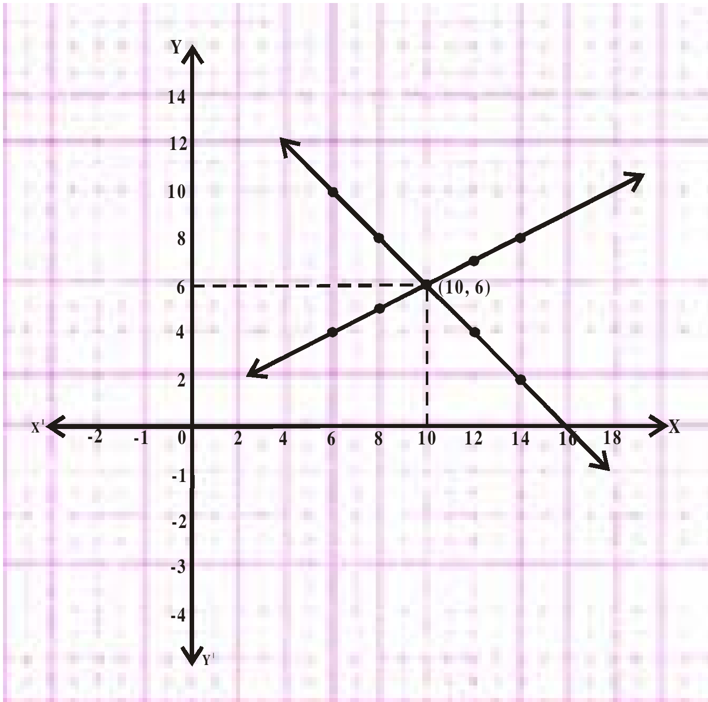
$l - 2b + 2 = 0$ ... (2)

For the equation $l + b - 16 = 0$		
l	$b = 16 - l$	(l, b)
6	$b = 16 - 6 = 10$	(6, 10)
8	$b = 16 - 8 = 8$	(8, 8)
10	$b = 16 - 10 = 6$	(10, 6)
12	$b = 16 - 12 = 4$	(12, 4)
14	$b = 16 - 14 = 2$	(14, 2)

For the equation $l - 2b + 2 = 0$		
l	$b = \frac{l+2}{2}$	(l, b)
6	$b = \frac{6+2}{2} = 4$	(6, 4)
8	$b = \frac{8+2}{2} = 5$	(8, 5)
10	$b = \frac{10+2}{2} = 6$	(10, 6)
12	$b = \frac{12+2}{2} = 7$	(12, 7)
14	$b = \frac{14+2}{2} = 8$	(14, 8)

So, original length of the plot is 10m and its breadth is 6m.

Taking measures of length on X-axis and measure of breadth on Y-axis, we get the graph



Exercise - 4.1

1. By comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$, find out whether the lines represented by the following pairs of linear equations intersect at a point, are parallel or are coincident.

a) $5x - 4y + 8 = 0$
$7x + 6y - 9 = 0$

b) $9x + 3y + 12 = 0$
$18x + 6y + 24 = 0$

c) $6x - 3y + 10 = 0$
$2x - y + 9 = 0$

2. Check whether the following equations are consistent or inconsistent. Solve them graphically.

a) $3x + 2y = 5$
$2x - 3y = 7$

b) $2x - 3y = 8$
$4x - 6y = 9$

c) $\frac{3}{2}x + \frac{5}{3}y = 7$
$9x - 10y = 14$

d) $5x - 3y = 11$
$-10x + 6y = -22$

e) $\frac{4}{3}x + 2y = 8$
$2x + 3y = 12$

f) $x + y = 5$
$2x + 2y = 10$

g) $x - y = 8$
$3x - 3y = 16$

h) $2x + y - 6 = 0$
$4x - 2y - 4 = 0$

i) $2x - 2y - 2 = 0$
$4x - 4y - 5 = 0$

3. Neha went to a 'sale' to purchase some pants and skirts. When her friend asked her how many of each she had bought, she answered "The number of skirts are two less than twice the number of pants purchased. Also the number of skirts is four less than four times the number of pants purchased."

