CHAPTER 1

Real Numbers

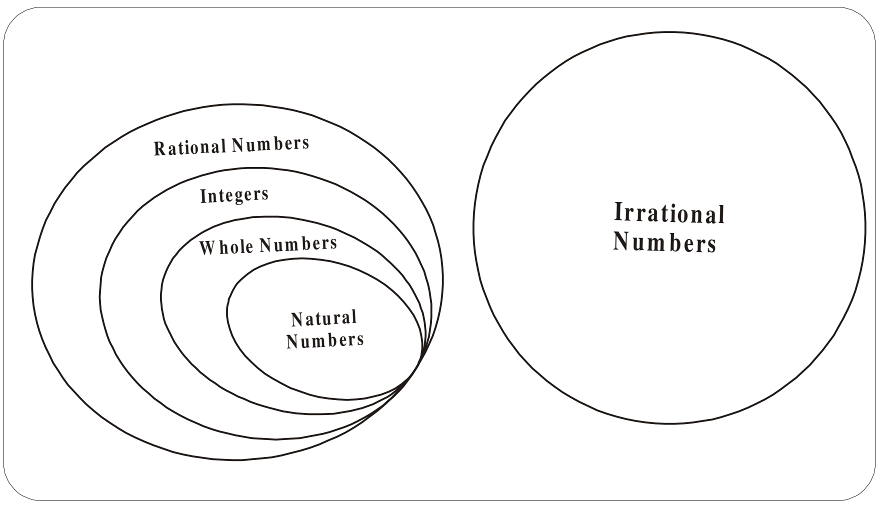
1.1 INTRODUCTION

We have studied different types of numbers in earlier classes. We have learnt about natural numbers, whole numbers, integers, rational numbers and irrational numbers. Let us recall a little bit about rational numbers and irrational numbers.

Rational numbers are numbers which can be written in the form of $\frac{p}{q}$ where both p and q are integers and $q \neq 0$. They are a bigger collection than integers as there can be many rational numbers between two integers. All rational numbers can be written either in the form of terminating decimals or non-terminating repeating decimals.

Numbers which cannot be expressed in the form of $\frac{p}{q}$ are irrational. These include numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and mathematical quantities like π. When these are written as decimals, they are non-terminaing, non-recurring. For example, $\sqrt{2}$ = 1.41421356... and π = 3.14159... These numbers can be located on the number line.

The set of rational and irrational numbers together are called real numbers. We can show them in the form of a diagram:



Real Numbers

In this chapter, we will see some theorems and the different ways in which we can prove them. We will use the theorems to explore properties of rational and irrational numbers. Finally, we will study about a type of function called logarithms (in short logs) and see how they are useful in science and everyday life.

But before exploring real numbers a little more, let us solve some questions.

Exercise - 1.1

1. Which of the following rational numbers are terminating and which are non-terminating, repeating in their deimenal form?

   (i) $\frac{2}{5}$ (ii) $\frac{17}{18}$ (iii) $\frac{15}{16}$ (iv) $\frac{7}{40}$ (v) $\frac{9}{11}$

2. Find any rational number between the pair of numbers given below:

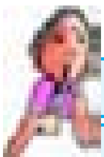
   (i) $\frac{1}{2}$ and $\sqrt{1}$ (ii) $3\frac{1}{3}$ and $3\frac{2}{3}$ (iii) $\sqrt{\frac{4}{9}}$ and $\sqrt{2}$

3. Classify the numbers given below as rational or irrational.

   (i) $2\frac{1}{2}$ (ii) $\sqrt{24}$ (iii) $\sqrt{16}$ (iv) $7.\overline{7}$ (v) $\sqrt{\frac{4}{9}}$ (vi) $-\sqrt{30}$ (vii) $-\sqrt{81}$

4. Represent the following real numbers on the number line. (If necessary make a seperate number line for each number).

   (i) $\frac{3}{4}$ (ii) $\frac{-9}{10}$ (iii) $\frac{27}{3}$ (iv) $\sqrt{5}$ (v) $-\sqrt{16}$



Think - Discuss

Are all integers also in real numbers? Why?

