MATHEMATICS PAPER IIA

TIME: 3hrs **Max. Marks.**75

Note: This question paper consists of three sections A, B and C.

SECTION- A

Very Short Answer Type Questions. 10X2 =20

1. Prove that roots of (x – a)(x – b) = h^2 are always real.

2. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$ *and* $r \neq 0$, then find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

3. Find the 3rd term from the end in the expansion of $\left(x^{-2/3} - \frac{3}{x^2}\right)^8$.

4. Find the eccentricity of the ellipse whose equation is $|z-4| + \left|z - \frac{12}{5}\right| = 10$

5. If $1, \omega, \omega^2$ are the cube roots of units then prove that $\frac{1}{2+\omega} + \frac{1}{1+2\omega} = \frac{1}{1+\omega}$

6. Find all values of $(-i)^{\frac{1}{6}}$

7. If ${}^{18}P_{r-1} : {}^{17}P_{r-1} = 9:7$, find r

8. Find the number of ways of selecting 3 vowels and 2 consonants from the letters of the word equation.

9. Find the constant C, so that $F(x) = C\left(\frac{2}{3}\right)^x$, x = 1, 2, 3.......... is the p.d.f of a discrete random variable X.

10. Find the mean deviation about the mean for the following distribution.

x_i	10	30	50	70	90
f_i	4	24	28	16	8

SECTION- B

Short Answer Type Questions.

Answer Any Five of the Following 5 X 4 = 20

11. In the expression $\frac{x-p}{x^2-3x+2}$ takes all values of $x \in R$, then find the bounds for p.

12. Show that $\left\{\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right\}^{8/3} = -1$

13. Find the number of ways of arranging the letters of the word SINGING so that
    (i) They begin and end with I
    (ii) The two G's come together

14. Find the number of 5 digited numbers that can be formed using the digits. 1, 1, 2, 2, 3. How many of them are even?

15. Resolve $\frac{2x^2+3x+4}{(x-1)(x^2+2)}$ into partial fractions.

16. If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.

17. A, B, C are three horse in a race. The probability of A to win the race is twice that of B and probability of B is twice that of C. what are the probabilities of A , B and C to win the race ?

SECTION -C

Long Answer Type Questions.

Answer Any Five of the Following 5 X 7 =35

18. Given that the sum of two roots of $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ is zero, find the roots of the equation

19. State and prove binomial theorem.

20. If I, n are positive integers, 0 < f < 1 and if $(7 + 4\sqrt{3})^n = I + f$, then show that (i) I is an odd integer and (ii) (I + f)(I – f) = 1.

21. Show that the points in the Argand diagram represented by the complex numbers – 2 + 7i, $\frac{-3}{2} + \frac{1}{2}i$, 4 – 3i, $\frac{7}{2}$(1 + i) are the vertices of a rhombus.

22. The probabilities of three mutually exclusive events are respectively given as $\frac{1+3p}{3}, \frac{1-p}{4}, \frac{1-2p}{2}$. Prove that $\frac{1}{3} \le p \le \frac{1}{2}$.

23. Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Find the mean of the random variable.

24. The arithmetic mean and standard deviation of a set of 9 items are 43 and 5 respectively. If an item of value 63 is added to that set, find the new mean and standard deviation of 10 item set given.

Maths IIA paper 3 – Solutions

1. Prove that roots of (x – a)(x – b) = h^2 are always real.

Sol: $(x-a)(x-b)=h^2$

$$x^2-(a+b)x+(ab-h^2)=0$$

Discriminant $=(a+b)^2-4(ab-h^2)=0$

$$=(a+b)^2-4ab+4h^2$$

$$=(a-b)^2+4h^2$$

$$=(a-b)^2+(2h)^2>0$$

$\therefore$ Roots are real.

2. If a, b, c are the roots of $x^3-px^2+qx-r=0$ and $r\neq 0$, then find $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$

Sol: Given that a, b, c are the roots of

$x^3-px^2+qx-r=0$, then

$a+b+c=p, ab+bc+ca=q, abc=r$

Now $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2}$

$$=\frac{(ab+bc+ca)^2-2abc(a+b+c)}{a^2b^2c^2}$$

$$=\frac{q^2-2pr}{r^2}$$

3. Find the 3rd term from the end in the expansion of $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$.

Sol. Comparing with $(X + a)^n$, we get

$$X=x^{-2/3},\ a=\frac{-3}{x^2}, n=8$$

In the given expansion $\left(x^{-2/3} - \frac{3}{x^2}\right)^8$, we have n + 1 = 8 + 1 = 9 terms.