Help her friend to find how many pants and skirts Neha bought.

4. 10 students of Class-X took part in a mathematics quiz. If the number of girls is 4 more than the number of boys then, find the number of boys and the number of girls who took part in the quiz.

5. 5 pencils and 7 pens together cost ₹50 whereas 7 pencils and 5 pens together cost ₹46. Find the cost of one pencil and that of one pen.

6. Half the perimeter of a rectangular garden, whose length is 4m more than its width, is 36m. Find the dimensions of the garden.

7. We have a linear equation $2x + 3y - 8 = 0$. Write another linear equation in two variables such that the geometrical representation of the pair so formed is intersecting lines.

Now, write two more linear equations so that one forms a pair of parallel lines and the second forms coincident line with the given equation.

8. The area of a rectangle gets reduced by 80 sq units if its length is reduced by 5 units and breadth is increased by 2 units. If we increase the length by 10 units and decrease the

breadth by 5 units, the area will increase by 50 sq units. Find the length and breadth of the rectangle.

9. In X class, if three students sit on each bench, one student will be left. If four students sit on each bench, one bench will be left. Find the number of students and the number of benches in that class.

4.3 Algebraic Methods of Finding the Solutions for a Pair of Linear Equations

We have learnt how to solve a pair of linear equations graphically. But, the graphical method is not convenient in cases where the point representing the solution has no integral co-ordinates. For example, when the solution is of the form ($\sqrt{3}$, $2\sqrt{7}$), (- 1.75, 3.3), ($\frac{4}{13}$, $\frac{1}{19}$) etc. There is every possibility of making mistakes while reading such co-ordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall discuss now.

4.3.1 Substitution Method

This method is useful for solving a pair of linear equations in two variables where one variable can easily be written in terms of the other variable. To understand this method, let us consider it step-wise

Step-1 : In one of the equations, express one variable in terms of the other variable. Say y in terms of x.

Step-2 : Substitute the value of y obtained in step 1 in the second equation.

Step-3 : Simplify the equation obtained in step 2 and find the value of x.

Step-4 : Substitute the value of x obtained in step 3 in either of the equations and solve it for y.

Step-5 : Check the obtained solution by substituting the values of x and y in both the original equations.

Example-6. Solve the given pair of equations using substitution method.

$2x - y = 5$

$3x + 2y = 11$

Solution : $2x - y = 5$ (1)

$3x + 2y = 11$ (2)

Equation (1) can be written as (Step 1)

$y = 2x - 5$

Substituting in equation (2) we get (Step 2)

$3x + 2(2x - 5) = 11$

$3x + 4x - 10 = 11$

$7x = 11 + 10 = 21$

$x = 21/7 = 3.$ (Step 3)

Substitute $x = 3$ in equation (1)

$2(3) - y = 5$ (Step 4)

$y = 6 - 5 = 1$

Substitute the values of x and y in equation (2), we get $3(3) + 2(1) = 9 + 6 = 11$

Both the equations are satisfied by $x = 3$ and $y = 1$. (Step 5)

Therefore, required solution is $x = 3$ and $y = 1$.

Do This

Solve each pair of equation by using the substitution method.

1) $3x - 5y = -1$
   $x - y = -1$

2) $x + 2y = -1$
   $2x - 3y = 12$

3) $2x + 3y = 9$
   $3x + 4y = 5$

4) $x + \frac{6}{y} = 6$
   $3x - \frac{8}{y} = 5$

5) $0.2x + 0.3y = 13$
   $0.4x + 0.5y = 2.3$

6) $\sqrt{2}x + \sqrt{3}y = 0$
   $\sqrt{3}x - \sqrt{8}y = 0$

4.3.2 Elimination Method

In this method, first we eliminate (remove) one of the two variables by equating its coefficients. This gives a single equation which can be solved to get the value of the other variable. To understand this method, let us consider it stepwise.

