Long Answers Questions

1. If $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, then for any integer $n \geq 1$ show that $A^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$.

Sol. Given $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

We shall prove the result by Mathematical induction.

$$A^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$$

When n = 1

$$A^1 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

The result is true for n = 1.

Suppose the result is true for n = k

i.e. $A^k = \begin{bmatrix} a_1^k & 0 & 0 \\ 0 & a_2^k & 0 \\ 0 & 0 & a_3^k \end{bmatrix}$

Now $A^{k+1} = A^k \cdot A$

$$= \begin{bmatrix} a_1^k & 0 & 0 \\ 0 & a_2^k & 0 \\ 0 & 0 & a_3^k \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^k \cdot a_1 + 0 + 0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0 + a_2^k \cdot a_2 + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+a_3^k \cdot a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^{k+1} & 0 & 0 \\ 0 & a_2^{k+1} & 0 \\ 0 & 0 & a_3^{k+1} \end{bmatrix}$$

$\therefore$ The given result is true for n = k + 1

By Mathematical induction, the given result is true for all positive integral values of n.

i.e. $A^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$, for any integer $n \geq 1$.

2. If $\theta - \phi = \frac{\pi}{2}$, show that

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = 0$$

Sol. Given $\theta - \phi = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2} + \phi$

$$\cos\theta = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$$

$$\sin\theta = \sin\left(\frac{\pi}{2} + \phi\right) = \cos\phi$$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{bmatrix}$$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi\cos^2\phi - \sin^2\phi\cos^2\phi & \sin^3\phi\cos\phi - \sin^3\phi\cos\phi \\ -\sin\phi\cos^3\phi + \sin\phi\cos^3\phi & -\sin^2\phi\cos^2\phi + \sin^2\phi\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

3. If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ **then show that** $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, n is a positive integer.

Sol. We shall prove the result by Mathematical Induction.

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$n = 1 \Rightarrow A' = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

The result is true for n = 1

Suppose the result is true for n = k

$$A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

$\therefore$ The given result is true for n = k + 1

By Mathematical Induction, given result is true for all positive integral values of n.

4. Give examples of two square matrices A and B of the same order for which AB = 0 and BA ≠ 0.

Sol. $A = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}$

Then $AB = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = 0$

$BA = \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}\begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ a^2+a^2 & 0+0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 2a^2 & 0 \end{bmatrix} \neq 0$$

$\therefore$ AB = 0 and BA ≠ 0

5. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Sol. L.H.S. $C_1 \to C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3 = \text{R.H.S}$$

6. Show that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

Sol. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a(bc-a^2) - b(b^2-ac) + c(ab-c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -(a^3+b^3+c^3-3abc) \qquad ...(1)$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \ldots(2)$$

From (1), (2) we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

7. Show that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

Sol. L.H.S. = $\begin{vmatrix} a^2-1 & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ $\begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$

$$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (a-1)^2[0(6-3) - 0[3(a+1)-3) + 1(a+1-2)$$

$$= (a-1)^2(a-1) = (a-1)^3 = \text{R.H.S.}$$

8. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ **= abc (a – b)(b – c)(c – a).**

Sol. L.H.S. $abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= abc\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= abc(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= abc(a-b)(b-c)[0(c^2 - c(b+c) - 0(c^2 - c(a+b) + 1(b+c-a-b)]$$
$$= abc(a-b)(b-c)(c-a)$$

9. Show that

$\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$ = 4(a + b)(b + c)(c + a).

Sol. Let $\Delta = \begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

Let a + b = 0, then $\Delta = \begin{vmatrix} -2a & 0 & c+a \\ 0 & 2a & -a+c \\ c+a & c-a & -2c \end{vmatrix}$

Apply $R_1 \to R_1 + R_3$, $R_3 \to R_3 + R_2$

$$\Delta = \begin{vmatrix} c-a & c-a & -c+a \\ 0 & 2a & -a+c \\ c+a & c+a & -c-a \end{vmatrix}$$

$$= (c-a)(c+a)\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2a & c-a \\ 1 & 1 & -1 \end{vmatrix} = 0 \ (\because R_1 \equiv R_3)$$

∴ (c + a) is a factor for Δ.

Similarly a + b, b + c are also factors of Δ.

∵ Δ is a third degree expression in a, b, c.

Δ ≡ k(a + b)(b + c)(c + a), where k is a non-zero scalar.

Put a = 1, b = 1, c = 1, then

$$\begin{vmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{vmatrix} = k(1 + 1)(1 + 1)(1 + 1)$$

⇒ –2(4 – 4) – 2(–4 – 4) + 2(4 + 4) = 8k

⇒ 16 + 16 = 8k ⇒ k = 4

∴ Δ = 4(a + b)(b + c)(c + a)

Hence

$$\begin{vmatrix} -2a & a+b & a+c \\ a+b & -2b & b+c \\ a+c & b+c & -2c \end{vmatrix} = 4(a + b)(b + c)(c + a).$$

10. Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$.

Sol. L.H.S. = $\begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

By $R_1 \to R_1 + (R_2 + R_3)$

11. Show that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$

Sol. L.H.S. $\begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-a^2+bc-ca \\ 0 & c-b & c^2-b^2+ac-ab \end{vmatrix}$

$$= (b-a)(c-b)\begin{vmatrix} 1 & a & a^2-bc \\ 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \end{vmatrix}$$

$= (b-a)(c-b)\cdot 0 \, (R_2, R_3$ are identical)

$= 0 = \text{R.H.S.}$

12. Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2$.

Sol. L.H.S. = $\begin{vmatrix} x+2a & a & a \\ x+2a & x & a \\ x+2a & a & x \end{vmatrix}$

By $C_1 \to C_1 + (C_2 + C_3)$

$$= (x+2a)\begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix}$$

$$= (x+2a)\begin{vmatrix} 1 & a & a \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= (x+2a)[1(x-a)^2 - a(0(x-a)-0)] + a[0-0(x-a)]$$

$$= (x+2a)(x-a)^2 = \text{R.H.S}$$

13. Examine whether the following systems of equations are consistent or inconsistent and if consistent find the complete solutions.

