# MATRICES

## Synopsis

1. A matrix is an arrangement of real or complex numbers into rows and columns so that all the rows (columns) contain equal no. of elements.
2. If a matrix consists of 'm' rows and 'n' columns, then it is said to be of order $m \times n$.
3. A matrix of order $n \times n$ is said to be a square matrix of order n.
4. A matrix $(a_{ij})_{m\times n}$ is said to be a null matrix if $a_{ij} = 0$ for all i and j.
5. Two matrices of the same order are said to be equal if the corresponding elements in the matrices are all equal.
6. A matrix $(a_{ij})_{n\times n}$ is a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.
7. A matrix $(a_{ij})_{n\times n}$ is a scalar matrix if $a_{ij}=0$ for all $i \neq j$ and $a_{ij}=k$ (constant) for $i = j$
8. A matrix $(a_{ij})_{n\times n}$ is said to be a unit matrix of order n, denoted by $I_n$ if $a_{ij}=1$, when $i= j$ and $a_{ij} = 0$ when $i \neq j$

   **Ex:** $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
9. If $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, then $A + B = (a_{ij} + b_{ij})_{m\times n}$
10. Matrix addition is commutative and associative
11. If $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$, then $AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)_{m\times p}$
12. Matrix multiplication is not commutative but associative

**13.** If A is a matrix of order $m \times n$, then $AI_n = I_mA = A(AI = IA = A)$

**14.** If AB = CA = I, then B = C

**15.** If $A = (a_{ij})_{m\times n}$, then $A^T = (a_{ji})_{n\times m}$

**16.** $(KA)^T = KA^T$, $(A + B)^T = A^T + B^T$, $(AB)^T = B^T.A^T$

**17.** A(B + C) = AB + AC, (A + B)C = AC + BC

**18.** **i)** A square matrix is said to be "non-singular" if $\det A \neq 0$

**ii)** A square matrix is said to be "singular" if $\det A = 0$

**19.** If AB = 0, where A and B are non-zero square matrices, then both A and B are singular.

**20.** A minor of any element in a square matrix is determinant of the matrix obtained by omitting the row and column in which the element is present.

**21.** In $(a_{ij})_{n\times n}$, the cofactor of $a_{ij}$ is $(-1)^{i+j} \times$ (minor of $a_{ij}$).

**22.** In a square matrix, the sum of the products of the elements of any row (column) and the corresponding cofactors is equal to the determinant of the matrix.

**23.** In a square matrix, the sum of the products of the elements of any row (column) and the corresponding cofactors of any other row (column) is always zero.

**24.** If A is any square matrix, then A adjA = adjA. A = detA. I

**25.** If A is any square matrix and there exists a matrix B such that AB = BA = I, then B is called the inverse of A and denoted by $A^{-1}$.

**26.** $AA^{-1} = A^{-1}A = I$.

**27.** If A is non-singular, then $A^{-1} = \frac{\text{adj}A}{\det A}$ (or) $\text{adj}A = |A|A^{-1}$

**28.** If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

**29.** $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}.A^{-1}$, $(A^{-1})^{T} = (A^{T})^{-1}$; $(ABC....)^{-1} = .....C^{-1}B^{-1}A^{-1}$.

**Theorem**: Matrix multiplication is associative. i.e. if conformability is assured for the matrices A, B and C, then (AB)C = A(BC).

Sol. **Proof**:

Let $A = (a_{ij})_{m\times n}, B = (b_{jk})_{n\times p}, C = (c_{kl})_{p\times q}$

$AB = (d_{ik})_{m\times p}$, where $d_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

$(AB)C = (f_{il})_{m\times q}$, where $f_{il} = \sum_{k=1}^{p} d_{ik}c_{kl}$

$BC = (g_{il})_{n\times q}$, where $g_{il} = \sum_{k=1}^{p} b_{jk}c_{kl}$

$A(BC) = (h_{il})_{m\times q}$, where $h_{il} = \sum_{j=1}^{n} a_{ij}g_{jl}$

$$f_{il} = \sum_{k=1}^{p} d_{ik}c_{kl} = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)c_{kl}$$

$$= \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}c_{kl}\right) = \sum_{j=1}^{n} a_{ij}h_{jl} = h_{il}$$

(AB)C = A(BC)

**Theorem:** Matrix multiplication is distributive over matrix addition i.e. if conformability is assured for the matrices A, B and C, then

i) A (B + C) = AB + AC

ii) (B + C) A = BA + CA

Sol. **Proof:**

Let $A = (a_{ij})_{m\times n}$, $B = (b_{jk})_{n\times p}$, $C = (c_{jk})_{n\times p}$