Step-1 : Write both the equations in the form of $ax + by = c$.

Step-2 : Make the coefficients of one of the variables, say 'x', numerically equal by multiplying each equation by suitable real numbers.

Step-3 : If the variable to be eliminated has the same sign in both equations, subtract the two equations to get an equation in one variable. If they have opposite signs then add.

Step-4 : Solve the equation for the remaining variable.

Step-5 : Substitute the value of this variable in any one of the original equations and find the value of the eliminated variable.

Example-7. Solve the following pair of linear equations using elimination method.

$3x + 2y = 11$

$2x + 3y = 4$

Solution : $3x + 2y = 11$ (1)

$2x + 3y = 4$ (2) (Step 1)

Let us eliminate 'y' from the given equations. The coefficients of 'y' in the given equations are 2 and 3. L.C.M.of 2 and 3 is 6. So, multiply equation (1) by 3 and equation (2) by 2.

Equation (1) × 3 $9x + 6y = 33$ (Step 2)

Equation (2) × 2 $4x + 6y = 8$

(-) (-) (-) (Step 3)

$5x = 25$

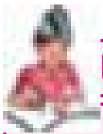
$x = \frac{25}{5} = 5$ (Step 4)

Substitute $x = 5$, in equation (1)

$3(5) + 2y = 11$

$2y = 11 - 15 = -4 \Rightarrow y = \frac{-4}{2} = -2$ (Step 5)

Therefore, the required solution is $x = 5, y = -2$.



Do this

Solve each of the following pairs of equations by the elimination method.

1. $8x + 5y = 9$, $3x + 2y = 4$
2. $2x + 3y = 8$, $4x + 6y = 7$
3. $3x + 4y = 25$, $5x - 6y = -9$

Try This

Solve the given pair of linear equations

$(a - b)x + (a + b)y = a^2 - 2ab - b^2$

$(a + b)(x + y) = a^2 + b^2$

Let us see some more examples:

Example-8. Tabita went to a bank to withdraw ₹2000. She asked the cashier to give the cash in ₹50 and ₹100 notes only. Snigdha got 25 notes in all. Can you tell how many notes each of ₹50 and ₹100 she received?

Solution : Let the number of ₹50 notes be x;

Let the number of ₹100 notes be y;

then, $x + y = 25$ (1)

and $50x + 100y = 2000$ (2)

Kavitha used the substitution method.

From equation (1)) $x = 25 - y$

Substituting in equation (2) $50(25 - y) + 100y = 2000$

$1250 - 50y + 100y = 2000$

$50y = 2000 - 1250 = 750$

$y = \frac{750}{50} = 15$

$x = 25 - 15 = 10$

Hence, Tabita received ten ₹50 notes and fifteen ₹100 notes.

Prathyusha used the elimination method to get the solution.

In the equations, coefficients of x are 1 and 50 respectively. So,

Equation (1) × 50 $\quad 50x + 50y = 1250$

Equation (2) × 1 $\quad 50x + 100y = 2000$ same sign, so subtract

$\quad$ (-) $\quad$ (-) $\quad$ (-)

$-50y = -750$

or $\quad y = \frac{-750}{-50} = 15$

Substitute y in equation (1) $x + 15 = 25$

$x = 25 - 15 = 10$

Hence Snigdha received ten ₹50 notes and fifteen ₹100 rupee notes.

Example-9. In a competitive exam, 3 marks are to be awarded for every correct answer and for every wrong answer, 1 mark will be deducted. Madhu scored 40 marks in this exam. Had 4 marks been awarded for each correct answer and 2 marks deducted for each incorrect answer, Madhu would have scored 50 marks. How many questions were there in the test? (Madhu attempted all the questions)

Solution : Let the number of correct answers be x;

and the number of wrong answers be y.

When 3 marks are given for each correct answer and 1 mark deducted for each wrong answer, his score is 40 marks.

$3x - y = 40 \quad (1)$

His score would have been 50 marks if 4 marks were given for each correct answer and 2 marks deducted for each wrong answer.