I. x + y + z = 4

2x + 5y – 2z = 3

X + 7y – 7z = 5

Sol. Augmented matrix = A = $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_2 \to R_3 - R_1$ we get

$A \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ we have $A \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

$\rho(A) = 2, \rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

$\therefore$ The given system of equations are in consistent.

II. $x + y + z = 6$

$x - y + z = 2$

$2x - y + 3z = 9$

Sol. Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 9 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 0 & -4 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

$\rho(A) = 3 = \rho(AB)$

$\therefore$ The given system of equations are consistent.

$$\therefore\ A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 0 & -4 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

By $R_2 \to R_2\left(-\frac{1}{2}\right)$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

By $R_1 \to R_1 - R_2$, $R_3 \to R_3 + 3R_2$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

By $R_1 \to R_1 - R_3$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ Solution is x = 1, y = 2, z = 3.

III. x + y + z = 1

2x + y + z = 2

x + 2y + 2z = 1

Sol. Augmented matrix is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$, we obtain $A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$, we have $A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$\rho(A) = 2 = \rho(AB) < 3$

The given system of equations are consistent and have infinitely many solutions.

The solutions are given by:

[(x, y, z)1x = y + z = 0]

IV. $\mathbf{x + y + z = 9}$

$\mathbf{2x + 5y + 7z = 52}$

$\mathbf{2x + y - z = 0}$

Sol. Augmented matrix is $A = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{bmatrix}$$

By $R_3 \to R_3 (-1)$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

By $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 3R_3$, we have

$$A \sim \begin{bmatrix} 1 & 0 & -2 & -9 \\ 0 & 0 & -4 & -20 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

By $R_2 \to R_2\left(-\frac{1}{4}\right)$, we obtain

$$A \sim \begin{bmatrix} 1 & 0 & -2 & -9 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

By $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - 3R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

By $R_2 \leftrightarrow R_3$, we obtain $A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

$\therefore \rho(A) = \rho(AB) = 3$

The given system of equations are consistent have a unique solution.

$\therefore$ Solution is given by x = 1, y = 3, z = 5.

V. x + y + z = 6

x + 2y + 3z = 10

x + 2y + 4z = 1

Sol. Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 4 & 1 \end{bmatrix}$

By $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

By $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 15 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

By $R_1 \to R_1 - R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

The given system of equations are consistent. They have a unique solution.

$\therefore$ Solution is given by x = –7, y = 22, z = –9.

VI. x – 3y – 8z = –10

3x + y – 4z = 0

2x + 5y + 6z = 13

Sol. The Augmented matrix

$$A = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$$

By $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{10}\right)$, $R_3 \to R_3\left(\frac{1}{11}\right)$ we have

$$A \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$, we obtain

$$A \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $R_1 \to R_1 + 4R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 2 < 3$

$\therefore$ The given system of equations are consistent have infinitely many solutions.

$x + y = 2$ and $y + 2z = 3$

Taking $z = k$, $y = 3 - 2z = 3 - 2k$

$x = 2 - y = 2 - (3 - 2k) = 2 - 3 + 2k = 2k - 1$

$\therefore$ The solutions are given by $x = -1 + 2k$,

$y = 3 - 2k$, $z = k$ where k is any scalar.

VII. $2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

Sol. Augmented matrix $A = \begin{bmatrix} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 8 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$, we get $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 9 \\ 3 & 1 & 2 & 8 \end{bmatrix}$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & -5 & -7 & -10 \end{bmatrix}$$

By $R_3 \to R_3 - 5R_2$, $R_1 \to R_1 + 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 18 & 5 \end{bmatrix}$$

By $R_3 \to R_3\left(\frac{1}{18}\right)$, $R_2 \to R_2(-1)$ we obtain

$$A \sim \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 5/18 \end{bmatrix}$$

By $R_1 \to R_1 + 7R_3$, $R_2 \to R_2 - 5R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 35/18 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 5/18 \end{bmatrix}$$

By $R_2 \to R_2 - 5R_3$, we obtain

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 35/18 \\ 0 & 1 & 0 & 29/18 \\ 0 & 0 & 1 & 5/18 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

The given system of equations are consistent have a unique solution.

$\therefore$ Solution is given by $x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$.

VIII. Augmented matrix A = $\begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$

By $R_2 \to R_2 - 3R_1$, $R_3 \to R_2 - 5R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}$$

By $R_3 \to R_3 - 4R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 38 & 19 \end{bmatrix}$$

By $R_2 \to R_2(-1)$, $R_3 \to R_3\left(\frac{1}{2}\right)$, we have

$$A \sim \begin{vmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & 1/2 \end{vmatrix}$$

By $R_1 \to R_1 - 4R_3$, $R_2 \to R_2 - 14R_3$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

$\therefore$ The given system of equations are consistent have a unique solution.

$\therefore$ Solution is given by x = 2, y = 2, z = 1/2.

14. Solve the following system of equations.

i) By using Cramer's rule and matrix inversion method when the coefficient matrix is non-singular.

ii) Using Gauss-Jordan method. Also determine whether the system has a unique solution or infinite number of solutions or no solution and find solutions if exist.