Hence the 3rd term from the end is 7th term from the beginning.

$\therefore T_7 = {}^nC_6\,(X)^{n-6}\,(a^6)$

$$= {}^8C_6 (x^{-2/3})^{8-6}\left(\frac{-3}{x^2}\right)^6 = {}^8C_6 x^{-4/3}\cdot\frac{3^6}{x^{12}}$$

$$= \frac{8\times 7}{1\times 2}\cdot 3^6\cdot x^{-4/3-12} = 28\cdot 3^6\cdot x^{-40/3}$$

4. Find the eccentricity of the ellipse whose equation is $|z-4| + \left|z - \frac{12}{5}\right| = 10$

Sol.

SP+S¹P =2a

$S(4,0) \qquad S'\left(\frac{12}{5},0\right)$

2a = 10 $\Rightarrow$ a=5

SS'= 2ae

$$\Rightarrow 4 - \frac{12}{5} = 5X5e \Rightarrow \frac{8}{5} = 10e \Rightarrow e = \frac{4}{5}$$

5. If $1, \omega, \omega^2$ **are the cube roots of units then prove that** $\frac{1}{2+\omega} + \frac{1}{1+2\omega} = \frac{1}{1+\omega}$

Solution : -

L.H.S $\frac{1}{2+\omega} + \frac{1}{1+2\omega}$

$$\frac{1+2\omega+2+\omega}{(2+\omega)(1+2\omega)} = \frac{3(1+\omega)}{2+4\omega+\omega+2\omega^2}$$

$$= \frac{3(1+\omega)}{2(1+\omega^2)+5\omega}$$

$$= \frac{3(-\omega^2)}{-2\omega+5\omega} \quad \because 1+\omega = -\omega^2$$

$$1+\omega^2=\omega$$

$$=\frac{-3\omega^2}{3\omega}=-\omega$$

$$=-\frac{1}{\omega^2}=\frac{1}{1+\omega}$$

6. Find all values of $(-i)^{\frac{1}{6}}$

Solution : -

$$(-i)^{\frac{1}{6}}=\left\{\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right\}^{\frac{1}{6}}$$

$$=cis\left(\frac{2k\pi-\pi/2}{6}\right)k=0,1,2,3,4,5$$

$$\therefore(-1)^{\frac{1}{6}}=cis(4k-1)\frac{\pi}{12}\quad k=0,1,2,3$$

7. If $^{18}P_{r-1}:{}^{17}P_{r-1}=9:7$, find r

Sol: $^{18}P_{r-1}:{}^{17}P_{r-1}=9:7\Rightarrow\frac{^{18}P_{r-1}}{^{17}P_{r-1}}=\frac{9}{7}$

$$\Rightarrow\frac{18!}{[18-(r-1)]!}\times\frac{[17-(r-1)]!}{17!}=\frac{9}{7}$$

$$\Rightarrow\frac{18!}{(19-r)!}\frac{(18-r)!}{17!}=\frac{9}{7}$$

$$\Rightarrow\frac{18\times17!(18-r)!}{(19-r)(18-r)!17!}=\frac{9}{7}$$

$$\Rightarrow18\times7=9(19-r)$$

$$\Rightarrow14=19-r\qquad\therefore r=19-14=5$$

8. Find the number of ways of selecting 3 vowels and 2 consonants from the letters of the word equation.

Sol: The word EQUATION contains 5 vowels and 3 consonants.