1.2 Exploring Real Numbers

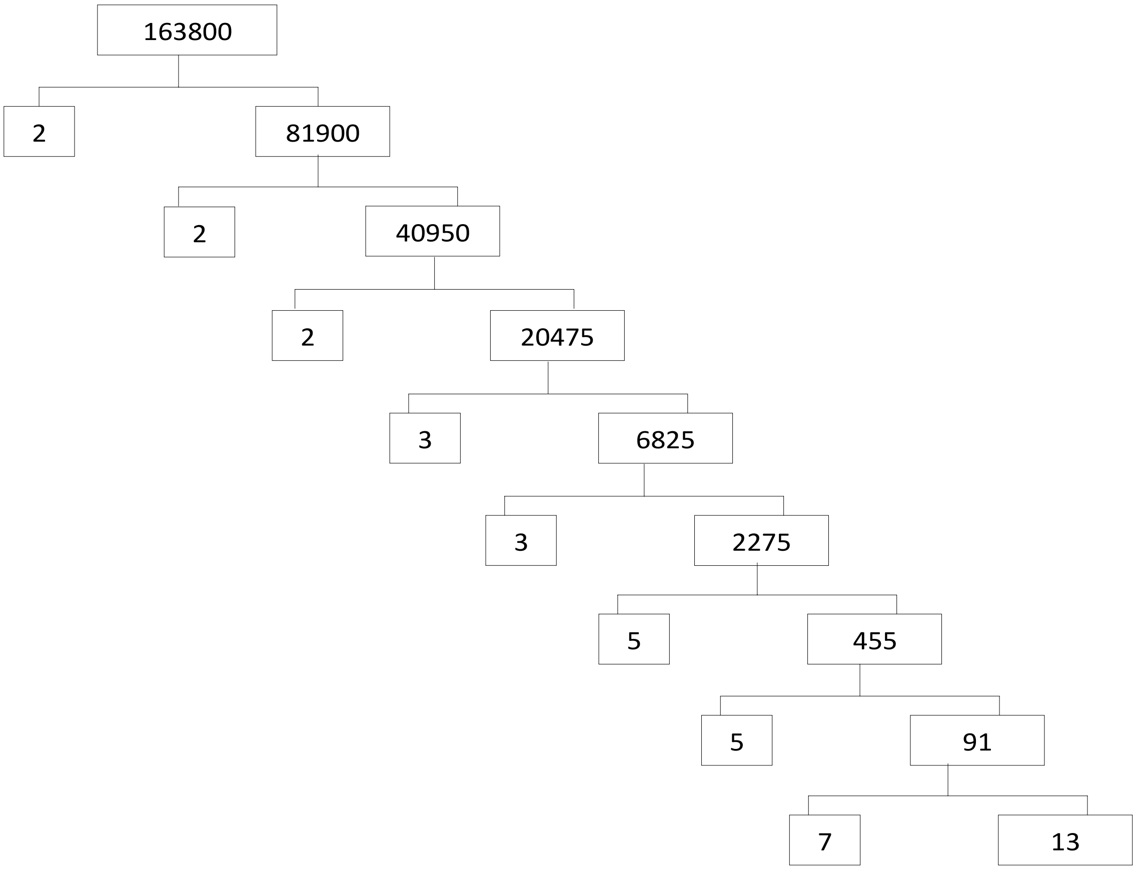
Let us explore real numbers more in this section. We know that natural numbers are also in real numbers. So, we will start with them.

1.2.1 The Fundamental Theorem of Arithmetic

In earlier classes, we have seen that all natural numbers, except 1, can be written as a product of their prime factors. For example, $3 = 3$, 6 as 2×3, 253 as 11×23 and so on. (Remember: 1 is neither a composite nor a prime).

Do you think that there may be a composite number which is not the product of the powers of primes? To answer this, let us factorize a natural number as an example.

We are going to use the factor tree which you all are familiar with. Let us take some large number, say 163800, and factorize it as shown :



So we have factorized 163800 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13$. So 163800 $= 2^3 \times 3^2 \times 5^2 \times 7 \times 13$, when we write it as a product of power of primes.

Try another number, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that ***every composite number can be written as the product of powers of primes.***

Now, let us try and look at natural numbers from the other direction. Let us take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce infinitely many large positive integers. Let us list a few :

$2 \times 3 \times 11 = 66$ $\qquad$ $7 \times 11 = 77$

$7 \times 11 \times 23 = 1771$

$3 \times 7 \times 11 \times 23 = 5313$

$2 \times 3 \times 7 \times 11 \times 23 = 10626$

$2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of primes or infinitely many? In fact, there are infinitely many primes. So, if we multiply all these primes in all possible ways, we will get an infinite collection of composite numbers.

This gives us the Fundamental Theorem of Arithmetic which says that every composite number can be factorized as a product of primes. Actually, it says more. It says that given any composite number it can be factorized as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. For example, when we factorize 210, we regard $2 \times 3 \times 5 \times 7$ as same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. Let us now formally state this theorem.

Theorem-1.1 : (Fundamental Theorem of Arithmetic) : Every composite number can be expressed (factorised) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

In general, given a composite number x, we factorize it as $x = p_1 p_2 ... p_n$, where $p_1, p_2, ..., p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq ... \leq p_n$. If we use the same primes, we will get powers of primes. Once we have decided that the order will be ascending, then the way the number is factorised, is unique. For example,

$$163800 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5^2 \times 7 \times 13$$

Try This

Express 2310 as a product of prime factors. Also see how your friends have factorized the number. Have they done it like you? Verify your final product with your friend's result. Try this for 3 or 4 more numbers. What do you conclude?

While this is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. Let us see two examples.

You have already learnt how to find the HCF (Highest Common Factor) and LCM (Lowest Common Multiple) of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realizing it! This method is also called the *prime factorization method*. Let us recall this method through the following example.

Example-1. Find the HCF and LCM of 12 and 18 by the prime factorization method.

Solution : We have $12 = 2 \times 2 \times 3 = 2^2 \times 3^1$

$18 = 2 \times 3 \times 3 = 2^1 \times 3^2$

Note that HCF (12, 18) $= 2^1 \times 3^1 = 6$ = **Product of the smallest power of each common prime factors in the numbers.**

LCM (12, 18) $= 2^2 \times 3^2 = 36$ = **Product of the greatest power of each prime factors, in the numbers.**

From the example above, you might have noticed that HCF (12, 18) $\times$ LCM (12, 18) $= 12 \times 18$. In fact, we can verify that for any two positive integers a and b, HCF (a,b) $\times$ LCM (a, b) = a $\times$ b. We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 2. Consider the numbers 4^n, where *n* is a natural number. Check whether there is any value of *n* for which 4^n ends with the digit zero?