1. 5x – 6y + 4z = 15

   7x + 4y – 3z = 19

   2x + y + 6z = 46

   Hint: $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

Sol. i) Cramer's rule:

$$\Delta = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$$

$$= 5(24+3)+6(42+6)+4(7-8)$$
$$= 135+288-4 = 419$$

$$\Delta_1 = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix}$$

$$= 15(24+3)+6(114+138)+4(19-184)$$
$$= 405+1512-660 = 1917-660 = 1257$$

$$\Delta_2 = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix}$$

$$= 5(114+138)-15(42+6)+4(322-38)$$
$$= 1260-720+1136 = 1676$$

$$\Delta_3 = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix}$$

$$= 5(184-19) + 6(322-38) + 15(7-8)$$
$$= 825 + 1074 - 15 = 2529 - 15 = 2514$$

$$x = \frac{\Delta_1}{\Delta} = \frac{1527}{419} = 3$$
$$y = \frac{\Delta_2}{\Delta} = \frac{1676}{419} = 4$$
$$z = \frac{\Delta_3}{\Delta} = \frac{2514}{419} = 6$$

Solution is x = 3, y = 4, z = 6.

ii) Matrix inversion method:

Hint: $A^{-1} = \frac{\text{Adj}A}{\det A}$

$$A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$$

$$A_1 = \begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} = 24 + 3 = 27$$
$$B_1 = -\begin{vmatrix} 7 & -3 \\ 1 & 6 \end{vmatrix} = -(42+6) = -48$$
$$C_1 = \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = 7 - 8 = -1$$

$$A_2 = -\begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix} = -(-36-4) = 40$$
$$B_2 = \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 30 - 8 = 22$$
$$C_2 = -\begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} = -(5+12) = -17$$

$$A_3 = \begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix} = 18 - 16 = 2$$

$$B_3 = -\begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = -(-15-28) = 43$$

$$C_3 = \begin{vmatrix} 5 & -6 \\ 7 & 4 \end{vmatrix} = 20+42 = 62$$

$$\text{Adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

det A = Δ = 419

$$A^{-1} = \frac{\text{AdjA}}{\text{detA}} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ 48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$x = A^{-1}D = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$= \frac{1}{419}\begin{bmatrix} +405+760+92 \\ -720+418+1978 \\ -15-323+2852 \end{bmatrix}$$

$$= \frac{1}{419}\begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

∴ Solution is x = 3, y = 4, z = 6.

iii) Gauss-Jordan method:

Augmented matrix is $A = \begin{bmatrix} 5 & -6 & 4 & 15 \\ 7 & 4 & -3 & 19 \\ 2 & 1 & 6 & 46 \end{bmatrix}$

$R_2 \to 5R_2 - 7R_1$, $R_3 \to 5R_3 - 2R_1$

$$A \sim \begin{bmatrix} 5 & -6 & 4 & 15 \\ 0 & 62 & -43 & -10 \\ 0 & 17 & 22 & 200 \end{bmatrix}$$

$R_1 \to 31R_1 + 3R_2$, $R_3 \to 62R_3 - 17R_2$

$$A \sim \begin{bmatrix} 155 & 0 & -5 & 435 \\ 0 & 62 & -43 & -10 \\ 0 & 0 & 2095 & 12570 \end{bmatrix}$$

$$R_3 \to R_3\left(\frac{1}{2095}\right)$$

$$A \sim \begin{bmatrix} 155 & 0 & -5 & 435 \\ 0 & 62 & -43 & -10 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$R_1 \to R_1 + 5R_3$, $R_2 \to R_2 + 43R_3$

$$A \sim \begin{bmatrix} 155 & 0 & 0 & 465 \\ 0 & 62 & 0 & 248 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$R_1 \to R_1\left(\frac{1}{155}\right) R_2 \to R_2\left(\frac{1}{62}\right)$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$\therefore$ Unique solution exists.

Solution is x = 3, y = 4, z = 6.

2. x + y + z = 1

   2x + 2y + 3z = 6

   x + 4y + 9z = 3

I. i) Cramer's rule

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 1(18-12) - 1(18-3) + 1(8-2)$$

$$= 6 - 15 + 6 = -3$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix}$$

$$= 1(18-12) - 1(54-9) + 1(24-6)$$

$$= 6 - 45 + 18 = -21$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 9 \end{vmatrix}$$

$$= 1(54-9) - 1(18-3) + 1(6-6)$$

$$= 45 - 15 = 30$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 6 \\ 1 & 4 & 3 \end{vmatrix}$$

$$= 1(6-24) - 1(6-6) + 1(8-2)$$

$$= -18 - 0 + 6 = -12$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-21}{-3} = 7$$

$$y = \frac{\Delta_2}{\Delta} = \frac{30}{-3} = -10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-12}{-3} = 4$$

Solution is x = 7, y = –10, z = 4.

ii) Matrix inversion method:

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$B_1 = -\begin{vmatrix} 2 & 3 \\ 1 & 9 \end{vmatrix} = -(18-3) = -15$$

$$C_1 = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8 - 2 = 6$$

$A_2 = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$

$B_2 = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9-1 = 8$

$C_2 = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$

$A_3 = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1$

$B_3 = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -(3-2) = -1$

$C_3 = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2-2 = 0$

$$\text{AdjA} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$$

$\text{DetA} = \Delta = -3$

$$A^{-1} = \frac{\text{AdjA}}{\text{DetA}} = -\frac{1}{3}\begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$$

$$x = A^{-1}D = \frac{1}{3}\begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 6-30+3 \\ -15+48-3 \\ 6-18+0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -21 \\ 30 \\ -12 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 4 \end{bmatrix}$$

$\therefore$ Solution is x = 7, y = –10, z = 4.

iii) Gauss-Jordan method:

Augmented matrix is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 6 \\ 1 & 4 & 9 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 8 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 8R_2, R_1 \to R_1 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 0 & -30 \end{bmatrix}$$

$R_3 \to R_3\left(\frac{1}{3}\right)$

$$A \sim \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -10 \end{bmatrix}$$

$R_1 \to R_1 - R_3, R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Unique solution exists

∴ Solution is x = 7, y = –10, z = 4.

