INDEFINITE INTEGRATION

Formulae

1. a. $\int x^n dx = \frac{x^{n+1}}{n+1} + c.$ (n≠-1)

b. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ (n≠-1)

c. $\int [f(x)]^n f^1(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$

2. $\int \frac{dx}{x} = \log|x| + c$

$\int \frac{dx}{ax+b} = \frac{\log|(ax+b)|}{a} + c$

3. $\int \sin x dx = -\cos x + c$

$\int \sin(ax+b) dx = \frac{-\cos(ax+b)}{a} + c$

4. $\int \cos x dx = \sin x + c$

5. $\int \sec^2 x dx = \tan x + c$

6. $\int \operatorname{cosec}^2 x dx = -\cot x + c$

7. $\int \sec x \tan x dx = \sec x + c$

8. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

9. $\int e^x dx = e^x + c, \int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$

10. $\int a^x dx = \frac{a^x}{\log a} + c$

11. $\int \frac{f^1(x)}{f(x)} dx = \log|f(x)| + c$

12. $\int \frac{f^1(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

13. $\int \tan x dx = \log|\sec x| + c$

14. $\int \cot x dx = \log|\sin x| + c$

15. $\int \sec x dx = \log|(\sec x + \tan x)| + c$

16. $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + c$

17. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$ or $-\cos^{-1} x + c$

18. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$ or $-\cot^{-1} x + c$

19. $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + c \text{ or } -\operatorname{cosec}^{-1} x + c$

20. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

21. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

22. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$

23. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{x^2+a^2}) + c \text{ or } \sinh^{-1}\frac{x}{a} + c$

24. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}) + c \text{ or } \cosh^{-1}\frac{x}{a} + c$

25. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

26. $\int e^x\{f(x)+f^1(x)\}dx = e^x f(x) + c$

27. $\int e^x f(x)dx = e^x\{f(x)-f^1(x)+f^{11}(x)-f^{111}(x).....\}$ where f(x) is a polynomial in x.

28. $\int e^x[f(x)-f^{11}(x)]\,dx = e^x[f(x)-f^1(x)] + c$

29. $\int udv = uv - \int vdu$

30. $\int \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$

31. $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx = \left(\frac{ac+bd}{c^2+d^2}\right)x + \frac{ad-bc}{c^2+d^2}\log|c\cos x + d\sin x| + k.$

32. $\int \log x dx = x\log x - x + c$

33. $\int \sin^{-1} x dx = x\sin^{-1} x + \sqrt{1-x^2} + c$

34. $\int \tan^{-1} x dx = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) + c$

35. $\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$ (or) $\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

36. $\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$ (or) $\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

37 $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c \text{ or } \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) + c$

38. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c \text{ or } \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + c$

39. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

40. $\int (xf^1 + f)dx = xf + c$

41. $\int |x|dx = \frac{x|x|}{2} + c$

42. $\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx\ (n \in N)$

43. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx\ (n \in N)$

44. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx\ (n \geq 2, n \in N)$

(or)

If $I_n = \int \tan^n x\, dx$ then $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

45. $\int \frac{dx}{x(x^n+1)} = \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right) + c$

46. $\int x \tan^{-1} x dx = \frac{1}{2}\left[(1+x^2)\tan^{-1} x - x\right] + c$

47. $\int x \sin^{-1} x\, dx = \frac{1}{4}\left[(2x^2-1)\sin^{-1} x + x\sqrt{1-x^2}\right] + c$

48. $\int \sqrt{\frac{1+x}{1-x}}\, dx = \sin^{-1} x - \sqrt{1-x^2} + c$

49. $\int \frac{1}{a\cos x + b\sin x} dx = \frac{1}{\sqrt{a^2+b^2}} \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2} - \frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right] + c$

50. $\int \frac{\sec^2\theta}{(\sec\theta + \tan\theta)^n} d\theta = -\frac{1}{2}\left[\frac{1}{(n-1)t^{n-1}} + \frac{1}{(n+1)t^{n+1}}\right]$, t = secθ + tanθ.