The 3 vowels can be selected from 5 vowels in $^5C_3=10$ ways

The 2 consonants can be selected from 3 consonants in ${}^{3}C_{2} = 3$ ways

$\therefore$ The required number of ways of selecting 3 vowels and 2 consonants $= 10 \times 3 = 30$

9. Find the constant C, so that $F(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3..........$ **is the p.d.f of a discrete random variable X.**

Sol. Given $F(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3$

We know that $p(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3 \ldots$

$$\because \sum_{x=1}^{\infty} p(x) = 1 \Rightarrow \sum_{x=1}^{\infty} c\left(\frac{2}{3}\right)^x = 1 \quad \Rightarrow c\left[\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + ...\infty\right] = 1$$

$$\Rightarrow C\frac{2}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + ...\infty\right] = 1 \quad \Rightarrow \frac{2c}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1 \quad \Rightarrow \frac{2c}{3} \times 3 = 1 \Rightarrow c = \frac{1}{2}$$

10. Find the mean deviation about the mean for the following distribution.

x_i	**10**	**30**	**50**	**70**	**90**
f_i	**4**	**24**	**28**	**16**	**8**

Sol.

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
10	4	40	40	160
30	24	720	20	480
50	28	1400	0	0
70	16	1120	20	320
90	8	720	40	320

	N = 80	$\Sigma f_i x_i = 4000$		$\Sigma f_i \mid x_i - \overline{x} \mid = 1280$

$$\therefore \text{ Mean } (\overline{x}) = \frac{\Sigma f_i x_i}{N} = \frac{4000}{80} = 50$$

$$\therefore \text{ Mean Deviation about the Mean } = \frac{\sum_{i=1}^{5} f_i \mid x_i - \overline{x} \mid}{N} = \frac{1280}{80} = 16 .$$

11. In the expression $\dfrac{x-p}{x^2-3x+2}$ takes all values of $x \in R$, then find the bounds for p.

Sol: $y = \dfrac{x-p}{x^2-3x+2}$

$$y(x^2-3x+2) = x-p$$

$$x^2 y + x(-3y-1) + 2y + p = 0$$

Discriminant ≥ 0

$$(-3y-1)^2 - 4y(2y+p) \geq 0$$

$$9y^2 + 6y + 1 - 8y^2 - 4p \geq 0$$

$$y^2 + y(6-4p) + 1 \geq 0$$

Discriminant < 0

$$(6-4p)^2 - 4 < 0$$

$$16p^2 - 48p + 36 - 4 < 0$$

$$16p^2 - 48p + 32 < 0$$

$$p^2 - 48p + 32 < 0$$

$$p^2 - 3p + 2 < 0$$

$$(p-2)(p-1) < 0$$

$1 < p < 2.$

12. **Show that** $\left\{\dfrac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right\}^{8/3}=-1$

Solution : -

$$\text{LHS }=\left\{\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right\}^{8/3}$$

$$\left\{\frac{1+\cos\left(\frac{\pi}{2}-\pi/8\right)+i\sin\left(\frac{\pi}{2}-\pi/8\right)}{1+\cos\left(\frac{\pi}{2}-\pi/8\right)-1\sin\left(\frac{\pi}{2}-\pi/3\right)}\right\}^{8/3}$$

$$\left\{\frac{1+\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}}{1+\cos\frac{3\pi}{8}-i\sin\frac{3\pi}{8}}\right\}^{8/3}=\left\{\frac{2\cos^2\frac{3\pi}{16}+2i\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}{2\cos^2\frac{3\pi}{16}-2i\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}\right\}^{8/3}$$

$$\left[\frac{2\cos\frac{3\pi}{16}\left\{\cos\frac{3\pi}{16}+1\sin\frac{3\pi}{16}\right\}}{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)}\right]^{\frac{8}{3}}$$

$$\left[\frac{\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)}{\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)\left(\cos\left(\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\right)}\right]^{8/3}$$

$$\left[\frac{\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)^2}{\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}}\right]^{8/3}$$

$$\left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)^{8/3}$$

$$\cos\pi+i\sin\pi=-1$$

13. Find the number of ways of arranging the **letters of the word SINGING so that**

(i) they begin and end with I

(ii) the two G's come together

Sol: The word SINGING has 2 I's, 2 G's and 2 N's and one S. Total 7 letters.

I					I

(i) First, we fill the first and last places with I's in $\frac{2!}{2!} = 1$ way as shown below.

Now we fill the remaining 5 places with the remaining 5 letters in $\frac{5!}{2!2!} = 30$ ways.

Hence the number of required permutations = 30

(ii) Treat two G's as one unit. Then we have 5 letters 2 I's, 2 N's and one S + one unit of 2G's = 6 can be arranged in $\frac{6!}{2!2!} = \frac{720}{2 \times 2} = 180$ ways.