$4x - 2y = 50 \quad (2)$

Substitution method

From equation (1), $y = 3x - 40$

Substitute in equation (2) $4x - 2\,(3x - 40) = 50$

$4x - 6x + 80 = 50$

$-2x = 50 - 80 = -30$

$x = \frac{-30}{-2} = 15$

Substitute the value of x in equation (1)

$3(15) - y = 40$

$45 - y = 40$

$y = 45 - 40 = 5$

$\therefore$ Total number of questions = 15 + 5 = 20

DO THIS

Now use the elimination method to solve the above example-9.

Example-10. Mary told her daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." Find the present age of Mary and her daughter.

Solution : Let Mary's present age be x years and her daughter's age be y years.

Then, seven years ago Mary's age was $x - 7$ and daughter's age was $y - 7$.

$x - 7 = 7(y - 7)$

$x - 7 = 7y - 49$

$x - 7y + 42 = 0$ (1)

Three years hence, Mary's age will be $x + 3$ and daughter's age will be $y + 3$.

$x + 3 = 3\,(y + 3)$

$x + 3 = 3y + 9$

$x - 3y - 6 = 0$ (2)

Elimination method

Equation 1 $x - 7y = -42$

Equation 2 $x - 3y = 6$

(-) (+) (-) same sign for x, so subtract.

$-4y = -48$

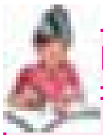
$$y = \frac{-48}{-4} = 12$$

Substitute the value of y in equation (2)

$$x - 3\,(12) - 6 = 0$$

$$x = 36 + 6 = 42$$

Therefore, Mary's present age is 42 years and her daughter's age is 12 years.`



Do This

Solve example-10 by the substitution method.

Example-11. A publisher is planning to produce a new textbook. The fixed costs (reviewing, editing, typesetting and so on) are ₹ 31.25 per book. Besides that, he also spends another ₹ 320000 in producing the book. The wholesale price (the amount received by the publisher) is ₹ 43.75 per book. How many books must the publisher sell to break even, i.e., so that the costs will equal revenues?

The point which corresponds to how much money you have to earn through sales in order to equal the money you spent in production is **break even point**.

Solution : The publisher breaks even when costs equal revenues. If x represents the number of books printed and sold and y be the breakeven point, then the cost and revenue equations for the publisher are

Cost equation is given by $\quad y = 320000 + 31.25x \quad$ (1)

Revenue equation is given by $\quad y = 43.75x \quad$ (2)

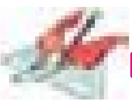
Using the second equation to substitute for y in the first equation, we have

$$43.75x = 3{,}20{,}000 + 31.25x$$

$$12.5x = 3{,}20{,}000$$

$$x = \frac{3{,}20{,}000}{12.5} = 25{,}600$$

Thus, the publisher will break even when 25,600 books are printed and sold.



Exercise - 4.2

Form a pair of linear equations for each of the following problems and find their solution.

1. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly income.
2. The sum of a two digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

3. The larger of two supplementary angles exceeds the smaller by 18°. Find the angles.
4. The taxi charges in Hyderabad are fixed, along with the charge for the distance covered. For a distance of 10 *km*., the charge paid is ₹220. For a journey of 15 *km*. the charge paid is ₹310.
   i. What are the fixed charges and charge per km?
   ii. How much does a person have to pay for travelling a distance of 25 km?
5. A fraction becomes $\frac{4}{5}$ if 1 is added to both numerator and denominator. If, however, 5 is subtracted from both numerator and denominator, the fraction becomes $\frac{1}{2}$. What is the fraction?
6. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time at different speeds. If the cars travel in the same direction, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?
7. Two angles are complementary. The larger angle is 3° less than twice the measure of the smaller angle. Find the measure of each angle.
8. An algebra textbook has a total of 1382 pages. It is broken up into two parts. The second part of the book has 64 pages more than the first part. How many pages are in each part of the book?
9. A chemist has two solutions of hydrochloric acid in stock. One is 50% solution and the other is 80% solution. How much of each should be used to obtain 100ml of a 68% solution.
10. Suppose you have ₹12000 to invest. You have to invest some amount at 10% and the rest at 15%. How much should be invested at each rate to yield 12% on the total amount invested?