$B+C=(d_{jk})_{n\times p}$, where $d_{jk}=b_{jk}+c_{jk}$

$A(B+C)=(x_{ik})_{m\times p}$, where $x_{ik}=\sum_{j=1}^{n}a_{ij}d_{jk}$

$AB=(f_{ik})_{m\times p}$, where $f_{ik}=\sum_{j=1}^{n}a_{ij}b_{jk}$

$AC=(g_{ik})_{m\times p}$, where $g_{ik}=\sum_{j=1}^{n}a_{ij}c_{jk}$

$AB+AC=(y_{ik})_{m\times p}$, where $y_{ik}=f_{ik}+g_{ik}$

$$x_{ik}=\sum_{j=1}^{n}a_{ij}d_{jk}=\sum_{j=1}^{n}a_{ij}(b_{jk}+c_{jk})$$

$$=\sum_{j=1}^{n}a_{ij}b_{jk}+\sum_{j=1}^{n}a_{ij}c_{jk}=f_{ik}+g_{ik}=y_{ik}$$

$\therefore A(B+C)=AB+AC$

Similarly we can prove that

(B + C) = BA + CA.

**Theorem**: If A is any matrix, then $(A^T)^T = A$.

Sol. Let $A = (a_{ij})_{m \times n}$

$A^T = (a'_{ji})_{n \times m}$, where $a'_{ji} = a_{ij}$

$(A^T)^T = (a''_{ji})_{m \times n}$, where $a''_{ij} = a_{ji}$

$a''_{ij} = a'_{ji} = a_{ij}$

$\therefore (A^T)^T = A$

**Theorem**: If A and B are two matrices o same type, then $(A + B)^T = A^T + B^T$.

**Proof:**

Let $A = (a_{ij})_{m \times n}, B = (b_{ij})_{m \times n}$

$A + B = (c_{ij})_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$

$(A + B)^T = (c'_{ji})_{n \times m}, c'_{ji} = c_{ij}$

$A^T = (a'_{ji})_{n \times m}$, where $a'_{ji} = a_{ij}$

$B^T = (b'_{ji})_{n \times m}$, where, $b'_{kj} = b_{jk}$

$A^T + B^T = (d_{ji})_{n \times m}$, where $d_{ji} = a'_{ji} + b'_{ji}$

$c'_{ji} = c_{ij} = a_{ij} + b_{ij} = a'_{ji} + b'_{ji} = d_{ji}$

$\therefore (A + B)^T = A^T + B^T$

**Theorem**: If A and B are two matrices for which conformability for multiplication is assured, then $(AB)^T = B^T A^T$.

**Sol.** **Proof:** Let $A = (a_{ij})_{m\times n}, B = (b_{ji})_{n\times p}$

$$AB = (c_{ik})_{m\times p}, \text{where } c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

$$(AB)^T = (c'_{ki})_{p\times m}, \text{where } c'_{ki} = c_{ik}$$

$$A^T = (a'_{ji})_{n\times m}, \text{where } a'_{ji} = a_{ij}$$

$$B^T = (b'_{kj})_{p\times n}, \text{where } b'_{kj} = b_{jk}$$

$$B^T \cdot A^T = (d_{ki})_{p\times m}, \text{ where } d_{ki} = \sum_{j=1}^{n} b'_{kj} a'_{ji}$$

$$c'_{ki} = c_{ik} = \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} b'_{kj} a'_{ji} d_{ki}$$

$$\therefore (AB)^T = B^T A^T$$

**Theorem**: If A and B are two invertible matrices of same type then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

**Sol.** **Proof:** A is invertible matrix $\Rightarrow A^{-1}$ exists and $AA^{-1} = A^{-1}A = I$.

B is an invertible matrix $\Rightarrow B^{-1}$ exists and

$BB^{-1} = B^{-1}B = I$

$$\text{Now } (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB = B^{-1}B = I$$

$\therefore$ AB is invertible and

$$\therefore (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = 1$$

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**Theorem**: If A is an invertible matrix then $A^T$ is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Sol. **Proof:** A is invertible matrix $\Rightarrow A^{-1}$ exists and $AA^{-1} = A^{-1}A = I$

$$(AA^{-1})^T = (A^{-1}A)^T = I^T$$
$$\Rightarrow (A^{-1})^T A^T = A^T (A^{-1})^T = I$$
$$\Rightarrow \text{By def. } (A^T)^{-1} = (A^{-1})^T$$

**Theorem**: If A is a non-singular matrix then A is invertible and $A^{-1} = \frac{\text{Adj}A}{\det A}$.

Sol. **Proof**: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a non-singular matrix.

$\therefore \det A \neq 0.$

$$\text{Adj}\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A \cdot \text{Adj}\,A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \det A\, I$$

$$\therefore\ A \cdot \frac{AdjA}{\det A} = I$$

Similarly we can prove that $A \cdot \frac{AdjA}{\det A} = I$

$$\therefore\ A^{-1} = \frac{AdjA}{\det A}$$