Solution : For the number 4^n to end with digit zero for any natural number n, it should be divisible by 5. This means that the prime factorisation of 4^n should contain the prime number 5. But it is not possible because $4^n = (2)^{2n}$ so 2 is the only prime in the factorisation of 4^n. Since 5 is not present in the prime factorization, so there is no natural number n for which 4^n ends with the digit zero.

TRY THIS

Show that 12^n cannot end with the digit 0 or 5 for any natural number '*n*'.

EXERCISE - 1.2

1. Express each number as a product of its prime factors.

   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

2. Find the LCM and HCF of the following integers by the prime factorization method.

   (i) 12, 15 and 21 (ii) 17, 23, and 29 (iii) 8, 9 and 25

   (iv) 72 and 108 (v) 306 and 657

3. Check whether 6^n can end with the digit 0 for any natural number n.
4. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
5. How will you show that $(17 \times 11 \times 2) + (17 \times 11 \times 5)$ is a composite number? Explain.

Now, let us use the Fundamental Theorem of Arithmetic to explore real numbers further. First, we apply this theorem to find out when the decimal expansion of a rational number is terminating and when it is non-terminating, repeating. Second, we use it to prove the irrationality of many numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

1.2.2 Rational numbers and their decimal expansions

In this section, we are going to explore when their decimal expansions of rational numbers are terminating and when they are non-terminating, repeating.

Let us consider the following terminating decimal forms of some rational numbers:

(i) 0.375 (ii) 1.04 (iii) 0.0875 (iv) 12.5 (v) 0.00025

Now let us express them in the form of $\frac{p}{q}$.

(i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $1.04 = \frac{104}{100} = \frac{104}{10^2}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $12.5 = \frac{125}{10} = \frac{125}{10^1}$

(v) $0.00025 = \frac{25}{100000} = \frac{25}{10^5}$

We see that all terminating decimals taken by us can be expressed as rational numbers whose denominators are powers of 10. Let us now prime factorize the numerator and denominator and then express in the simplest rational form :

Now (i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3} = \frac{3}{8}$

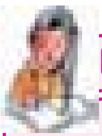
(ii) $1.04 = \frac{104}{10^2} = \frac{2^3 \times 13}{2^2 \times 5^2} = \frac{26}{5^2} = \frac{26}{25}$

(iii) $0.0875 = \frac{875}{10^4} = \frac{5^3 \times 7}{2^4 \times 5^4} = \frac{7}{2^4 \times 5}$

(iv) $12.5 = \frac{125}{10} = \frac{5^3}{2 \times 5} = \frac{25}{2}$

(v) $0.00025 = \frac{25}{10^5} = \frac{5^2}{2^5 \times 5^5} = \frac{1}{2^5 \times 5^3} = \frac{1}{4000}$

Do you see a pattern in the denominators? It appears that when the decimal expression is expressed in its simplest rational form then p and q are coprime and the denominator (i.e., q) has only powers of 2, or powers of 5, or both. This is because the powers of 10 can only have powers of 2 and 5 as factors.



Do This

Write the following terminating decimals in the form of $\frac{p}{q}$, $q \neq 0$ and p, q are co-primes

(i) 15.265 (ii) 0.1255 (iii) 0.4 (iv) 23.34 (v) 1215.8

What can you conclude about the denominators through this process?

Let us conclude

Even though, we have worked only with a few examples, you can see that any rational number which has a decimal expansion that terminates can be expressed as a rational number whose denominator is a power of 10. The only prime factors of 10 are 2 and 5. So, when we simplyfy the rational number, we find that the number is of the form $\frac{p}{q}$, where the prime factorization of q is of the form 2^n5^m, and n, m are some non-negative integers.

We can write our result formally :

Theorem-1.2 : Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.

You are probably wondering what happens the other way round. That is, if we have a rational number in the form $\frac{p}{q}$, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

So, it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{25}{2} = \frac{5^3}{2 \times 5} = \frac{125}{10} = 12.5$

(ii) $\frac{26}{25} = \frac{26}{5^2} = \frac{13 \times 2^3}{2^2 \times 5^2} = \frac{104}{10^2} = 1.04$

(iii) $\frac{3}{8} = \frac{3}{2^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} = 0.375$

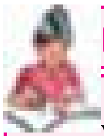
(iv) $\frac{7}{80} = \frac{7}{2^4 \times 5} = \frac{7 \times 5^3}{2^4 \times 5^4} = \frac{875}{10^4} = 0.0875$

(v) $\frac{1}{4000} = \frac{1}{2^5 \times 5^3} = \frac{5^2}{2^5 \times 5^5} = \frac{25}{10^5} = 0.00025$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Therefore, the decimal expansion of such a rational number terminates. We find that a rational number of the form $\frac{p}{q}$, where q is a power of 10, will have terminating decimal expansion.

So, we find that the converse of theorem 12 is also true and can be formally stated as :

Theorem 1.3 : Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.