Now the two G's among themselves can be arranged in one way. Hence the number of received permutations = 180 x 1 = 180.

14. Find the number of 5 digited numbers that can be formed using the digits. 1, 1, 2, 2, 3. How many of them are even?

Sol. In the given 5 digits, there are two 1's and two 2's.

Hence the number of 5 digited numbers that can beformed is $\frac{5!}{2!2!} = 30$

Now, for the number to be even, it should end with 2.

				2

After filling the units place with 2, the remaining4 places can be filled with the remaining4 digits 1,1, 2, 3 in $\frac{4!}{2!} = 12$ ways. Thus, the number of 5 digited even numbers that can be formed using the digits. 1, 1,2, 2, 3, is 12.

15. Resolve $\frac{2x^2+3x+4}{(x-1)(x^2+2)}$ into partial fractions.

Sol. Let $\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2}$

Multiplying with $(x-1)(x^2+2)$

$2x^2+3x+4=A(x^2+2)+(Bx+C)(x-1)$

$x=1\Rightarrow 2+3+4=A(1+2)$

$9=3A\Rightarrow A=3$

Equating the coefficients of x^2

$2 = A + B \Rightarrow B = 2 - A = 2 - 3 = -1$

Equating constants

$4 = 2A - C \Rightarrow C = 2A - 4 = 6 - 4 = 2$

$$\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{3}{x-1}+\frac{-x+2}{x^2+2}$$

16. If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.

Sol. i) Let A be the event that the sum of the numbers is even when two numbers are selected out of 20 consecutive natural numbers.

In 20 consecutive natural numbers, we have 10 odd and 10 even natural number.

$\because$ The sum of two odd natural numbers is an even number and the sum of two even natural numbers is also an even number

$n(A)={}^{10}C_2+{}^{10}C_2=\frac{2(10\times 9)}{1\times 2}=190$ $n(S)={}^{20}C_2=\frac{20\times 19}{1\times 2}=190$ $P(A)=\frac{n(A)}{n(S)}=\frac{90}{190}=\frac{9}{19}$

ii) Probability that the sum of two numbers is an odd number

$P(\overline{A})=1-P(A)=1-\frac{9}{19}=\frac{10}{19}$

17. A, B, C are three horse in a race. The probability of A to win the race is twice that of B and probability of B is twice that of C. what are the probabilities of A , B and C to win the race ?

Sol. Let A, B, C be the events that the horses A, B, C win the race respectively.

Given P(A) = 2P(B), P(B) = 2P(C)

$\therefore$P(A) = 2P(B) 2[2P(C)] = 4P(C)

Since the horses A, B and C run the race,

$A\cup B\cup C$ = S and A, B, C are mutually disjoint

$P(A \cup B\cup C) = P(A) + P(B) + P(C)$

$\Rightarrow$ P(S) = 4p(C) +2p(C) + P(C)

$\Rightarrow$ 1 = 7P(C)

$$\therefore P(C)=\frac{1}{7}$$

$$P(A)=4P(C)=4\times\frac{1}{7}=\frac{4}{7}$$

$$P(B)=2P(C)=2\times\frac{1}{7}=\frac{2}{7}$$

$$\therefore P(A)=\frac{4}{7},P(B)=\frac{2}{7},P(C)=\frac{1}{7}$$

18. Given that the sum of two roots of $x^4-2x^3+4x^2+6x-21=0$ is zero, find the roots of the equation

Sol: Let $\alpha, \beta, \gamma, \delta$ are the roots of given equation, since sum of two is zero

$\alpha+\beta=0$

Now $\alpha+\beta+\gamma+\delta=2 \Rightarrow \gamma+\delta=2$

Let $\alpha\beta=p, \gamma\delta=q$

The equation having the roots α, β is

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\therefore x^2+p=0$$

The equation having the roots γ, δ is

$$x^2-(\gamma+\delta)x+\gamma\delta=0$$

$\therefore x^2 - 2x + q = 0$

$\therefore x^4 - 2x^3 + 4x^2 + 6x - 21$

$= \left(x^2 + p\right)\left(x^2 - 2x + q\right)$

$= x^4 - 2x^3 + x^2\left(p + q\right) - 2px + pq$

Comparing the like terms

$p + q = 4, -2p = 6$

$-3 + q = 4 \; p = -3$

$q = 7$

$\therefore x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3} \; and \; x^2 - 2x + 7 = 0 \; \Rightarrow x = \dfrac{2 \pm \sqrt{4-28}}{2}$