3. x – y + 3z = 5

4x + 2y – z = 0

–x + 3y + z = 5

Sol. (i) Cramer's rule:

$$\Delta = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 1(2+3) + 1(4-1) + 3(12+2)$$

$$= 5 + 3 + 42 = 50$$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 3 \\ 0 & 2 & -1 \\ 5 & 3 & 1 \end{vmatrix}$$

$$= 5(2+3) + 1(0+5) + 3(0-10)$$

$$= 25 + 5 - 30 = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 0 & -1 \\ -1 & 5 & 1 \end{vmatrix}$$

$$= 1(0+5) - 5(4-1) + 3(20-0)$$

$$= 5 - 15 + 60 = 50$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 5 \\ 4 & 2 & 0 \\ -1 & 3 & 5 \end{vmatrix}$$

$$= 1(10-0) + 1(20-0) + 5(12+2)$$

$$= 10 + 20 + 70 = 100$$

$$x = \frac{\Delta_1}{\Delta} = \frac{0}{50} = 0$$

$$y = \frac{\Delta_2}{\Delta} = \frac{50}{50} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{100}{50} = 2$$

$\therefore$ Solution is x = 0, y = 1, z = 2

ii) Matrix inversion method:

Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 + 3 = 5$$

$$B_1 = -\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix} = -(4-1) = -3$$

$$C_1 = \begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix} = 12 + 2 = 14$$

$$A_2 = -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = -(-1-9) = 10$$

$$B_2 = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1+3 = 4$$

$$C_2 = -\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_3 = \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = 1-6 = -5$$

$$B_3 = -\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} = -(-1-12) = 13$$

$$C_3 = \begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} = 2+4 = 6$$

$$\therefore \text{Adj}A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 5 & 10 & -5 \\ -3 & 4 & 13 \\ 14 & -2 & 6 \end{bmatrix}$$

$$\begin{aligned} \det A &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= 1\cdot 5 - 1\cdot(-30+3\cdot 14 \\ &= 5+3+42 = 50 \end{aligned}$$

$$A^{-1} = \frac{\text{Adj}A}{\det A} = \frac{1}{50}\begin{bmatrix} 5 & 10 & -5 \\ -3 & 4 & 13 \\ 14 & -2 & 6 \end{bmatrix}$$

$$X = A^{-1}D = \frac{1}{50}\begin{bmatrix} 5 & 10 & -5 \\ -3 & 4 & 13 \\ 14 & -2 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 25+0-25 \\ -15+0+65 \\ 70+0+30 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 \\ 50 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ Solution is x = 0, y = 1, z = 2.

iii) Gauss Jordan method:

Augmented matrix is $\begin{bmatrix} 1 & -1 & 3 & 5 \\ 4 & 2 & -1 & 0 \\ -1 & 3 & 1 & 5 \end{bmatrix}$

By $R_2 \to R_2 - 4R_1, R_3 \to R_3 + R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 6 & -13 & -20 \\ 0 & 2 & 4 & 10 \end{bmatrix}$$

$R_3 \to R_3\left(\frac{1}{2}\right)$

$$A \sim \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 6 & -13 & -20 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$R_1 \to R_1 + R_3, R_2 \to R_2 - 6R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 5 & 10 \\ 0 & 0 & -25 & -50 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{-1}{25}\right)$

$$A \sim \begin{bmatrix} 1 & 0 & 5 & 10 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2, R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Unique solution

$\therefore$ Solution is x = 0, y = 1, z = 2.

4. 2x + 6y + 11 = 0

6x + 20y – 6z + 3 = 0

6y – 18z + 1 = 0

Sol. $\Delta = \begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{vmatrix}$

$= 2(-360+36) - 6(-108-0)$

$= -648 + 648 = 0$

$\therefore$ Cramer's rule and matrix inversion method cannot be used. $\because \Delta = 0$

ii) Gauss Jordan method:

Augmented matrix is $\begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$A \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$

$A \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -93 \end{bmatrix}$

$\rho(A) = 2, \rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

$\therefore$ The given system of equations do not have a solution.

5. 2x – y + 3z = 9

x + y + z = 6

x – y + z = 2

Sol. i) Cramer's rule:

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 2(1+1) + 1(1-1) + 3(-1-1)$$

$$= 4 + 0 - 6 = -2$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 9(1+1) + 1(6-2) + 3(-6-2)$$

$$= 18 + 4 - 24 = -2$$

$$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 2(6-2) - 9(1-1) + 3(2-6)$$

$$= 8 - 0 - 12 = -4$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(2+6) + 1(2-6) + 9(-1-1)$$

$$= 16 - 4 - 18 = -6$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1,\ y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3$$

Solution is x = 1, y = 2, z = 3.

ii) Matrix inversion method:

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1+1 = 2$$

$$B_1 = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_1 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2$$

$$A_2 = -\begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} = -(-1+3) = -2$$

$$B_2 = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1$$

$$C_2 = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -(-2+1) = 1$$

$$A_3 = \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -1-3 = -4$$

$$B_3 = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2-3) = 1$$

$$C_3 = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2+1 = 3$$

$$\text{Adj}A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} \text{Det}A &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= 2(2) - 1\cdot 0 + 3(-2) \\ &= 4-6 = -2 \end{aligned}$$

$$A^{-1} = \frac{\text{Adj}A}{\text{Det}A} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$X = A^{-1}D = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 18-12-8 \\ -6+2 \\ -18+6+6 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution is x = 1, y = 2, z = 3.

iii) Gauss-Jordan method:

Augmented matrix $A = \begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix}$

$R_1 \to R_1 - 2R_2, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 0 & -3 & 1 & -3 \\ 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \end{bmatrix}$$

$R_3 \to R_3\left(-\frac{1}{2}\right)$ gives $A \sim \begin{bmatrix} 0 & -3 & 1 & -3 \\ 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$R_1 \to R_1 + 3R_3, R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ gives $A \sim \begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

By $R_1 \leftrightarrow R_2, R_2 \leftrightarrow R_2 - R_3$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ The given equations have a unique solution.

Solution is x = 1, y = 2, z = 3.

