Short Answers Questions

1. If $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$ **then find the values of x, y, z and a**

Sol. Given $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$

$\therefore x - 1 = 1 \Rightarrow x = 1 + 1 = 2$

$5 - y = 3 \Rightarrow y = 5 - 3 = 2$

$z - 1 = 4 \Rightarrow z = 4 + 1 = 5$

$a - 5 = 0 \Rightarrow a = 5$

2. Find trace of A if A = $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$**.**

Sol. Trace of A = Sum of the diagonal elements

$= 1 - 1 + 1 = 1.$

3. If A = $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ find B – A and 4A – 5B.

Sol. Given A = $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$, B = $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$B - A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1-0 & 2-1 & 3-2 \\ 0-2 & 1-3 & 0-4 \\ 0-4 & 0-5 & -1+6 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & -2 & -4 \\ -4 & -5 & 5 \end{bmatrix}$$

$$4A - 5B = 4\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix} - 5\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 8 \\ 8 & 12 & 16 \\ 16 & 20 & -24 \end{bmatrix} - \begin{bmatrix} -5 & 10 & 15 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+5 & 4-10 & 8-15 \\ 8-0 & 12-5 & 16-0 \\ 16-0 & 20-0 & -24+5 \end{bmatrix} = \begin{bmatrix} 5 & -6 & 7 \\ 8 & 7 & 16 \\ 16 & 20 & -19 \end{bmatrix}$$

4. If A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ find 3B – 2A.

Sol. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$3B - 2A = 3\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-2 & 6-4 & 3-6 \\ 3-6 & 6-4 & 9-2 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{bmatrix}$$

5. If A = $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then find A^4.

Note: A is diagonal matrix.

Sol. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then

$$A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}, n \in N$$

$$A^4 = \begin{bmatrix} 3^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

6. If A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ then find A^3.

Sol. $A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. If A= $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$**, then find** $A^3 - 3A^2 - A - 3I$.

Sol Given $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & -2-2-1 & 1+2+1 \\ 0+0-3 & 0+1+1 & 0-1-1 \\ 3-0+3 & -6-1-1 & 3+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}$$

$$A^3 = A^2\,A = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+12 & -8-5-4 & 4+5+4 \\ -3+0-6 & 6+2+2 & -3-3-2 \\ 6+0+15 & -12-8-5 & 6+8+5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix}$$

Now $A^3 - 3A^2 - A - 3I$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix} - 3\begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16-12-1-3 & -17+15+2+0 & 13-12-1-0 \\ -9+9+0-0 & 10-6-1-3 & -7+6+1+0 \\ 21-18-3+0 & -25+24+1+0 & 19-15-1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3\times 3}$$

$\therefore A^3 - 3A^2 - A - 3I = 0$

8. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bE)^3 = a^3I + 3a^2bE$.

Sol. $aI + bE = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$$(aI + bE)^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$(aI + bE)^3 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}$$

$$= a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= a^3I + 3a^2bE$$

9. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$**, show that** $A \cdot A^T = A^T A = I_2$**.**

Sol. $A \cdot A^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad \text{...(1)}$$

$$A^T \cdot A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad \text{...(2)}$$

From (1), (2) we get $A \cdot A^T = A^T \cdot A = I_2$.

10. If $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 0 \\ 3 & -1 & -5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \\ 1 & 2 & 0 \end{bmatrix}$ **then find** $3A - 4B^T$**.**

Sol. $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \\ 1 & 2 & 0 \end{bmatrix}$

$$\Rightarrow B^T = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \\ 1 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 1 \\ -1 & -2 & 2 \\ 0 & 5 & 0 \end{bmatrix}$$

$$3A - 4B^T = 3\begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 0 \\ 3 & -1 & -5 \end{bmatrix} - 4\begin{bmatrix} 2 & 0 & 1 \\ -1 & -2 & 2 \\ 0 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 15 & 9 \\ 6 & 12 & 0 \\ 9 & -3 & -15 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 4 \\ -4 & -8 & 8 \\ 0 & 20 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3-8 & 15-0 & 9-4 \\ 6+4 & 12+8 & 0-8 \\ 9-0 & -3-20 & -15-0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 15 & 5 \\ 10 & 20 & -8 \\ 9 & -23 & -15 \end{bmatrix}$$

11. $A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$ **then find AB^T and BA^T.**

Sol. $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$

$$\Rightarrow B^T = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -14+2 & 28-4 & -7+0 \\ 2-2 & -4+4 & 1+0 \\ -10-3 & 20+6 & -5+0 \end{bmatrix} = \begin{bmatrix} -12 & 24 & -7 \\ 0 & 0 & 1 \\ -13 & 26 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow A^T = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$BA^T = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -14+2 & 2-2 & -10-3 \\ 28-4 & -4+4 & 20+6 \\ -7+0 & 1+0 & -5+0 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -13 \\ 24 & 0 & 26 \\ -7 & 1 & -5 \end{bmatrix}$$

12. For any square matrix A, show that AA′ is symmetric.

Sol. A is a square matrix

$(AA')' = (A')'A' = A \cdot A'$

$\because (AA')' = AA'$

$\Rightarrow AA'$ is a symmetric matrix.