51. $\int \frac{\sec x\, dx}{(\sec x + \tan x)^n} = -\frac{1}{n(\sec x + \tan c)^n} + C$

52. $\int x\sin x\, dx = -x\cos x + \sin x + C$

53. $\int x \cos x\, dx = -x\sin x + \cos x + C$

54. $\int x\, e^x\, dx = e^x(x-1) + C$

55. $\int \frac{dx}{a\cos x + b\sin x} = \frac{1}{\sqrt{a^2+b^2}} \left[\log\tan\left(\frac{\pi}{4} + \frac{x}{2} - \frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right]$

Theorem:

$\int f(x)dx = g(x)$, $a \neq 0$ అయితే అప్పుడు $\int f(ax+b)dx = \frac{1}{a}g(ax+b)+c$.

Proof:

$ax + b = t$ గా తీసుకొనగా

$$\frac{d}{dx}(ax+b) = \frac{dt}{dx} \Rightarrow a \cdot dx = dt \Rightarrow dx = \frac{1}{a}dt$$

$$\therefore \int f(ax+b)dx = \int f(t) \cdot \frac{1}{a}dt$$

$$= \frac{1}{a}\int f(t)dt = \frac{1}{a}g(t)+c = \frac{1}{a}g(ax+b)+c$$

E.g. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, (n \neq -1)$

Theorem:

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c .$$

Proof:

$f(x) = t$ గా తీసుకొనగా $\Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$

$$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{1}{t}dt = \log|t|+c = \log|f(x)|+c$$

Theorem:

$\int \tan x\, dx = \log|\sec x|+c$ for $x \neq (2n+1)\frac{\pi}{2}, n \in Z$.

Proof :

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d(\cos x)}{\cos x}dx$$

$$= -\log|\cos x|+c = \log\frac{1}{|\cos x|}+c = \log|\sec x|+c$$

Theorem:

$$\int \cot x\, dx = \log|\sin x| + c \text{ for } x \neq n\pi,\ n \in Z.$$

Proof:

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + c$$

Theorem:

$$\int \sec x\, dx = \log|\sec x + \tan x| + c = \log|\tan(\pi/4 + x/2) + c \text{ for } x \neq (2n+1)\frac{\pi}{2}, n \in Z.$$

Proof:

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \log|\sec x + \tan x| + c$$

$$= \log\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c = \log\left|\frac{1+\sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{1-\cos(\pi/2+x)}{\sin(\pi/2+x)}\right| + c$$

$$= \log\left|\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right| + c$$

$$= \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$$

Theorem:

$$\int \csc x\, dx = \log|\csc x - \cot x| + c = \log|\tan x/2| + c \text{ for } x \neq n\pi, n \in Z.$$

Proof:

$$\int \csc x\, dx = \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} dx$$

$$= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx = \log|\csc x - \cot x| + c$$

$$= \log\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| + c = \log\left|\frac{1-\cos x}{\sin x}\right| + c$$

$$= \log\left|\frac{2\sin^2 x/2}{2\sin x/2\cos x/2}\right| + c = \log|\tan x/2| + c$$

Theorem:

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c\,.$$

Proof:

f(x) = t గా తీసుకొనగా $\Rightarrow$ f′(x) dx = dt

$$\therefore \int [f(x)]^n f'(x)dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c$$

Note: $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

Theorem:

If $\int f(x)dx = F(x)$ and g(x) is a differentiable function then $\int (fog)(x)g'(x)dx = F[g(x)] + c$.