Do This

Write the following rational numbers in the form of $\frac{p}{q}$, where q is of the form 2^n5^m where n, m are non-negative integers and then write the numbers in their decimal form

(i) $\frac{3}{4}$ (ii) $\frac{7}{25}$ (iii) $\frac{51}{64}$ (iv) $\frac{14}{23}$ (v) $\frac{80}{81}$

1.2.3 Non-terminating, recurring decimals in rational numbers

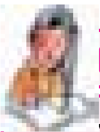
Let us now consider rational numbers whose decimal expansions are non-terminating and recurring. Once again, let us look at an example to see what is going on-

Let us look at the decimal conversion of $\frac{1}{7}$.

$\frac{1}{7} = 0.1428571428571$ ..... which is a non-terminating and recurring decimal. Notice, the block of digits '142857' is repeating in the quotient.

Notice that the denominator here, i.e., 7 is not of the form 2^n5^m.

$$
\begin{array}{r|l}
 & 0.1428571 \\
7 & 1.0000000 \\
 & 7 \\ \hline
 & 30 \\
 & 28 \\ \hline
 & 20 \\
 & 14 \\ \hline
 & 60 \\
 & 56 \\ \hline
 & 40 \\
 & 35 \\ \hline
 & 50 \\
 & 49 \\ \hline
 & 10 \\
 & 7 \\ \hline
 & 30
\end{array}
$$



Do This

Write the following rational numbers as decimals and find out the block of digits, repeating in the quotient.

(i) $\frac{1}{3}$ (ii) $\frac{2}{7}$ (iii) $\frac{5}{11}$ (iv) $\frac{10}{13}$

From the 'do this exercise' and from the example taken above, we can formally state:

Theorem-1.4 : Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).

From the discussion above, we can conclude that the decimal form of every rational number is either terminating or non-terminating repeating.

Example-3. Using the above theorems, without actual division, state whether the following rational numbers are terminating or non-terminating, repeating decimals.

(i) $\frac{16}{125}$ (ii) $\frac{25}{32}$ (iii) $\frac{100}{81}$ (iv) $\frac{41}{75}$

Solution : (i) $\frac{16}{125} = \frac{16}{5 \times 5 \times 5} = \frac{16}{5^3}$ is terminating decimal.

(ii) $\frac{25}{32} = \frac{25}{2 \times 2 \times 2 \times 2 \times 2} = \frac{25}{2^5}$ is terminating decimal.

(iii) $\frac{100}{81} = \frac{100}{3 \times 3 \times 3 \times 3} = \frac{10}{3^4}$ is non-terminating, repeating decimal.

(iv) $\frac{41}{75} = \frac{41}{3 \times 5 \times 5} = \frac{41}{3 \times 5^2}$ is non-terminating, repeating decimal.

Example-4. Write the decimal expansion of the following rational numbers without actual division.

(i) $\frac{35}{50}$ (ii) $\frac{21}{25}$ (iii) $\frac{7}{8}$

Solution : (i) $\frac{35}{50} = \frac{7 \times 5}{2 \times 5 \times 5} = \frac{7}{2 \times 5} = \frac{7}{10^1} = 0.7$

(ii) $\frac{21}{25} = \frac{21}{5 \times 5} = \frac{21 \times 2^2}{5 \times 5 \times 2^2} = \frac{21 \times 4}{5^2 \times 2^2} = \frac{84}{10^2} = 0.84$

(iii) $\frac{7}{8} = \frac{7}{2 \times 2 \times 2} = \frac{7}{2^3} = \frac{7 \times 5^3}{(2^3 \times 5^3)} = \frac{7 \times 25}{(2 \times 5)^3} = \frac{875}{(10)^3} = 0.875$

Exercise - 1.3

1. Write the following rational numbers in their decimal form and also state which are terminating and which have non-terminating, repeating decimal.

   (i) $\frac{3}{8}$ (ii) $\frac{229}{400}$ (iii) $4\frac{1}{5}$ (iv) $\frac{2}{11}$ (v) $\frac{8}{125}$

2. Without actually performing division, state whether the following rational numbers will have a terminating decimal form or a non-terminating, repeating decimal form.

(i) $\frac{13}{3125}$ (ii) $\frac{11}{12}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$ (v) $\frac{29}{343}$

(vi) $\frac{23}{2^3 5^2}$ (vii) $\frac{129}{2^2 5^7 7^5}$ (viii) $\frac{9}{15}$ (ix) $\frac{36}{100}$ (x) $\frac{77}{210}$

3. Write the following rationals in decimal form using Theorem 1.1.

(i) $\frac{13}{25}$ (ii) $\frac{15}{16}$ (iii) $\frac{23}{2^3.5^2}$ (iv) $\frac{7218}{3^2.5^2}$ (v) $\frac{143}{110}$

4. The decimal form of some real numbers are given below. In each case, decide whether the number is rational or not. If it is rational, and expressed in form $\frac{p}{q}$, what can you say about the prime factors of q?

(i) 43.123456789 (ii) 0.120120012000120000… (iii) $43.\overline{123456789}$

1.3 More about irrational numbers

Recall, a real number ("Q'" or "S") is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2},\ \sqrt{3},\ \sqrt{15},\ \pi,\ -\frac{\sqrt{2}}{\sqrt{3}},\ 0.10110111011110\ldots, \text{etc.}$$

In this section, we will prove some real numbers are irrationals with the help of the fundamental theorem of arithmetic. We will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and in general, $\sqrt{p}$ is irrational, where p is a prime.