$= \dfrac{2 \pm 2\sqrt{6i}}{2}$

$= 1 \pm \sqrt{6i}$

$\therefore$ Roots are $-\sqrt{3}, \sqrt{3}, 1 - i\sqrt{6} \; and \; 1 + i\sqrt{6}$

19. State and prove binomial theorem.

Let *n* be *a* positive integer and *x*, *a* be real numbers, then

$$(x+a)^n = {}^nC_0.x^n a^0 + {}^nC_1.x^{n-1} a^1 + {}^nC_2.x^{n-2} a^2 + ... + {}^nC_r.x^{n-r} a^r + ... + {}^nC_n.x^0 a^n$$

Proof:

We prove this theorem by using the principle of mathematical induction (on *n*).

When $n = 1, (x+a)^n = (x+a)^1 = x + a = {}^1C_0 x^1 a^0 + {}^1C_1 x^0 a^1$

Thus the theorem is true for $n = 1$

Assume that the theorem is true for $n = k \geq 1$ (where *k* is a positive integer). That is

$$(x+a)^k = {}^kC_0.x^k.a^0 + {}^kC_1.x^{k-1}.a^1 + {}^kC_2.x^{k-2}.a^2 + ... + {}^kC_r.x^{k-r}.a^r + ... + {}^kC_k.x^0.a^k$$

Now we prove that the theorem is true when

$n = k + 1$ also

$(x+a)^{k+1} = (x+a)(x+a)^k$

$$= (x+a)({}^kC_0.x^k.a^0 + {}^kC_1.x^{k-1}.a^1 + {}^kC_2.x^{k-2}.a^2 + ... + {}^kC_r.x^{k-r}.a^r + ... + {}^kC_k.x^0.a^k)$$

$$= {}^kC_0.x^{k+1}.a^0 + {}^kC_1.x^k.a^1 + {}^kC_2.x^{k-1}.a^2 + ... +$$
$${}^kC_r.x^{k-r+1}.a^r + ... + {}^kC_k.x^1.a^k + {}^kC_0.x^k.a^1 + {}^kC_1.x^{k-1}.a^2 + ...$$
$$+ {}^kC_{r-1}.x^{k-r+1}.a^r + ... + {}^kC_{k-1}.x^1.a^k + {}^kC_k.x^0.a^{k+1}$$

$$= {}^kC_0.x^{k+1}.a^0 + ({}^kC_1 + {}^kC_0).x^k.a^1 + ({}^kC_2 + {}^kC_1).x^{k-1}.a^2 +$$
$$.... + ({}^kC_r + {}^kC_{r-1}).x^{k-r+1}.a^r + ... + ({}^kC_k + {}^kC_{k-1}).x^1.a^k + {}^kC_k.x^0.a^{k+1}$$

since ${}^kC_0 = 1 = {}^{k+1}C_0, {}^kC_r + {}^kC_{r-1} = {}^{(k+1)}C_r$ for $1 \le r \le k$, ${}^kC_k = 1 = {}^{(k+1)}C_{(k+1)}$

$$(x + a)^{k+1} = {}^{(k+1)}C_0.x^{k+1}.a^0 + {}^{(k+1)}C_1.x^k.a^1 + {}^{(k+1)}C_2.x^{k-1}.a^2 + ... + {}^{(k+1)}C_r.x^{k-r+1}.a^r + ... +$$
$${}^{(k+1)}C_k.x^1.a^k + {}^{k+1}C_{k+1}.x^0.a^{k+1}$$

Therefore the theorem is true for $n = k+1$

Hence, by mathematical induction, it follows that the theorem is true of all positive integer n

20. If I, n are positive integers, 0 < f < 1 and if $(7+4\sqrt{3})^n = I + f$, then show that (i) I is an odd integer and (ii) (I + f)(I – f) = 1.