Proof :

g(x) = t $\Rightarrow$ g′(x) dx = dt

$$\therefore \int (fog)(x)g'(x)dx = \int f[g(x)]g'(x)dx$$

$$= \int f(t)dt = F(t) + c = F[g(x)] + c$$

Theorem:

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c \text{ for } x \in (-a, a).$$

Proof:

x = a sin θ అనుకోండి అప్పుడు dx = a cos θ dθ

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}dx = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} a\cos\theta\, d\theta$$

$$= \int \frac{1}{a\sqrt{1 - \sin^2\theta}} a\cos\theta\, d\theta = \int \frac{1}{\cos\theta}\cos\theta\, d\theta$$

$$= \int d\theta = \theta + c = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c$$

Theorem:

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = \text{Sinh}^{-1}\left(\frac{x}{a}\right)+c \text{ for } x \in R.$$

Proof:

x = a sinh θ అనుకోండి అప్పుడు dx = a cos h θ dθ

$$\therefore \int \frac{1}{\sqrt{a^2+x^2}}dx = \int \frac{1}{\sqrt{a^2+a^2\sinh^2\theta}} a\cosh\theta\, d\theta$$

$$= \int \frac{a\cosh\theta}{a\cosh\theta} = \int d\theta = \theta + c = \text{Sinh}^{-1}\left(\frac{x}{a}\right)+c$$

Theorem:

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \text{Cosh}^{-1}\left(\frac{x}{a}\right)+c \text{ for } x \in (-\infty, -a) \cup (a, \infty).$$

Proof:

x = a cos h θ అనుకోండి అప్పుడు dx = a sinh θ dθ

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{a^2\cosh^2\theta - a^2}} a\sinh\theta\, d\theta$$

$$= \int \frac{a\sinh\theta}{a\sinh\theta}d\theta = \int d\theta = \theta + c = \text{Cosh}^{-1}\left(\frac{x}{a}\right)+c$$

Theorem:

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c \text{ for } x \in R.$$

Proof:

x = a tan θ అనుకోండి అప్పుడు dx = a $\sec^2$ θ dθ

$$\therefore \int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2+a^2\tan^2\theta} a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{a^2(1+\tan^2\theta)} a\sec^2\theta\, d\theta = \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c$$

Theorem:

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c \text{ for } x \neq \pm a$$

Proof:

$$\int \frac{1}{a^2 - x^2} dx = \int \frac{1}{(a+x)(a-x)} dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{a+x} + \frac{1}{a-x} \right) dx = \frac{1}{2a} [\log|a+x| - \log|a-x|] + c$$

$$= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

Theorem:

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c \text{ for } x \neq \pm a$$

Proof:

$$\int \frac{1}{x^2 - a^2} dx = \int \frac{1}{(x-a)(x+a)} dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx = \frac{1}{2a} [\log|x-a| - \log|x+a|] + c$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

Theorem:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c \text{ for } x \in (-a, a).$$

Proof:

$x = a \sin\theta$ అనుకోండి అప్పుడు $dx = a \cos\theta \, d\theta$

$$\therefore \int \sqrt{a^2 - x^2} dx = \int \sqrt{a^2 - a^2 \sin^2\theta} \, a\cos\theta \, d\theta$$

$$= \int a\sqrt{1 - \sin^2\theta} \, a\cos\theta d\theta = a^2 \int \cos^2\theta \, d\theta$$

$$= a^2\int\frac{1+\cos 2\theta}{2}d\theta = \frac{a^2}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\frac{1}{2}2\sin\theta\cos\theta\right]+c = \frac{a^2}{2}\left[\theta+\sin\theta\sqrt{1-\sin^2\theta}\right]$$

$$= \frac{a^2}{2}\left[\operatorname{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+c$$

$$= \frac{a^2}{2}\operatorname{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Theorem:

$$\int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\operatorname{Sinh}^{-1}\left(\frac{x}{a}\right)+c \text{ for } x\in R.$$

Proof:

$x = \sinh\theta$ అనుకోండి అప్పుడు $dx = a\cosh\theta\, d\theta$

$$\therefore \int\sqrt{a^2+x^2}\,dx = \int\sqrt{a^2+a^2\sinh^2\theta}\;a\cosh\theta d\theta$$

$$= \int a\sqrt{1+\sinh^2\theta}\;a\cosh\theta\, d\theta = a^2\int\cosh^2\theta\, d\theta$$

$$= a^2\int\frac{1+\cosh 2\theta}{2}d\theta = \frac{a^2}{2}\left[\theta+\frac{1}{2}\sinh 2\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\frac{1}{2}2\sinh\theta\cosh\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\sinh\theta\sqrt{1+\sinh^2\theta}\right]+c$$

$$= \frac{a^2}{2}\left[\operatorname{Sinh}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}\right]+c$$

$$= \frac{a^2}{2}\operatorname{Sinh}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{a^2+x^2}+c$$