Before we prove that $\sqrt{2}$ is irrational, we will look at a statement, the proof of which is based on the Fundamental Theorem of Arithmetic.

Statement-1 : ***Let p be a prime number. If p divides a^2, (where a is a positive integer), then p divides a.***

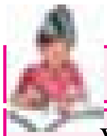
Proof : Let a be any positive integer. Then the prime factorization of a is as follows :

$a = p_1 p_2 \dots p_n$, where $p_1, p_2, \dots, p_n$ are primes, not necessarily distinct.

Therefore $a^2 = (p_1 p_2 \dots p_n)(p_1 p_2 \dots p_n) = p^2_1 p^2_2 \dots p^2_n$.

Now, here we have been given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. Also, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1\ p_2 \dots p_n$. So p is one of $p_1, p_2, \dots p_n$.

Now, since p is one of $p_1 p_2 \dots p_n$, it divides a.



Do This

Verify the statement proved above for p= 2, p = 5 and for a^2 = 1, 4, 9, 25, 36, 49, 64 and 81.

We are now ready to give a proof that $\sqrt{2}$ is irrational. We will use a technique called proof by contradiction.

Example-5. Prove that $\sqrt{2}$ is irrational.

Proof : Since we are using proof by contradiction, let us assume the contrary, i.e., $\sqrt{2}$ is rational.

If it is rational, then there must exist two integers r and s ($s \neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are co-prime.

So, $b\sqrt{2} = a$.

On squaring both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by statement 1, it follows that if 2 divides a^2 it also divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using statement 1 with $p= 2$).

Therefore, both a and b have 2 as a common factor.

But this contradicts the fact that *a* and *b* are co-prime and have no common factors other than 1.

This contradiction has arisen because of our assumption that $\sqrt{2}$ is rational. So, we conclude that $\sqrt{2}$ is irrational.

In general, it can be shown that $\sqrt{d}$ is irrational whenever d is a positive integer which is not the square of an integer. As such, it follows that $\sqrt{6}$, $\sqrt{8}$, $\sqrt{15}$ $\sqrt{24}$ etc. are all irrational numbers.

In earlier classes, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product or quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example-6. Show that $5 - \sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5 - \sqrt{3}$ is rational.

That is, we can find coprimes *a* and *b* ($b \neq 0$) such that $5 - \sqrt{3} = \frac{a}{b}$.

Therefore, $5 - \frac{a}{b} = \sqrt{3}$

Rearranging this equation, we get $\sqrt{3} = 5 - \frac{a}{b} = \frac{5b - a}{b}$

Since a and b are integers, we get $5 - \frac{a}{b}$ is rational so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5 - \sqrt{3}$ is rational.

So, we conclude that $5 - \sqrt{3}$ is irrational.

Example-7. Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, the contrary, that $3\sqrt{2}$ is rational.

i.e., we can find co-primes *a* and *b* ($b \neq 0$) such that $3\sqrt{2} = \frac{a}{b}$.

Rearranging, we get $\sqrt{2} = \frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that 3 $\sqrt{2}$ is irrational.

Example-8. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution : Let us suppose that $\sqrt{2} + \sqrt{3}$ is rational.

Let $\sqrt{2} + \sqrt{3} = \frac{a}{b}$, where a, b are integers and $b \neq 0$

Therefore, $\sqrt{2} = \frac{a}{b} - \sqrt{3}$.

Squarring on both sides, we get

$$2 = \frac{a^2}{b^2} + 3 - 2\frac{a}{b}\sqrt{3}$$

Rearranging

$$\frac{2a}{b}\sqrt{3} = \frac{a^2}{b^2} + 3 - 2$$

$$= \frac{a^2}{b^2} + 1$$

$$\sqrt{3} = \frac{a^2 + b^2}{2ab}$$

Since a, b are integers, $\frac{a^2+b^2}{2ab}$ is rational, and so, $\sqrt{3}$ is rational.

This contradicts the fact that $\sqrt{3}$ is irrational. Hence, $\sqrt{2} + \sqrt{3}$ is irrational.

Note :

1. The sum of the two irrational numbers need not be irrational.

   For example, if a = $\sqrt{2}$ and b = $-\sqrt{2}$, then both a and b are irrational, but a + b = 0 which is rational.

2. The product of two irrational numbers need not be irrational.

For example, $a = \sqrt{2}$ and $b = \sqrt{8}$, then both a and b are irrational, but $ab = \sqrt{16} = 4$ which is rational.

Exercise - 1.4

1. Prove that the following are irrational.

(i) $\frac{1}{\sqrt{2}}$ (ii) $\sqrt{3} + \sqrt{5}$ (iii) $6 + \sqrt{2}$ (iv) $\sqrt{5}$ (v) $3 + 2\sqrt{5}$

2. Prove that $\sqrt{p} + \sqrt{q}$ is irrational, where p, q are primes.

Try This

Properties of real numbers

In this chapter, you have seen many examples to show whether a number is rational or irrational. Now assuming that a, b and c represent real numbers, use your new knowledge to find out whether all the properties listed below hold for real numbers. Do they hold for the operations of subtraction and division? Take as many real numbers you want and investigate.