4.4 Equations Reducible to a Pair of Linear Equations in Two Variables

Now we shall discuss the solution of pairs of equations which are not linear but can be reduced to linear form by making suitable substitutions. Let us see an example:

Example-12. Solve the following pair of equations.

$$\frac{2}{x}+\frac{3}{y} = 13$$

$$\frac{5}{x}-\frac{4}{y} = -2$$

Solution : Observe the given pair of equations. They are not linear equations. (Why?)

We have $2\left(\frac{1}{x}\right) + 3\left(\frac{1}{y}\right) = 13$ (1)

$5\left(\frac{1}{x}\right) - 4\left(\frac{1}{y}\right) = -2$ (2)

If we substitute $\frac{1}{x} = p$ and $\frac{1}{y} = q$, we get the following pair of linear equations:

$2p + 3q = 13$ (3)

$5p - 4q = -2$ (4)

Coefficients of q are 3 and 4 and their l.c.m. is 12. Using the elimination method:

Equation (3) × 4 $8p + 12q = 52$

Equation (4) × 3 $\underline{15p - 12q = -6}$ 'q' terms have opposite sign, so we add the two equations.

$23p = 46$

$$p = \frac{46}{23} = 2$$

Substitute the value of p in equation (3)

$2(2) + 3q = 13$

$3q = 13 - 4 = 9$

$$q = \frac{9}{3} = 3$$

But, $\frac{1}{x} = p = 2$ $\Rightarrow x = \frac{1}{2}$

$\frac{1}{y} = q = 3$ $\Rightarrow y = \frac{1}{3}$

Example-13. Kavitha thought of constructing 2 more rooms in her house. She enquired about the labour. She came to know that 6 men and 8 women could finish this work in 14 days. But she wanted the work completed in only 10 days. When she enquired, she was told that 8 men and 12 women could finish the work in 10 days. Find out that how much time would be taken to finish the work if one man or one woman worked alone?

Solution : Let the time taken by one man to finish the work = x days.

Work done by one man in one day $= \frac{1}{x}$

Let the time taken by one woman to finish the work $= y$ days.

Work done by one woman in one day $= \frac{1}{y}$

Now, 8 men and 12 women can finish the work in 10 days.

So work done by 8 men and 12 women in one day $= \frac{1}{10}$ (1)

Also, work done by 8 men in one day is $8 \times \frac{1}{x}$. $= \frac{8}{x}$

Similarly, work done by 12 women in one day is $12 \times \frac{1}{y}$ $= \frac{12}{y}$

Total work done by 8 men and 12 women in one day $= \frac{8}{x} + \frac{12}{y}$ (2)

Equating equations (1) and (2) $\left(\frac{8}{x} + \frac{12}{y}\right) = \frac{1}{10}$

$$10\left(\frac{8}{x} + \frac{12}{y}\right) = 1$$

$$\frac{80}{x} + \frac{120}{y} = 1 \qquad (3)$$

Also, 6 men and 8 women can finish the work in 14 days.

Work done by 6 men and 8 women in one day $= \frac{6}{x} + \frac{8}{y} = \frac{1}{14}$

$$\Rightarrow \qquad 14\left(\frac{6}{x} + \frac{8}{y}\right) = 1$$

$$\left(\frac{84}{x} + \frac{112}{y}\right) = 1 \qquad (4)$$

Observe equations (3) and (4). Are they linear equations? How do we solve them then? We can convert them into linear equations by substituting $\frac{1}{x} = u$ and $\frac{1}{y} = v$.