Theorem:

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\,\text{Cosh}^{-1}\left(\frac{x}{a}\right) + c \ \text{ for } x \in [a, \infty).$$

Proof:

$x = a\cosh\theta$ అనుకోండి అప్పుడు $dx = a\sinh\theta\, d\theta$

$$\therefore \int \sqrt{x^2 - a^2}\,dx = \int \sqrt{a^2\cosh^2\theta - a^2}\; a\sinh\theta\, d\theta$$

$$= \int a\sqrt{\cosh^2\theta - 1}\; a\sinh\theta\, d\theta = a^2\int \sinh^2\theta\, d\theta$$

$$= a^2\int \frac{\cosh 2\theta - 1}{2}\, d\theta = \frac{a^2}{2}\left[\frac{1}{2}\sinh 2\theta - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\frac{1}{2}2\sinh\theta\cosh\theta - \theta\right] + c$$

$$= \frac{a^2}{c}\left[\cosh\theta\sqrt{\cosh^2\theta - 1} - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1} - \text{Cosh}^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\,\text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$$

Theorem:

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$$

Proof: Let I =

$$\int e^{ax}\cos bx\, dx = \cos bx \int e^{ax}dx - \int\left[d(\cos bx)\int e^{ax}dx\right]dx$$

$$= \cos bx\,\frac{e^{ax}}{a} - \int (-b\sin bx)\frac{e^{ax}}{a}\,dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\, dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\sin bx\,\frac{e^{ax}}{a} - \int b\cos bx\,\frac{e^{ax}}{a}\,dx\right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I$$

$$\Rightarrow I\left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a^2} e^{ax} [a \cos bx + b \sin bx]$$

$$\Rightarrow I\left(\frac{a^2 + b^2}{a^2}\right) = \frac{1}{a^2} e^{ax} [a \cos bx + b \sin bx]$$

$$\therefore I = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] + c$$

Theorem:

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

Proof: Let I =

$$\int e^{ax} \sin bx \, dx = \sin bx \int e^{ax} dx - \int \left[d(\sin bx) \int e^{ax} dx \right] dx$$

$$= \sin bx \frac{e^{ax}}{a} - \int b \cos bx \frac{e^{ax}}{a} dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left[\cos bx \frac{e^{ax}}{a} - \int (-b \sin bx) \frac{e^{ax}}{a} dx \right]$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} I$$

$$\Rightarrow I\left(1 + \frac{a^2}{b^2}\right) = \frac{1}{a^2} e^{ax} [a \sin bx - b \cos bx]$$

$$\Rightarrow I\left(\frac{a^2 + b^2}{b^2}\right) = \frac{e^{ax}}{a^2} [a \sin bx - b \cos bx]$$

$$\therefore I = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] + c$$

Theorem: $\int e^x [f(x)+f'(x)]dx = e^x f(x)+c$

Proof:

$$\int e^x [f(x)+f'(x)]dx = \int e^x f(x)dx + \int e^x f'(x)dx$$

$$= f(x)\int e^x dx - \int\left[d[f(x)]\int e^x dx\right]dx + \int e^x f'(x)dx$$

$$= f(x)e^x - \int f'(x)e^x dx + \int e^x f'(x)dx = e^x f(x)+c$$

Note: $\int e^{-x}[f(x)-f'(x)]dx = -e^{-x}f(x)+c$

Reduction Formulae

Theorem:

$I_n = \int x^n e^{ax}\,dx$ అయితే $I_n = \frac{e^{ax}}{a}x^n - \frac{n}{a}I_{n-1}$ అని చూపుము

Proof:

$$I_n = \int x^n e^{ax}\,dx = x^n\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}n\,x^{n-1}\,dx$$

$$= \frac{e^{ax}}{a}x^n - \frac{n}{a}\int e^{ax}x^{n-1}dx = \frac{e^{ax}}{a}x^n - \frac{n}{a}I_{n-1}$$