	Property	Addition	Multiplication
1.	Closure	$a + b = c$	$a . b = c$
2.	Commutative	$a + b = b + a$	$a . b = b.a$
3.	Associative	$a + (b + c) = (a + b) + c$	$a(bc) = (ab).c$
4.	Identity	$a + 0 = 0 + a = a$	$a.1 = 1.a = a$
5.	Inverse	$a + (-a) = 0$	$a. \frac{1}{a} = 1, (a \neq 0)$
6.	Distributive	$a (b + c) = ab + ac$	

1.5 Understanding logarithms

In this section, we are going to learn about logarithms. Logarithms are used for all sorts of calculations in engineering, science, business, economics and include calcuating compound interest, exponential growth and decay, pH value in chemistry, measurement of the magnitude of earthquakes etc.

However, before we can deal with logarithms, we need to revise the laws of exponents as logarithms and laws of exponents are closely related.

1.5.1 Exponents revisted

We know that when 81 is written as 3^4 it is said to be written in its exponential form. That is, in $81 = 3^4$, the number 4 is the exponent or **index** and 3 is the **base**. We say that -

81 is the 4^{th} **power** of the **base** 3 or 81 is the 4^{th} power of 3. Similarly, $27 = 3^3$.

Now, suppose we want to multiply 27 and 81; one way of doing this is by directly multiplying. But multiplication could get long and tedious if the numbers were much larger than 81 and 27. Can we use powers to makes our work easier?

We know that $81 = 3^4$. We also know that $27 = 3^3$.

Using the Law of exponents $a^m \times a^n = a^{m+n}$, we can write

$$27 \times 81 = 3^3 \times 3^4 = 3^7$$

Now, if we had a table containing the values for the powers of 3, it would be straight forward task to find the value of 3^7 and obtain the result of $81 \times 27 = 2187$.

Similaly, if we want to divide 81 by 27 we can use the law of exponents $a^m \div a^n = a^{m-n}$ where $m > n$. Then, $81 \div 27 = 3^4 \div 3^3 = 3^1$ or simply 3

Notice that by using powers, we have changed a multiplication problem into one involving addition and a division problem into one of subtration i.e., the addition of powers, 4 and 3 and the subtraction of the powers 4 and 3.

Do this

Try to write the numbers 10, 100, 1000, 10000 and 100000 in exponential forms. Identify the base and index in each case.

Try this

(i) Find 16 x 64, without actual multiplication, using exponents.

(ii) Find 25 x 125, without actual multiplication, using exponents.

(iii) Express 128 and 32 as powers of 2 and find $128 \div 32$.

1.5.2 Writing exponents as logarithms

We know that $10000 = 10^4$. Here, 10 is the base and 4 is the exponent. Writing a number in the form of a base raised to a power is known as exponentiation. We can also write this in another way called logarithms as

$$\log_{10} 10000 = 4.$$

This is stated as "log of 10000 to the base 10 is equal to 4".

We observe that the base in the original expression becomes the base of the logarithmic form. Thus,

$10000 = 10^4$ is the same as $\log_{10} 10000 = 4$.

In general, if $a^n = x$; we write it as $\log_a x = n$ where a and x are positive numbers and $a \neq 1$.

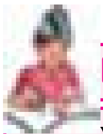
Let us understand this better through examples.

Example-9. Write i) $64 = 8^2$ ii) $64 = 4^3$ in logarithmic form.

Solution : (i) The logarithmic form of $64 = 8^2$ is $\log_8 64 = 2$.

(ii) The logarithmic form of $64 = 4^3$ is $\log_4 64 = 3$.

In this example, we find that log base 8 of 64 is 2 and log base 4 of 64 is 3. So, the logarithms of the same number to different bases are different.



Do this

Write $16 = 2^4$ in logarithmic form. Is it the same as $\log_4 16$?

Example-10. Write the exponential form of the following .

(i) $\log_{10} 100 = 2$ (ii) $\log_5 25 = 2$ (iii) $\log_2 2 = 1$ (iv) $\log_{10} 10 = 1$

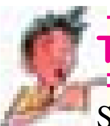
Solution : (i) Exponential form of $\log_{10} 100 = 2$ is $10^2 = 100$.

(ii) Exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

(iii) Exponential form of $\log_2 2 = 1$ is $2^1 = 2$.

(iv) Exponential form of $\log_{10} 10 = 1$ is $10^1 = 10$.

In cases (iii) and (iv), we notice that $\log_{10} 10 = 1$ and $\log_2 2 = 1$. In general, for any base a, $\boldsymbol{a^1 = a}$ so $\boldsymbol{\log_a a = 1}$



Try This

Show that $\boldsymbol{a^0 = 1}$ so $\boldsymbol{\log_a 1 = 0}$.

Do This

1. Write the following in logarithmic form.

   (i) $11^2 = 121$ (ii) $(0.1)^2 = 0.01$ (iii) $a^x = b$

2. Write the following in exponential form.

   (i) $\log_5 125 = 3$ (ii) $\log_4 64 = 3$ (iii) $\log_a x = b$ (iv) $\log_2 2 = 1$

Example-11. Determine the value of the following logarithms.