13. Show that $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol. L.H.S. = $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix}$

$= \begin{vmatrix} bc & b+c & 1 \\ c(a-b) & a-b & 0 \\ b(a-c) & a-c & 0 \end{vmatrix}$ by $\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$= (a-b)(a-c)\begin{vmatrix} bc & b+c & 1 \\ c & 1 & 0 \\ b & 1 & 0 \end{vmatrix}$

= (a – b) (a – c) (c – b)

[Expanding on 3rd column]

= (a – b) (b – c) (c – a) = R.H.S.

14. Show that $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix}$ **= $a^3 + b^3 + c^3 - 3abc$.**

Sol. $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix}$

$R_1 \to R_1 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -c & -a & -b \\ a & b & c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ -c & -a & -b \\ a & b & c \end{vmatrix}$$

$$= (a+b+c)[(-ac+b^2)-(-c^2+ab)+(-bc+a^2)]$$

$$= (a+b+c)(-ac+b^2+c^2-ab-bc+a^2)$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= a^3+b^3+c^3-3abc$$

15. Show that $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$ **.**

Sol. L.H.S. = (y + z)[(z + x) (x + y) – yz] – x[y(x + y) – yz] + x[yz – z(z + x)]

$$= (y+z)(zx+yz+x^2+xy-yz)-x(xy+y^2-yz)+x(yz-z^2-zx)$$

$$= (y+z)(zx+x^2+xy)-x(xy+y^2-yz)+x(yz-z^2-zx)$$

$$= xyz+x^2y+xy^2+xz^2+x^2z+xyz-x^2y-xy^2+xyz+xyz-xz^2-x^2z$$

$$= 4xyz$$

16. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$**, then show that abc = –1.**

Hint: If each element in row (column) of a square matrix is the sum of two numbers, then its discriminant can be expressed as the sum of discriminant of two square matrices.

Sol. L.H.S = $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\therefore$ 1 + abc = 0 $\Rightarrow$ abc = –1

17. If A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then show that $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$ for all positive integers n.

Sol. Let S(n) be the statement that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Given A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\Rightarrow A^1 = \begin{bmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{bmatrix} \Rightarrow$ S(1) is true.

Assume that S(k) is true.

$$\therefore A^k = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$$

Now $A^{k+1} = A^k A$

$$= \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta+\theta) & -\sin(k\theta+\theta) \\ \sin(k\theta+\theta) & \cos(k\theta+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

$\therefore$ S(k + 1) is true.

By principle of Mathematical induction S(n) is true for all $n \in N$.

$$\therefore\ A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$ for all positive integers n.

18. If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ **then show that** $A^2 - 4A - 5I = 0$.

Sol. $$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$A^2 - 5A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

19) For any $n \times n$ matrix A, prove that A can be uniquely expressed as a sum of a symmetric and a skew symmetric matrix.

Sol. For A square matrix of order n,

A + A′ is symmetric and A – A′ is a skew symmetric matrix and

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

To prove uniqueness, let B be a symmetric matrix and C be a skew-symmetric matrix, such that A = B + C.

Then A′ = (B + C)′ = B′ + C′

= B + (–C) = B – C

and hence $B = \frac{1}{2}(A + A'), C = \frac{1}{2}(A - A')$

20. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

Sol. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

We get $\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$

$= (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(c-b)$

$\therefore \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

21. Show that the value of the determinant of skew-symmetric matrix of order three is always zero.

Sol. Let us consider a skew-symmetric matrix of order 3×3, say

$$A = \begin{bmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} \because |B| = |B^T|$$

$$= -|A| \Rightarrow 2|A| = 0$$

Hence |A| = 0.

22. Find the value of x if

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$$

Sol. $$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_2 - R_1 \end{matrix} \Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0$$

$$(-2)(-6)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(30-24) - (2x-3)(10-6) + (3x-4)(4-3) = 0$$

$$\Rightarrow 6x - 12 - 8x + 12 + 3x - 4 = 0$$

$$x - 4 = 0 \Rightarrow x = 4$$