Theorem:

$I_n = \int \sin^n x\,dx$ అయితే $I_n = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$, అని చూపుము

Theorem:

$I_n = \int \cos^n x\,dx$ అయితే $I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$. అని చూపుము

Proof:

$$I_n = \int \cos^{n-1}x\cos x\,dx$$

$$= \cos^{n-1}x\sin x - \int \sin x(n-1)\cos^{n-2}x(-\sin x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x(1-\cos^2 x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\,dx - (n-1)\int\cos^n x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = \cos^{n-1} x \sin x + (n-1)I_{n-2}$$

$$\therefore I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Theorem:

$I_n = \int \tan^n x\, dx$ అయితే $\quad I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$. అని చూపుము

Proof:

$$I_n = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2}(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} dx = \frac{\tan^{n-1} dx}{n-1} - I_{n-2}$$

Theorem:

$I_n = \int \cot^n x\, dx$ అయితే $\quad I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$. అని చూపుము

Proof:

$$I_n = \int \cot^{n-2} x \cot^2 x\, dx = \int \cot^{n-2} x(\csc^2 x - 1)dx$$

$$= \int \cot^{n-2} x \csc^2 x\, dx - \int \cot^{n-2} x\, dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

Theorem:

$I_n = \int \sec^n x\, dx$ అయితే $\quad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$. అని చూపుము

Proof:

$$I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$$

$$= \sec^{n-2} x \tan x - \int \tan x (n-2)\sec^{n-3} x \sec x \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Theorem:

$I_n = \int \csc^n x\,dx$ అయితే $I_n = \dfrac{-\csc^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$. అని చూపుము

Proof:

$$I_n = \int \csc^{n-2} x \csc^2 x\,dx$$

$$= -\csc^{n-2} x \cot x - \int -\cot x (n-2)\csc^{n-3} x(-\csc x \cot x)\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^{n-2} x \cot^2 x\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^{n-2} x(\csc^2 x - 1)dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2} x\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = -\csc^{n-2} x \cot x + (n-2)I_{n-2}$$

$$I_n = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Theorem:

$I_n = \int (\log x)^n dx$ అయితే $I_n = x(\log x)^n - nI_{n-1}$. అని చూపుము

Proof:

$$I_n = \int (\log x)^n dx = x(\log x)^n - \int x\,n(\log x)^{n-1}\frac{1}{x}dx$$

$$x(\log x)^n - n\int (\log x)^{n-1}dx = x(\log x)^n - nI_{n-1}.$$

Theorem:

$I_{m,n} = \int \sin^m x \cos^n x\, dx$ అయితే

i) $I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$

ii) $I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$ అని చూపుము

Proof: i)

$$I_{m,n} = \int \sin^m x \cos^n x\, dx = \int (\sin^m x \cos^{n-1} x) \cos x\, dx$$

$$= \sin^m x \cos^{n-1} x \sin x - \int \begin{bmatrix} \sin^m x (n-1) \cos^{n-2} x(-\sin x) + \\ \cos^{n-1} x\, m \sin^{m-1} x \cos x \end{bmatrix} \sin x\, dx$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) \int \sin^m x \cos^{n-2} x \sin^2 x dx - m \int \sin^m x \cos^n x\, dx$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) \int \sin^m x \cos^{n-2} x (1-\cos^2 x) dx - m I_{m,n}$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) \int \sin^m x \cos^{n-2} x\, dx$$
$$(n-1) \int \sin^m x \cos^n x\, dx - m I_{m,n}$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2} - (n-1) I_{m,n} - m I_{m,n}$$

$$\Rightarrow (m+n) I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}$$

$$\Rightarrow I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

ii) $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$= \int \sin^{m-1} x \cos^n x \sin x\, dx$$