(i) $\log_3 9$ (ii) $\log_8 2$ (iii) $\log_c \sqrt{c}$

Solution : (i) Let $\log_3 9 = x$, then the exponential form is $3^x = 9 \Rightarrow 3^x = 3^2 \Rightarrow x=2$

(ii) Let $\log_8 2=y$, then the exponential form is $8^y=2 \Rightarrow (2^3)^y=2 \Rightarrow 3y=1 \Rightarrow y = \frac{1}{3}$

(iii) Let $\log_c \sqrt{c} = z$, then the exponential form is $c^z = \sqrt{c} \Rightarrow c^z = c^{\frac{1}{2}} \Rightarrow z = \frac{1}{2}$

1.5.3 Laws of Logarithms

Just like we have rules or laws of exponents, we have three laws of logarithms. We will try to prove them in the coming sections

1.5.3a The first law of logarithms

Suppose $x = a^n$ and $y = a^m$ where a>0 and $a \neq 1$. Then we know that we can write:

$\log_a x = n$ and $\log_a y = m$ .... (1)

Using the first law of exponents we know that $a^n \times a^m = a^{n+m}$

So, $xy = a^n \times a^m = a^{n+m}$ i.e. $xy = a^{n+m}$

Writing in the logarithmic form, we get

$\log_a xy = n+m$ .... (2)

But from (1), $n = \log_a x$ and $m=\log_a y$.

So, $\log_a xy = \log_a x + \log_a y$

So, if we want to multiply two numbers and find the logarithm of the product, we can do this by adding the logarithms of the two numbers. This is the first law of logarithms.

$$\mathbf{\log_a xy = \log_a x + \log_a y}$$

1.5.3b The second law of logarithms states $\log_a \frac{x}{y} = \log_a x - \log_a y$

TRY THIS

Prove the second law of logarithms by using the law of exponents $\frac{a^n}{a^m} = a^{n-m}$

1.5.3c The third law of logarithms

Let $x = a^n$ so $\log_a x = n$. Suppose, we raise both sides of $x = a^n$ to the power m, we get-

$x^m = (a^n)^m$

Using the laws of exponents-

$x^m = a^{nm}$

If we think of x^m as a single quantity, the logarithmic form of it, is

$\log_a x^m = nm$

$\log_a x^m = m \log_a x$ $\quad\quad (a^n = x \text{ so } \log_a x = n)$

This is the third law. It states that the logarithm of a power number can be obtained by multiplying the logarithm of the number by that power.

$$\log_a x^m = m \log_a x$$

Example-12. Expand log15

Solution : As you know, $\log_a xy = \log_a x + \log_a y$.

So, $\log 15 = \log(3 \times 5)$

$= \log 3 + \log 5$

Example-13. Expand $\log \frac{343}{125}$

Solution : As you know, $\log_a \frac{x}{y} = \log_a x - \log_a y$

So, $\log \frac{343}{125} = \log 343 - \log 125$

$= \log 7^3 - \log 5^3$

Since, $\log_a x^m = m \log_a x$

$= 3\log 7 - 3\log 5$

So $\log \frac{343}{125} = 3(\log 7 - \log 5)$.

Example-14. Write $2\log 3 + 3\log 5 - 5\log 2$ as a single logarithm.

Solution : $2\log 3 + 3\log 5 - 5\log 2$

$= \log 3^2 + \log 5^3 - \log 2^5$ (since in $m \log_a x = \log_a x^m$)

$= \log 9 + \log 125 - \log 32$

$= \log (9 \times 125) - \log 32$ (Since $\log_a x + \log_a y = \log_a xy$)

$= \log 1125 - \log 32$

$= \log \frac{1125}{32}$ (Since $\log_a x - \log_a y = \log_a \frac{x}{y}$)

Do This

1. Write the logarithms following in the form $\log_a x + \log_a y$

   (i) 8×32 (ii) 49×343 (iii) 81×729

2. Write the logarithms following in the form $\log_a x - \log_a y$

   (i) $8 \div 64$ (ii) $81 \div 27$

3. Write the logarithms following in logarithmic forms

   (i) $4^3 = (2^2)^3$ (ii) $36^2 = (6^2)^2$

Exercise - 1.5

1. Write the following in logarithmic form.

   (i) $3^5 = 243$ (ii) $2^{10} = 1024$ (iii) $10^6 = 1000000$

   (iv) $10^{-3} = 0.001$ (v) $3^{-2} = \frac{1}{9}$ (vi) $6^0 = 1$

   (vii) $5^{-1} = \frac{1}{5}$ (viii) $\sqrt{49} = 7$ (ix) $27^{\frac{2}{3}} = 9$ (x) $32^{-\frac{2}{5}} = \frac{1}{4}$

2. Write the following in exponemtial form

(i) $\log_{18}324 = 2$ (ii) $\log_{10}10000 = 4$ (iii) $\log_a \sqrt{x} = b$

(iv) $\log_4{}^8 = x$ (v) $\log_3\left(\frac{1}{27}\right) = y$

3. Determine the value of the following.

(i) $\log_{25}5$ (ii) $\log_{81}3$ (iii) $\log_2\left(\frac{1}{16}\right)$

(iv) $\log_7 1$ (v) $\log_x \sqrt{x}$ (vi) $\log_2 512$

(vii) $\log_{10}0.01$ (viii) $\log_{\frac{2}{3}}\left(\frac{8}{27}\right)$

4. Write each of the following expressions as log N. Determine the value of N. (You can assume the base is 10, but the results are identical which ever base is used).