Sol. Given I, n are positive integers and

$(7+4\sqrt{3})^n = I + f$, $0 < f < 1$

Let $7 - 4\sqrt{3} = F$

Now $6 < 4\sqrt{3} < 7 \Rightarrow -6 > -4\sqrt{3} > -7$

$\Rightarrow 1 > 7 - 4\sqrt{3} > 0 \Rightarrow 0 < (7-4\sqrt{3})^n < 1$

$\therefore\ 0 < F < 1$

1 + f + F = $(7+4\sqrt{3})^n (7-4\sqrt{3})^n$

$$= \left[\begin{array}{r}{}^nC_0 7^n + {}^nC_1 7^{n-1}(4\sqrt{3}) + {}^nC_2 7^{n-2}(4\sqrt{3})^2 \\ + ... + {}^nC_n (4\sqrt{3})^n\end{array}\right]$$

$$= \left[\begin{array}{r}{}^nC_0 7^n - {}^nC_1 7^{n-1}(4\sqrt{3}) + {}^nC_2 7^{n-2}(4\sqrt{3})^2 \\ + ... + {}^nC_n (-4\sqrt{3})^n\end{array}\right]$$

$$= 2\left[{}^nC_0 7^n + {}^nC_2 7^{n-2}(4\sqrt{3})^2 ......\right]$$

= 2k where k is an integer.

$\therefore$ 1 + f + F is an even integer.

$\Rightarrow$ f + F is an integer since I is an integer.

But 0 < f < 1 and 0 < F < 1 $\Rightarrow$ f + F < 2

$\therefore$ f + F = 1 …(1)

$\Rightarrow$ I + 1 is an even integer.

$\therefore$ I is an odd integer.

(I + f)(I – f) = (I + f)F, by (1)

$$= (7+4\sqrt{3})^n (7-4\sqrt{3})^n$$

$$= \left[(7+4\sqrt{3})(7-4\sqrt{3})\right]^n = (49-48)^n = 1$$

21. Show that the points in the Argand diagram represented by the complex numbers – 2 + 7i, $\frac{-3}{2}+\frac{1}{2}i$, 4 – 3i, $\frac{7}{2}$(1 + i) are the vertices of a rhombus.

Sol: $A(-2, 7),\ B\left(\frac{-3}{2},\frac{1}{2}\right), C(4,-3), D\left(\frac{7}{2},\frac{7}{2}\right)$

$$AB = \sqrt{\left(-2+\frac{3}{2}\right)^2 + \left(7-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{169}{4}} = \frac{\sqrt{170}}{2}$$

$$BC = \sqrt{\left(4+\frac{3}{2}\right)^2 + \left(-3-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{121}{4}+\frac{49}{4}} = \frac{\sqrt{170}}{2}$$

$$CD = \sqrt{\left(4-\frac{7}{2}\right)^2 + \left(-3-\frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{169}{4}} = \frac{\sqrt{170}}{2}$$

$$AD = \sqrt{\left(-2-\frac{7}{2}\right)^2 + \left(7-\frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{121}{4}+\frac{49}{4}} = \frac{\sqrt{170}}{2}$$

Slope of AC = $\frac{7+3}{-2-4} = \frac{10}{-6} = \frac{-5}{3}$

Slope of BD = $\frac{\frac{7}{2}-\frac{1}{2}}{\frac{7}{3}+\frac{3}{2}} = \frac{3}{5}$

$AC \perp BD$

$\therefore$ ABCD is rhombus.

22. The probabilities of three mutually exclusive events are respectively given as $\frac{1+3p}{3}, \frac{1-p}{4}, \frac{1-2p}{2}$. Prove that $\frac{1}{3} \le p \le \frac{1}{2}$.

Sol. Suppose A, B, C are exclusive events such that

$$P(A) = \frac{1+3p}{3}$$

$$P(B) = \frac{1-p}{4}$$

$$P(C) = \frac{1-2p}{2}$$

We know that

$0 \le P(A) \le 1$

$0 \le \frac{1+3p}{3} \le 1$

$0 \le 1+3p \le 3$

$-1 \le 3p \le 3-1$

$\frac{-1}{3} \le p \le \frac{2}{3}$ ...(1)

$0 \le P(B) \le 1$

$0 \le \frac{1-p}{4} \le 1$

$0 \le 1-p \le 4$

$-1 \le -p \le 4-1$

$1 \ge p \ge -3$

$-3 \le p \le 1$ ...(2)