Equation (3) becomes $80u + 120v = 1$ (5)

Equation (4) becomes $84u + 112v = 1$ (6)

L.C.M. of 80 and 84 is 1680. Using the elimination method,

Equation (3) × 21 $(21 \times 80)u + (21 \times 120)v = 21$

Equation (4) × 20 $(20 \times 84)u + (20 \times 112)v = 20$

$$\begin{array}{r} 1680u + 2520v = 21 \\ 1680u + 2240v = 20 \\ (-) \quad (-) \quad (-) \\ \hline 280v = 1 \end{array}$$

Same sign for u, so subtract

$$v = \frac{1}{280}$$

Substitute in equation (5) $80u + 120 \times \frac{1}{280} = 1$

$$80u = 1 - \frac{3}{7} = \frac{7-3}{7} = \frac{4}{7}$$

$$u = \frac{4}{7} \times \frac{1}{80} = \frac{1}{140}$$

So one man alone can finish the work in 140 days and one woman alone can finish the work in 280 days.

Example-14. A man travels 370 km partly by train and partly by car. If he covers 250 km by train and the rest by car, it takes him 4 hours. But if he travels 130 km by train and the rest by car, it takes 18 minutes more. Find the speed of the train and that of the car.

Solution : Let the speed of the train be x *km*. per hour and that of the car be y *km*. per hour.

Also, we know that time $= \frac{\text{Distance}}{\text{Speed}}$

In situation 1, time spent travelling by train $= \frac{250}{x}$ hrs.

And time spent travelling by car $= \frac{120}{y}$ hrs.

So, total time taken = time spent in train + time spent in car = $\frac{250}{x} + \frac{120}{y}$

But, total time of journey is 4 hours, so

$$\frac{250}{x} + \frac{120}{y} = 4$$

$$\frac{125}{x} + \frac{60}{y} = 2 \qquad \rightarrow (1)$$

Again, when he travels 130 km by train and the rest by car

Time taken by him to travel 130 km by train = $\frac{130}{x}$ hrs.

Time taken by him to travel 240 km (370 - 130) by car = $\frac{240}{y}$ hrs.

Total time taken = $\frac{130}{x} + \frac{240}{y}$

But given, time of journey is 4 hrs 18 min i.e., $4\frac{\cancel{18}^{3}}{\cancel{60}_{10}}$ hrs. = $4\frac{3}{10}$ hrs.

So, $$\frac{130}{x} + \frac{240}{y} = \frac{43}{10} \qquad (2)$$

Substitute $\frac{1}{x} = a$ and $\frac{1}{y} = b$ in equations (1) and (2)

$125a + 60b = 2$ (3)

$130a + 240b = 43/10$ (4)

For 60 and 240, l.c.m. is 240. Using the elimination method,

Equation (3) × 4 $\quad 500a + \cancel{240b} = 8$

Equation (4) × 1 $\quad 130a + \cancel{240b} = \frac{43}{10}$ (Same sign, so subtract)

$\quad (-) \quad (-) \quad (-)$

$$370a = 8 - \frac{43}{10} = \frac{80-43}{10} = \frac{37}{10}$$

$$a = \frac{\cancel{37}}{10} \times \frac{1}{\underset{10}{\cancel{370}}} = \frac{1}{100}$$

Substitute $a = \frac{1}{100}$ in equation (3)

$$\left(\overset{5}{\cancel{125}} \times \frac{1}{\underset{4}{\cancel{100}}}\right) + 60b = 2$$

$$60b = 2 - \frac{5}{4} = \frac{8-5}{4} = \frac{3}{4}$$

$$b = \frac{\cancel{3}}{4} \times \frac{1}{\underset{20}{\cancel{60}}} = \frac{1}{80}$$

So $a = \frac{1}{100}$ and b = $\frac{1}{80}$

So $\frac{1}{x} = \frac{1}{100}$ and $\frac{1}{y} = \frac{1}{80}$

$x = 100$ km/hr and $y = 80$ km/hr.

So, speed of train was 100 km/hr and speed of car was 80 km/hr.