$$= \sin^{m-1} x \cos^n x (-\cos x) - \int [\sin^{m-1} x\, n \cos^{n-1} x$$
$$(-\sin x) - \cos^n x (m-1) \sin^{m-2} x \cos x](-\cos x) dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - n \int \sin^m x \cos^n x\, dx + (m-1) \int \sin^{m-2} x \cos^n x \cos^2 x\, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1) \int \sin^{m-2} x \cos^{n} x (1 - \sin^{2} x) dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1) \int \sin^{m-2} x \cos^{n} x \, dx - (m-1) \int \sin^{m} x \cos^{n} x \, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1) I_{m-2,n} - (m-1) I_{m,n}$$

$$\Rightarrow (m+n) I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}$$

$$\Rightarrow I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}.$$

EXERCISE – 6(A)

I. క్రింది సమాకలనులను గణించండి

1. $\int (x^3 - 2x^2 + 3)dx$

Sol. $\int (x^3 - 2x^2 + 3)dx = \int x^3 dx - \int 2x^2 dx + 3\int dx = \frac{x^4}{4} - \frac{2}{3}x^3 + 3x + c$

2. $\int 2x\sqrt{x}\, dx$

Sol. $\int 2x\sqrt{x}\, dx = 2\int x^{3/2}dx = \frac{2x^{5/2}}{(5/2)} + c = \frac{4}{5}x^{5/2} + c$

3. $\int \sqrt[3]{2x^2}\, dx$

Sol. $\int \sqrt[3]{2x^2}\, dx = \int 2^{1/3} \cdot x^{2/3}\, dx$

$= 2^{1/2} \cdot \frac{x^{5/3}}{(5/3)} + c = \sqrt[3]{2} \cdot \frac{3}{5}x^{5/3} + c$

4. $\int \frac{x^2 + 3x - 1}{2x} dx, x \in I \subset R \setminus \{0\}$

Sol. $\int \frac{x^2 + 3x - 1}{2x} dx = \int \left(\frac{x^2}{2x} + \frac{3}{2} - \frac{1}{2x} \right) dx$

$= \int \frac{x}{2} dx + \frac{3}{2}\int dx - \frac{1}{2}\int \frac{1}{x} dx$

$= \frac{x^2}{4} + \frac{3}{2}x - \frac{1}{2}\log|x| + c$

5. $\int \frac{1-\sqrt{x}}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{1-\sqrt{x}}{x}dx = \int \frac{dx}{x} - \int \frac{\sqrt{x}}{x}dx$

$= \log|x| - \frac{x^{-\frac{1}{2}+1}}{(1/2)} + c = \log|x| - 2\sqrt{x} + c$

6. $\int \left(1+\frac{2}{x}-\frac{3}{x^2}\right)$ **dx on I ⊂ R \ {0}.**

Sol. $\int \left(1+\frac{2}{x}-\frac{3}{x^2}\right)dx = \int dx + 2\int \frac{dx}{x} - 3\int x^{-2}dx$

$= x + 2\log|x| + \frac{3}{x} + c$

7. $\int \left(x+\frac{4}{1+x^2}\right)dx$ **on R.**

Sol. $\int \left(x+\frac{4}{1+x^2}\right)dx = \int xdx + 4\int \frac{1}{1+x^2}dx$

$= \frac{x^2}{2} + 4\tan^{-1}x + c$

8). $\int \left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2-1}}\right)dx$ **.**

Sol. $\int \left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2-1}}\right)dx$

$= \int e^x dx - \int \frac{1}{x}dx + 2\int \frac{1}{\sqrt{x^2-1}}dx$

$= e^x - \log|x| + 2\log|x+\sqrt{x^2-1}| + c$

9. $\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx$

Sol. $\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx = \int\frac{1}{1-x^2}dx+\int\frac{1}{1+x^2}dx$

$$= \tanh^{-1}x+\tan^{-1}x+c$$

10. $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1+x^2}}\right)dx$ **on (–1, 1).**