$0 \le P(C) \le 1$

$0 \le \frac{1-2p}{2} \le 1$

$0 \le 1-2p \le 2$

$-1 \le -2p \le 2-1$

$1 \ge 2p \ge -1$

$\frac{1}{2} \ge p \ge -\frac{1}{2}$

$\frac{-1}{2} \le p \le \frac{1}{2}$ ...(3)

Since A, B, C are exclusive events,

$0 \le P(A \cup B \cup C) \le 1$

$\Rightarrow 0 \le P(A)+P(B)+P(C) \le 1$

$\Rightarrow 0 \le \frac{4+12P+3-3P+6-12P}{12} \le 1$

$\Rightarrow 0 \le \frac{13-3P}{12} \le 1$

$\Rightarrow 0 \le 13-3P \le 12$

$\Rightarrow -13 \le -3P \le 12-13$

$\Rightarrow 13 \ge 3P \ge 1$

$\Rightarrow \frac{13}{3} \ge P \ge \frac{1}{3}$

$\Rightarrow \frac{1}{3} \le P \le \frac{13}{3}$ ...(4)

Max. of $\left\{\frac{-1}{3}, -3, \frac{-1}{2}, \frac{1}{3}\right\} = \frac{1}{3}$

Min. of $\left\{\frac{2}{3}, 1, \frac{1}{2}, \frac{13}{3}\right\} = \frac{1}{2}$

(1), (2), (3) and (4) holds if $\frac{1}{3} \le p \le \frac{1}{2}$.

23. Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Find the mean of the random variable.

Sol. When two dice are rolled, the sample space S contains $6 \times 6 = 36$ sample points.

S = {(1, 1), (1, 2)…(1, 6), (2, 1), (2, 2)…(6,6)}

Let x denote the sum of the numbers on the two dice

Then the range x = {2, 3, 4, …12}

Probability Distribution of x is given by the following table.

$X=x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$X_i.p(x_i)$	2/36	6/36	12/36	20/36	30/36	42/36	40/36	36/36	30/36	22/36	12/36

Mean of x = $\mu = \sum_{1-2}^{12} x_1 p(X = x_1)$

$$= 2.\frac{1}{36} + 3.\frac{2}{36} + 4.\frac{3}{36} + 5.\frac{4}{36} + 6.\frac{5}{36} + 7.\frac{6}{36}$$

$$+8.\frac{5}{36} + 9.\frac{4}{36} + 10.\frac{3}{36} + 11.\frac{2}{36} + 12.\frac{1}{36}$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40$$

$$+36 + 30 + 22 + 12)$$

$$= \frac{252}{36} = 7$$

24. The arithmetic mean and standard deviation of a set of 9 items are 43 and 5 respectively. If an item of value 63 is added to that set, find the new mean and standard deviation of 10 item set given.

Sol. $\bar{X} = \frac{\sum_{i=1}^{9} x_i}{n}$

$$43 = \frac{\sum_{i=1}^{9} x_i}{9}$$

$$\sum_{i=1}^{9} x_i = 43 \times 9 = 387$$

$$\text{New Mean} = \frac{\sum_{i=1}^{10} x_i}{n} = \frac{\sum_{i=1}^{9} x_i + x_{10}}{10} = \frac{387 + 63}{10} = 45$$

$$S.D^2 = \frac{\sum_{i=1}^{9} {x_i}^2}{9} - (\overline{x})^2 \Rightarrow 5^2 = \frac{\sum_{i=1}^{9} {x_i}^2}{9} - (43)^2$$

$$\frac{\sum_{i=1}^{9} {x_i}^2}{9} = 25 + 1849 \Rightarrow \frac{\sum_{i=1}^{9} {x_i}^2}{9} = 1874$$

$$\sum_{i=1}^{9} {x_i}^2 = 1874 \times 9 = 16866$$

$$\sum_{i=1}^{10} {x_i}^2 = \sum_{i=1}^{9} {x_i}^2 + x_{10}^2 = 16866 + 3969 = 20835$$

$$\text{New S.D.} = \sqrt{\frac{\sum_{i=1}^{10} {x_i}^2}{10} - (\overline{x})^2} = \sqrt{\frac{20835}{10} - (45)^2}$$

$$= \sqrt{2083.5 - 2025} = \sqrt{58.5} = 7.6485 .$$