6. 2x – y + 8z = 13

3x + 4y + 5z = 18

5x – 2y + 7z = 20

Sol. i) Cramer's rule:

$$\Delta = \begin{vmatrix} 2 & -1 & 8 \\ 3 & 4 & 5 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= 2(28+10)+1(21-25)+8(-6-20)$$
$$= 76-4-208 = -136$$

$$\Delta_1 = \begin{vmatrix} 13 & -1 & 8 \\ 18 & 4 & 5 \\ 20 & -2 & 7 \end{vmatrix}$$
$$= 13(28+10)+1(126-100)+8(-36-80)$$
$$= 494+26-928 = -408$$

$$\Delta_2 = \begin{vmatrix} 2 & 13 & 8 \\ 3 & 18 & 5 \\ 2 & 20 & 7 \end{vmatrix}$$
$$= 2(126-100)-13(21-25)+8(60-90)$$
$$= 52+52-240 = -136$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 13 \\ 3 & 4 & 18 \\ 5 & -2 & 20 \end{vmatrix}$$
$$= 2(80+36)+1(60-90)+13(-6-20)$$
$$= 232-30-338 = -136$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-408}{-136} = 3$$
$$y = \frac{\Delta_2}{\Delta} = \frac{-136}{-136} = 1$$
$$z = \frac{\Delta_3}{\Delta} = \frac{-136}{-136} = 1$$

∴ Solution is x = 3, y = 1, z = 1.

ii) Matrix inversion method:

Let $A = \begin{bmatrix} 2 & -1 & 8 \\ 3 & 4 & 5 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 13 \\ 18 \\ 20 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = 28 + 10 = 38$$

$$B_1 = -\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -(21 - 25) = 4$$

$$C_1 = \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = -6 - 20 = -26$$

$$A_2 = -\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = -(-7 + 16) = -9$$

$$B_2 = \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = (14 - 40) = -26$$

$$C_2 = -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -(-4 + 5) = -1$$

$$A_3 = \begin{vmatrix} -1 & 8 \\ 4 & 5 \end{vmatrix} = -5 - 32 = -37$$

$$B_3 = -\begin{vmatrix} 2 & 8 \\ 3 & 5 \end{vmatrix} = -(10 - 24) = 14$$

$$C_3 = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8 + 3 = 11$$

$$\text{Adj}\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}$$

$$\begin{aligned} \det A &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= 2 \cdot 38 + (-1)4 + 8(-26) \\ &= 76 - 4 - 208 = -136 \end{aligned}$$

$$A^{-1} = \frac{\text{Adj } A}{\text{Det } A} = -\frac{1}{136}\begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}$$

$$X = A^{-1}D = -\frac{1}{280}\begin{bmatrix} 38 & -9 & -37 \\ 4 & -26 & 14 \\ -26 & 1 & 11 \end{bmatrix}\begin{bmatrix} 13 \\ 18 \\ 20 \end{bmatrix}$$

$$= -\frac{1}{136}\begin{bmatrix} 494 & -162 & -740 \\ 52 & -468 & +280 \\ -338 & -18 & +220 \end{bmatrix}$$

$$= -\frac{1}{136}\begin{bmatrix} -408 \\ -136 \\ -136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Solution is x = 3, y = 1, z = 1.

iii) Gauss Jordan method:

Augmented matrix is $A = \begin{bmatrix} 2 & -1 & 8 & 13 \\ 3 & 4 & 5 & 18 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_2 \to 2R_2 - 3R_1, R_3 \to 2R_3 - 5R_2$ we get

$$A \sim \begin{bmatrix} 2 & -1 & 8 & 13 \\ 0 & 11 & -14 & -3 \\ 0 & 1 & -26 & -25 \end{bmatrix}$$

$R_1 \to R_1 + R_3, R_2 \to R_2 - 11R_3$, we get

$$A \sim \begin{bmatrix} 2 & 0 & -18 & -12 \\ 0 & 0 & 272 & 272 \\ 0 & 1 & -26 & -25 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{272}\right)$ we get

$$A \sim \begin{bmatrix} 2 & 0 & -18 & -12 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -26 & -25 \end{bmatrix}$$

$R_1 \to R_1 + 18R_2, R_3 \to R_3 + 26R_2$, we get

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3, R_1\left(\frac{1}{2}\right) \text{we get } A \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

∴ The given equations have a unique solution and solution is x = 3, y = 1, z = 1

7. 2x – y + 3z = 8

–x + 2y + z = 4

3x + y – 4z = 0

Sol. i) Cramer's rule:

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix}$$
$$= 2(-8-1)+1(4-3)+3(-1-6)$$
$$= -18+1-21 = -38$$

$$\Delta_1 = \begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix}$$
$$= 8(-8-1)+1(-16-0)+3(4-0)$$
$$= -72-16+12 = -76$$

$$\Delta_2 = \begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix}$$
$$= 2(-16-0)-8(4-3)+3(-0-12)$$
$$= -32-8-36 = -76$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix}$$
$$= 2(0-4)+1(0-12)+8(-1-6)$$
$$= -8-12-56 = -76$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-76}{-38} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-76}{-38} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-76}{-38} = 2$$

∴ Solution is x = 2, y = 2, z = 2.

ii) Matrix inversion method:

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} = -8-1 = -9$$

$$B_1 = -\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} = -(4-3) = -1$$

$$C_1 = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -1-6 = -7$$

$$A_2 = -\begin{vmatrix} -1 & 3 \\ 1 & -4 \end{vmatrix} = -(4-3) = -1$$

$$B_2 = \begin{vmatrix} 2 & 3 \\ 3 & -4 \end{vmatrix} = -8-9 = -17$$

$$C_2 = -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = -(2+3) = -5$$

$$A_3 = \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = -1-6 = -7$$

$$B_3 = -\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = -(4+3) = -7$$

$$C_3 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$$

$$\text{Adj}A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -7 \\ -7 & -5 & 3 \end{bmatrix}$$