(i) $\log 2 + \log 5$ (ii) $\log 16 - \log 2$ (iii) $3 \log 4$

(iv) $2 \log 3 - 3 \log 2$ (v) $\log 243 + \log 1$ (vi) $\log 10 + 2 \log 3 - \log 2$

5. Expand the following.

(i) $\log 1000$ (ii) $\log\left(\frac{128}{625}\right)$ (iii) $\log x^2y^3z^4$

(iv) $\log \frac{p^2q^3}{r}$ (v) $\log \sqrt{\frac{x^3}{y^2}}$

1.5.4 Standard bases of a logarithm (not meant for examination purpose)

There are two bases which are used more commonly than any others and deserve special mention. They are **base 10** and **base e**

Usually the expression log x implies that the base is 10. In calculators, the button marked log is pre-programmed to evaluate logarithms to base '10'.

For example,

$\log 2 = 0.301029995664\ldots$

$\log 3 = 0.4771212547197\ldots$

Are log 2 and log 3 irrational?

The second common base is 'e'. The symbol 'e' is called the exponential constant. This is an irrational number with an infinite, non-terminating non-recurring decimal expansion. It is usually approximated as 2.718. Base 'e' is used frequently in scientific and mathematical applications. Logarithms to base e or $\log_e$, are often written simply as 'ln'. So, "ln x" implies the base is 'e'. Such logarithms are also called natural logarithms. In calculators, the button marked 'ln' gives natural logs.

For example

$\ln 2 = 0.6931471805599\ldots$

$\ln 3 = 1.0986122886681\ldots$

Are ln(2) and ln(3) irrational?

1.5.5 Application of logarithms (Not meant for examination purpose)

Let us understand applications of logarithms with some examples.

Example-15. The magnitude of an earthquake was defined in 1935 by Charles Richer with the expression $M=\log\frac{I}{S}$; where I is the intensity of the earthquake tremor and S is the intensity of a "threshold earthquake".

(a) If the intensity of an earthquake is 10 times the intensity of a threshold earthquake, then what is its magnitude?

(b) If the magnitude of an earthquake registers 10 on the Richter scale, how many times is the intensity of this earthquake to that of a threshold earthquake?

Solution :

(a) Let the intensity of the earthquake be I, then we are given

$I = 10\ S$

The magnitude of an earthquake is given by-

$$M = \log\frac{10S}{S}$$

$\therefore$ The magnitude of the Delhi earthquake will be-

$$M = \log\frac{I}{S}$$

$$= \log 10$$

$$= 1$$

(b) Let x be the number of times the intensity of the earthquake to that of a threshold earthquake. So the intensity of earthquake is-

$$I = xS$$

We know that-

$$M = \log \frac{I}{S}$$

So, the magnitude of the earthquake is-

$$M = \log \frac{xs}{s}$$

or $M = \log x$

We know that $M = 10$

So $\log x = 10$ and therefore $x = 10^{10}$

Try This

The formula for calculating pH is pH = $-\log_{10}$ [H+] where pH is the acidity or basicity of the solution and [H^+] is the hydrogen ion concentration.

(i) If Shankar's Grandma's Lux Soap has a hydrogen ion concentration of $9.2 \times 10^{(-12)}$. What is its pH?

(ii) If the pH of a tomato is 4.2, what is its hydrogen ion concentration?

Optional Exercise

[This exercise is not meant for examination]

1. Can the number 6^n, n being a natural number, end with the digit 5? Give reason.
2. Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.
3. Check whether 12^n can end with the digit 0 for any natural number n?
4. Show that one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3, where n is any positive integer.
5. Prove that $\left(2\sqrt{3}+\sqrt{5}\right)$ is an irrational number. Also check whether $\left(2\sqrt{3}+\sqrt{5}\right)\left(2\sqrt{3}-\sqrt{5}\right)$ is rational or irrational.

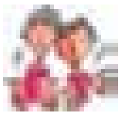
6. Without actual division, find after how many places of decimals in the decimal expansion of the following rational numbers terminates. Verify by actual division. What do you infer?

(i) $\frac{5}{16}$ (ii) $\frac{13}{2^2}$ (iii) $\frac{17}{125}$ (iv) $\frac{13}{80}$ (v) $\frac{15}{32}$ (vi) $\frac{33}{2^2 \times 5}$

7. If $x^2 + y^2 = 6xy$, prove that $2 \log (x + y) = \log x + \log y + 3 \log 2$

8. Find the number of digits in 4^{2013}, if $\log_{10} 2 = 0.3010$.

Note : Ask your teacher about integral part and decimal part of a logarithm of number.



What we have discussed

1. The Fundamental Theorem of Arithmetic states that every composite number can be expressed (factorized) as a product of its primes, and this factorization is unique, apart from the order in which the prime factors occur.

2. If p is a prime and p divides a^2, where a is a positive integer, then p divides a.

3. Let x be a rational number whose decimal expansion terminates. Then we can express x in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.

4. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

5. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which is non-terminating, repeating (recurring).

6. We define $\log_a x = n$, if $a^n = x$, where a and x are positive numbers and $a \neq 1$.

7. Laws of logarithms :

(i) $\log_a xy = \log_a x + \log_a y$ (ii) $\log_a \frac{x}{y} = \log_a x - \log_a y$

(iii) $\log_a x^m = m \log_a x$

8. Logarithms are used for all sorts of calculations in engineering, science, business and economics.