Sol. $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1+x^2}}\right)dx$

$$=\int\frac{1}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1+x^2}}dx$$

$$=\sin^{-1}x+2\sinh^{-1}x+c$$

11. $\int e^{\log(1+\tan^2 x)}dx$

Sol. $\int e^{\log(1+\tan^2 x)}dx==\int e^{\log(\sec^2 x)}dx$

$$=\int \sec^2 x\,dx=\tan x+c$$

12. $\int\frac{\sin^2 x}{1+\cos 2x}dx$

$$\int\frac{\sin^2 x}{1+\cos 2x}dx$$

$$=\int\frac{\sin^2 x}{\cos^2 x}dx=\int \tan^2 x dx$$

$$=\int (1+\sec^2 x)dx=x+\tan x+c$$

II. క్రింది సమాకలనులను గణించండి

1. $\int (1-x^2)^3 dx$

Sol. $\int (1-x^2)^3 dx = \int (1-3x^2+3x^4-x^6)dx$

$$= x - x^3 + \frac{3}{5}x^5 - \frac{x^7}{7} + c$$

2. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right)dx$

Sol. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right)dx =$

$$= 3\int \frac{dx}{\sqrt{x}} - 2\int \frac{dx}{x} + \frac{1}{3}\int x^{-2}dx$$

$$= 3(2\sqrt{x}) - 2\log|x| - \frac{1}{3x} + c$$

$$= 6\sqrt{x} - 2\log|x| - \frac{1}{3x} + c$$

3. $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx$

Sol. $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx = \int \frac{x+1+2\sqrt{x}}{x^2}dx$

$$= \int \frac{x}{x^2}dx + \int \frac{dx}{x^2} + 2\int \frac{\sqrt{x}}{x^2}dx$$

$$= \int \frac{dx}{x} + \int \frac{dx}{x^2} + 2\int x^{-3/2}dx$$

$$= \log|x| - \frac{1}{x} + \frac{2x^{-1/2}}{(-1/2)} + c$$

$$= \log|x| - \frac{1}{x} - \frac{4}{\sqrt{x}} + c$$

4. $\int \frac{(3x+1)^2}{2x}dx$

Sol. $\int \frac{(3x+1)^2}{2x}dx = \int \frac{9x^2+6x+1}{2x}dx$

$= \frac{9}{2}\int x dx + 3\int dx + \frac{1}{2}\int \frac{1}{x}dx$

$= \frac{9}{2}\cdot\frac{x^2}{2} + 3x + \frac{1}{2}\log|x| + c$

$= \frac{9}{4}x^2 + 3x + \frac{1}{2}\log|x| + c$

5. $\int \left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx$

Sol. $\int \left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx = \int \frac{4x^2-4x+1}{9x}dx$

$= \frac{4}{9}\int x\,dx - \frac{4}{9}\int dx + \frac{1}{9}\int \frac{dx}{x}$

$= \frac{4}{9}\frac{x^2}{2} - \frac{4}{9}x + \frac{1}{9}\log|x| + c$

$= \frac{4}{18}x^2 - \frac{4}{9}x + \frac{1}{9}\log|x| + c$

6. $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right)dx$ **on (1, ∞)**

Sol. $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right)dx$

$= \int \frac{1}{\sqrt{x}}dx + 2\int \frac{1}{\sqrt{x^2-1}}dx + \frac{3}{2}\int \frac{1}{x^2}dx$

$= 2\sqrt{x} + 2\cosh^{-1}x + \frac{3}{2x} + c$

7. $\int (\sec^2 x - \cos x + x^2)dx,$

Sol. $\int (\sec^2 x - \cos x + x^2)dx$

$= \int \sec^2 x dx - \int \cos x\, dx + \int x^2 dx$

$= \tan x - \sin x + \dfrac{x^3}{3} + c$

8. $\int \left(\sec x \tan x + \dfrac{3}{x} - 4 \right) dx$

Sol. $\int \left(\sec x \tan x + \dfrac{3}{x} - 4 \right) dx$

$= \int \sec x \tan x\, dx + 3\int \dfrac{dx}{x} - 4\int dx$

$= \sec x + 3\log|x| - 4x + c$

9. $\int \left(\sqrt{x} - \dfrac{2}{1-x^2} \right) dx$ **on (0, 1).**

Sol. $\int \left(\sqrt{x} - \dfrac{2}{1-x^2} \right) dx = \int \sqrt{x}\, dx - 2\int \dfrac{dx}{1-x^2}$