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$
$$= 2(-9) - 1(-1) + 3(-7)$$
$$= -18 + 1 - 21 = -38$$

$$A^{-1} = \frac{\text{Adj}A}{\det A} = -\frac{1}{38}\begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -7 \\ -7 & -5 & 3 \end{bmatrix}$$

$$X = A^{-1}D = -\frac{1}{38}\begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -7 \\ -7 & -5 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{38}\begin{bmatrix} -72-4 \\ -8-68 \\ -56-20 \end{bmatrix} = -\frac{1}{38}\begin{bmatrix} -76 \\ -76 \\ -76 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Solution is x = 2, y = 2, z = 2.

iii) Gauss Jordan method:

Augmented matrix is $A = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

$R_1 \to R_1 + 2R_2$, $R_3 \to R_3 + 3R_2$, we get

$$A \sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -1 & 2 & 1 & 4 \\ 0 & 7 & -1 & 12 \end{bmatrix}$$

$R_2 \to 3R_2 - 2R_1$, $R_3 \to 3R_3 - 7R_1$, we have

$$A \sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -3 & 0 & -7 & -20 \\ 0 & 0 & -38 & -76 \end{bmatrix}$$

$R_3 \to R_3\left(-\frac{1}{38}\right)$, we get

$$A \sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -3 & 0 & -7 & -20 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2, R_2 \to R_2 + 7R_3$, we get

$$A \sim \begin{bmatrix} 0 & 3 & 0 & 6 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1\left(\frac{1}{3}\right), R_2 \to R_2\left(-\frac{1}{3}\right), R_1 \to R_2$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore$ The given equations have a unique solution and solution is x = 2, y = 2, z = 2.

8. Solve x + y + z = 9

2x + 5y + 7z = 52

2x + y – z = 0

Sol. i) Cramer's rule:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & -7 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1(-5-7) - 1(-2-14) + 1(2-10)$$

$$= -12 + 16 - 8 = -4$$

$$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 9(-5-7) - 1(-52-0) + 1(52-0)$$

$$= -108 + 52 + 52 = -4$$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= 1(-52-0) - 9(-2-14) + 1(0-104)$$

$$= -52 + 144 - 104 = -20$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 1(0-52) - 1(0-104) + 9(2-10)$$

$$= -52 + 104 - 72 = -20$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-12}{-4} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$$

Solution is x = 1, y = 3, z = 5

ii) Matrix inversion method:

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$

$$A_1 = \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} = -5 - 7 = -12$$

$$B_1 = -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} = -(-2 - 14) = 16$$

$$C_1 = \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} = 2 - 10 = -8$$

$$A_2 = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -(-1 - 1) = 2$$

$$B_2 = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3$$

$$C_2 = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1 - 2) = 1$$

$$A_3 = \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} = 7 - 5 = 2$$

$$B_3 = -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} = -(7 - 2) = -5$$

$$C_3 = \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = 5 - 2 = 3$$

$$\text{Adj}\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} \det A &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= 1(-12) + 1(16) + 1(-8) \\ &= -12 + 16 - 8 = -4 \end{aligned}$$

$$A^{-1} = \frac{\text{Adj}A}{\det A} = -\frac{1}{4}\begin{bmatrix} -12 & +2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$X = A^{-1}D = -\frac{1}{4}\begin{bmatrix} -12 & +2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -108+104 \\ 144-156 \\ -72+52 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 \\ -12 \\ -20 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$\therefore$ Solution is x = 1, y = 3, z = 5.

iii) Gauss Jordan method:

Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -4 & -8 & -52 \end{bmatrix}$$

$R_1 \to 3R_1 - R_2, R_3 \to 3R_3 + 4R_2$

$$A \sim \begin{bmatrix} 3 & 0 & -2 & -7 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{bmatrix}$$

$R_3 \to R_3\left(-\frac{1}{4}\right)$, we obtain

$$A \sim \begin{bmatrix} 3 & 0 & -2 & -7 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$R_1 \to R_1 + 2R_3, R_2 \to R_2 - 5R_3$, we get

$$A \sim \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$R_1 \to R_1\left(\frac{1}{3}\right), R_2 \to R_2\left(\frac{1}{3}\right)$ we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$\therefore$ The given equations have a unique solution and solution is x = 1, y = 3, z = 5.

9. Solve the following system of homogeneous equations.

i. 2x + 3y – z = 0

x – y – 2z = 0

3x + y + 3z = 0.

Sol. The coefficient matrix is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$

$$\det \text{ of } \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix}$$

$$= 2(-3+2) - 3(3+6) - 1(1+3)$$

$$= -2 - 27 - 4 = -33 \neq 0, \ \rho(A) = 3$$

Hence the system has the trivial solution

x = y = z = 0 only.

ii. 3x + y – 2z = 0

x + y + z = 0

x – 2y + z = 0

Hint: If the determinant of the coefficient matrix ≠ 0 then the system has trivial solution. i.e. $\rho(A) = 3$.

Sol. The coefficient matrix is $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} = 3(1+2) - 1(1-1) - 2(-2-1)$$

$$= 9 + 6 = 15 \neq 0, \ \rho(A) = 3$$

Hence the system has the trivial solutions

$x = y = z = 0$ only.

iii. x + y – 2z = 0

2x + y – 3z = 0

5x + 4y – 9z = 0

Sol. The coefficient matrix is $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & 9 \end{bmatrix}$ =A (say)

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & 9 \end{vmatrix} = 1(-9+12) - 1(-18+15) - 2(8-5)$$

$$= 3 + 3 - 6 = 0$$

∴ Rank of A = 2 as the sub matrix $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is non-singular, $\rho(A) < 3$.

Hence the system has non-trivial solution.

$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & 9 \end{bmatrix}$ the system of equation is equivalent to the given system of equations are

$x + y - 2z = 0$

$-y + z = 0$

Let $z = k \Rightarrow y = k, x = k$

$\therefore x = y = z = k$ for real number k.