EXERCISE - 4.3

Solve each of the following pairs of equations by reducing them to a pair of linear equations.

i) $\frac{5}{x-1} + \frac{1}{y-2} = 2$

$\frac{6}{x-1} - \frac{3}{y-2} = 1$

ii) $\frac{x+y}{xy} = 2$

$\frac{x-y}{xy} = 6$

iii) $\frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 2$

$\frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} = -1$

iv) 6x+3y = 6xy

2x + 4y = 5xy

v) $\frac{5}{x+y} - \frac{2}{x-y} = -1$

vi) $\frac{2}{x} + \frac{3}{y} = 13$

$\frac{15}{x+y} + \frac{7}{x-y} = 10$ where $x\neq0, y\neq0$

$\frac{5}{x} - \frac{4}{y} = -2$ where $x\neq0, y\neq0$

vii) $\frac{10}{x+y} + \frac{2}{x-y} = 4$

$\frac{15}{x+y} - \frac{5}{x-y} = -2$

viii) $\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$

$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$

2. Formulate the following problems as a pair of equations and then find their solutions.

i. A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours it can go 40 km upstream and 55 km downstream. Determine the speed of the stream and that of the boat in still water.

ii. Rahim travels 600 km to his home partly by train and partly by car. He takes 8 hours if he travels 120 km by train and rest by car. He takes 20 minutes more if he travels 200 km by train and rest by car. Find the speed of the train and the car.

iii. 2 women and 5 men can together finish an embroidery work in 4 days while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone and 1 man alone to finish the work.

Optional Exercise

[This exercise is not meant for examination]

1. Solve the following equations:-

(i) $\frac{2x}{a} + \frac{y}{b} = 2$

$\frac{x}{a} - \frac{y}{b} = 4$

(ii) $\frac{x+1}{2} + \frac{y-1}{3} = 8$

$\frac{x-1}{3} + \frac{y+1}{2} = 9$

(iii) $\frac{x}{7} + \frac{y}{3} = 5$

$\frac{x}{2} - \frac{y}{9} = 6$

(iv) $\sqrt{3}x + \sqrt{2}y = \sqrt{3}$

$\sqrt{5}x + \sqrt{3}y = \sqrt{3}$

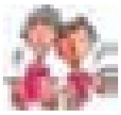
(v) $\frac{ax}{b} - \frac{by}{a} = a + b$

$ax - by = 2ab$

(vi) $2^x + 3^y = 17$

$2^{x+2} - 3^{y+1} = 5$

2. Animals in an experiment are to be kept on a strict diet. Each animal is to receive among other things 20g of protein and 6g of fat. The laboratory technicians purchased two food mixes, A and B. Mix A has 10% protein and 6% fat. Mix B has 20% protein and 2% fat. How many grams of each mix should be used?



What We Have Discussed

1. Two linear equations in the same two variables are called a pair of linear equations in two variables.

   $a_1x + b_1y + c_1 = 0 \ (a_1^2 + b_1^2 \neq 0)$

   $a_2x + b_2y + c_2 = 0 \ (a_2^2 + b_2^2 \neq 0)$

   Where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers.

2. A pair of linear equations in two variables can be solved using various methods.

3. The graph of a pair of linear equations in two variables is represented by two lines.

   i. If the lines intersect at a point then the point gives the unique solution of the two equations. In this case, the pair of equations is consistent.

   ii. If the lines coincide, then there are infinitely many solutions - each point on the line being a solution. In this case, the pair of equations is dependent.

   iii. If the lines are parallel then the pair of equations has no solution. In this case, the pair of equations is inconsistent.

4. We have discussed the following methods for finding the solution(s) of a pair of linear equations.

   i. Model Method.

   ii. Graphical Method

   iii. Algebraic methods - Substitution method and Elimination method.

5. There exists a relation between the coefficients and nature of system of equations.

   i. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then the pair of linear equations is consistent.

   ii. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then the pair of linear equations is inconsistent.

   iii. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ then the pair of linear equations is dependent and consistent.

6. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we can alter them so that they will be reduced to a pair of linear equations.