IV. $x + y - z = 0$

$x - 2y + z = 0$

$3x + 6y - 5z = 0$

Sol. Coefficient matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 3 & 6 & -5 \end{bmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 3 & -2 \end{bmatrix}$$

$\Rightarrow \det A = 0$ as R_2, R_3 are identical.

and rank(A) = 2 as the sub matrix $\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ is non-singular. Hence the system has non-trivial solution, $\rho(A) < 3$.

The system of equations equivalent to the given system of equations are

$x + y - z = 0$

$3y - 2z = 0$

Let $z = k \Rightarrow y = \frac{2k}{3}, x = \frac{k}{3}$

$\therefore x = \frac{k}{3}, y = \frac{2k}{3}, z = k$ for any real number of k.

10. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$.

Sol. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -5 - 6 = -11 \neq 0$

Hence A is invertible.

The cofactor matrix of A = $\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$\therefore \text{Adj} A = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{\text{Adj} A}{\det A} = -\frac{1}{11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$

11. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Sol. $|A| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$= 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$

$= 7 - 3 - 3 = 1 \neq 0$

$\therefore$ A is invertible.

The factor of A is B = $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$\therefore \text{ Adj A} = B^T = \begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{AdjA}}{\det A} = \begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \because |A| = 1$$

12. Find the inverse of A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

det A = 1(4 – 3) – 2(6 – 3) + 1(3 – 2)

= 1 – 6 + 1 = –4

The cofactors of elements of A are

$A_{11} = +(4-3) = 1$

$A_{12} = -(6-3) = -3$

$A_{13} = +(3-2) = 1$

$A_{21} = -(4-1) - 3$

$A_{22} = +(2-1) = 1$

$A_{23} = -(1-2) = 1$

$A_{31} = +(6-2) = 4$

$A_{32} = -(3-3) = 0$

$A_{33} = +(2-6) = -4$

$$\therefore \text{Adj A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{AdjA}}{\det A} = -\frac{1}{4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

13. Find the rank of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ using elementary transformations.

Sol. $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ (on interchanging R_1 and R_2)

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{bmatrix} R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 4R_2 \end{matrix}$

The last matrix is singular and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a non-singular sub matrix of it. Hence its rank is 2.

$\therefore$ Rank (A) = 2.

14. Find the rank A = $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ using elementary transformations.

Sol. $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 + 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix}$

$R_1 \to R_1 + R_2, R_3 \to 2R_3 + 7R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 11 & 41 \end{bmatrix}$$

$R_1 \to 11R_1 - R_3, R_2 \to 11R_2 - R_3$

$$\sim \begin{bmatrix} 11 & 0 & 0 & 3 \\ 0 & -22 & 0 & 13 \\ 0 & 0 & 11 & 41 \end{bmatrix}$$

Now det $\begin{bmatrix} 11 & 0 & 0 \\ 0 & -22 & 0 \\ 0 & 0 & 11 \end{bmatrix} = 11(-232) \neq 0.$

Hence ran (A) = 3.

15. a) Apply the test of rank to examine whether the following equations are consistent. 2x – y + 3z = 8, –x + 2y + z = 4, 3x + y = 4z = 0 and, if consistent, find the complete solution.

Sol. The augmented matrix is $\begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

(On interchange R_1 and R_2)

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 + 3R_1$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix}$$

$R_3 \to 3R_3 - 7R_2$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix}$$

Now det $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38 \end{bmatrix} = (-1)(3)(-38) = 144$

Hence rank (A) = rank [AD] = 3

∴ The system has a unique solution.

We write the equivalent system of the equations from (F) : i.e.

–x + 2y + z = 4

3y + 5z = 16

–38z = –76

∴ z = 2, y = 2, x = 2 is the solution.

b) Show that the following system of equations is consistent and solve it completely x + y + z = 3, 2x + 2y – z = 3, x + y – z = 1.

Sol. Let A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and D = $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

Then the given equations can be written as

AX = D

Augmented Matrix $[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly all the sub matrices of order 3×3 of the above matrix are singular.

Hence rank of A $\neq$ 3, and rank of [AD] $\neq$ 3

Now consider the sub matrix $\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ of both A and AD is non-singular.

Hence Rank(A) = 2 = Rank [AD]

$\therefore$ The system of equations is consistent and has infinitely many solutions.

From the above [AD] matrix

$x + y + z = 3$

$-3z = -3 \Rightarrow z = 1$

and x + y = 2

Hence x = k, y = 2 – k, z = 1, k $\in$ R is a solution set.

16. Solve the following simultaneous linear equations by using Cramer's rule.

3x + 4y + 5z = 18

2x – y + 8z = 13

5x – 2y + 7z = 20

Sol. Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

i.e. AX = D

$$\Delta = \det A = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= 3(-7+16) - 4(14-40) + 5(-4+5)$$

$$= 27 + 104 + 5 = 136 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408$$

$$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$\Delta_3 = \begin{vmatrix} 2 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136$$

Hence by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1$$

∴ The solution of the given system of equation is x = 3, y = 1, z = 1.

17. Solve 3x + 4y + 5z = 18, 2x – y + 8z = 13 and 5x – 2y + 7z = 20 by using Matrix inversion method.

Sol. Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

Given can be written as AX = 8

By matrix inversion method $X = A^{-1}B$ is the solution.

det A = 3(–7 + 16) – 4(14 – 40) + 5(–4 + 5)

= 27 + 104 + 5 = 136

The cofactors of elements of A are

$A_{11} = +(-7+16) = 9$

$A_{12} = -(-14-40) = 26$

$A_{13} = +(-4+5) = 1$

$A_{21} = -(28+10) = -38$

$A_{22} = +(21-25) = -4$

$A_{23} = -(-6-20) = 26$

$A_{31} = +(32+5) = 37$

$A_{32} = -(24-10) = -14$

$A_{33} = (-3-8) = -11$

$$\therefore \text{Adj}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}A}{\det A} = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 162 & -494 & +740 \\ 468 & -52 & -280 \\ 18 & +338 & -220 \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ x = 3, y = 1, z = 1 is the solution.

18. Solve the following equations by Gauss-Jordan method.

3x + 4y + 5z = 18

2x – y + 8z = 13

5x – 2y + 7z = 20

Sol. The augmented matrix is $\begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_1 \to R_1 - R_2$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_2 \to R_2 - 2R_3, R_3 \to R_3 - 5R_1$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{bmatrix}$

$R_2 \to -5R_2 + 2R_3$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & 1 & -26 & 25 \\ 0 & -27 & 22 & -5 \end{bmatrix}$

$R_1 \to R_1 - 5R_2, R_3 \to R_3 + 27R_2$

$= \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & -680 & -680 \end{bmatrix}$

$R_3 \to R_3 + (-680)$

$= \begin{bmatrix} 1 & 0 & 127 & 135 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$R_1 \to R_1 - 127R_3, R_2 \to R_2 + 26R_3$

$= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Hence the solution is x = 3, y = 1, z = 1.

19. Solve the following system of equation by Gauss-Jordan method,

x + y + z = 3

2x + 2y – z = 3

x + y – z = 1.

Sol. The matrix equation is AX = D, where

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

The augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{2}{3}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the following is the system of equations equivalent to the given system of equations:

x + y + z = 3, –3z = –3

Hence z = 1, x + y = 1

∴ The solution set is x = k, y = 2 – k, z = 1 where k ∈ R.

20. By using Gauss-Jordan method, show that 2x + 4y – z = 0, x + 2y + 2z = 5, 3x + 6y – 7z = 2 has no solution.

Sol. $\begin{bmatrix} 2 & 4 & -1 & 0 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & -7 & 2 \end{bmatrix}$

$R_2 \to 2R_2 - R_1, R_3 \to 2R_3 - 3R_1$

$\sim \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -11 & 4 \end{bmatrix}$

$R_3 \to 5R_3 + 11R_2$

$\sim \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 130 \end{bmatrix}$ (Final matrix)

Hence the given system of equations is equivalent to the following system of equations

$2x + 4y - z = 0, 5z = 10 \Rightarrow z = 2$ and

$0(x) + 0(y) + 0(z) = 130$

Clearly no x, y, z satisfy the last equation

$\therefore$ Given system of equations has no solution.

21. Find the non-trivial solutions, if any, for the following equations.

2x + 5y + 6z = 0, x – 3y – 8z = 0 3x – y – 4z = 0

Sol. The coefficient matrix A = $\begin{bmatrix} 2 & 5 & 6 \\ 1 & -3 & -8 \\ 3 & 1 & -4 \end{bmatrix}$

On interchanging R_1 and R_2, we get

$$A \sim \begin{bmatrix} 1 & -3 & 8 \\ 2 & 5 & 6 \\ 3 & 1 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$ we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 11 & 22 \\ 0 & 10 & 20 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 10 & 20 \end{bmatrix}$$

$R_3 \to R_3 + 10$

$$A \sim \begin{bmatrix} 1 & -3 & 8 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

det A = 0 as R_2 and R_3 are identical.

Clearly rank (A) = 2, as the sub-matrix $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ is non-singular.

Hence the system has non-trivial solution.

The following system of equations is equivalent to the given system of equations.

$x - 3y - 8z = 0$

$y + 2z = 0$

On giving an arbitrary value k to z, we obtain the solution set is

$x = 2k, y = -2k, z = k, k \in R$ for $k \neq 0$.

We obtain non-trivial solutions.

$$\Rightarrow x+4-2y=0 \qquad ...(1)$$

$$2x+2-2y=0 \qquad ...(2)$$

$$x^2+4+y^2=9$$

$$x+2y+4=0$$

$$x-y+1=0$$

$$3y+3=0 \Rightarrow y=-1$$

$$\therefore x=y-1=-1-1=-2$$

$$\therefore x=-2, y=-1$$

22. Prove that $\begin{bmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{bmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$

$$= \frac{1}{abc} abc \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$\underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - R_3}{=}} \begin{vmatrix} 0 & a^2-c^2 & a^3-c^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$\underset{R_2 \to R_2 - R_1}{\overset{(a-c)(b-c)}{=}} \begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-c & b^2+bc+c^2 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-c & b^2-a^2+bc-ac \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & 1 & c+a+b \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} a+c & a^2+ac+c^2 \\ 1 & a+b+c \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)(ab+bc+ca)$$

23. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Sol. $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ 1/c & 0 & 1 \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

24. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol. L.H.S. = $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

By applying $R_1 \Rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

By applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$$= (2)(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S.}$$

25. Show that

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

Sol. L.H.S = $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^2-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2+ba+a^2 \\ 0 & c+a & c^2+ca+a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$= -(a-b)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b-c & b^2-c^2+a(b-c) \\ 0 & c+a & c^2+ca+a^2 \end{vmatrix}$$

$$= -(a-b)(c-a)(b-c)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & 1 & b+c+a \\ 0 & c+a & c^2+ca+a^2 \end{vmatrix}$$

$= -(a-b)(b-c)(c-a)$

$[(c^2+ca+a^2)-(b+c+a)(c+a)]$

$= -(a-b)(b-c)(c-a)$

$[c^2+ca+a^2-b(c+a)-(c+a)^2 =$

$= -(a-b)(b-c)(c-a)$

$[c^2+ca+a^2-bc-ab-c^2-2ca-a^2]$

$= -(a-b)(b-c)(c-a)[-ab-bc-ca]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

26. If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$.

Sol. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} [\because 1+\omega+\omega^2 = 0]$$

$= 0$

27. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Sol. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a-b-c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3$$