$= \dfrac{x^{3/2}}{(3/2)} - 2\tanh^{-1} x + c$

$= \dfrac{2}{3} x\sqrt{x} - 2\tanh^{-1} x + c$

10. $\int \left(x^3 - \cos x + \dfrac{4}{\sqrt{x^2+1}} \right) dx, x \in R$

Sol. $\int \left(x^3 - \cos x + \dfrac{4}{\sqrt{x^2+1}} \right) dx$

$= \int x^3 dx - \int \cos x\, dx + 4\int \dfrac{1}{\sqrt{x^2+1}} dx$

$= \dfrac{x^4}{4} - \sin x + 4\sinh^{-1} x + c$

11. $\int \left(\cosh x + \frac{1}{\sqrt{x^2+1}} \right) dx, x \in R$

Sol. $\int \left(\cosh x + \frac{1}{\sqrt{x^2+1}} \right) dx$

$= \int \cosh x dx + \int \frac{dx}{\sqrt{x^2+1}}$

$= \sinh x + \sinh^{-1} x + c$

12. $\int \left(\sinh x + \frac{1}{(x^2-1)^{1/2}} \right) dx,$

Sol. $\int \left(\sinh x + \frac{1}{(x^2-1)^{1/2}} \right) dx$

$= \int \sinh x dx + \int \frac{dx}{\sqrt{x^2-1}}$

$= \cosh x + \log(x + \sqrt{x^2-1}) + c$

13. $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$

Sol. $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$

$= \int \frac{a^{2x} + b^{2x} - 2a^x b^x}{a^x \cdot b^x} dx$

$= \int \frac{a^{2x}}{a^x \cdot b^x} dx + \int \frac{b^{2x}}{a^x \cdot b^x} dx - 2\int \frac{a^x b^x}{a^x \cdot b^x} dx$

$= \int \left(\frac{a}{b} \right)^x dx + \int \left(\frac{b}{x} \right)^2 dx - 2\int dx$

$= \frac{(a/b)^x}{\log(a/b)} + \frac{(b/a)^x}{\log(b/a)} - 2x + c$

$= \frac{1}{(\log a - \log b)} \left[\left(\frac{a}{b} \right)^x - \left(\frac{b}{a} \right)^x \right] - 2x + c$

14). $\int \sec^2 x \csc^2 x\, dx$ **.**

Sol. $\int \sec^2 x \csc^2 x\, dx = \int \frac{1}{\cos^2 x \sin^2 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx = \int \sec^2 x\, dx + \int \csc^2 x\, dx = \tan x - \cot x + C$$

15). $\int \frac{1+\cos^2 x}{1-\cos 2x} dx$ **.**

Sol. $\int \frac{1+\cos^2 x}{1-\cos 2x} dx = \int \frac{1+\cos^2 x}{2\sin^2 x} dx$

$$= \frac{1}{2}\int \frac{1}{\sin^2 x} dx + \frac{1}{2}\int \cot^2 x\, dx$$

$$= \frac{1}{2}\int \operatorname{cosec}^2 x\, dx + \frac{1}{2}\int (\csc^2 x - 1) dx$$

$$= \int \csc^2 x\, dx - \frac{1}{2}\int dx = -\cot x - \frac{x}{2} + C$$

16. $\int \sqrt{1-\cos 2x}\, dx$

Sol. $\int \sqrt{1-\cos 2x}\, dx = \int \sqrt{2\sin^2 x}\, dx$

$$= \int \sqrt{2} \sin x dx = -\sqrt{2}\cos x + C$$

17. $\int \frac{1}{\cosh x + \sinh x} dx$ **on R.**

Sol. $\int \frac{1}{\cosh x + \sinh x} dx = \int \frac{(\cosh x - \sinh x)}{(\cosh x + \sinh x)(\cosh x - \sinh x)} dx = \int \frac{\cosh x - \sinh x}{\cosh^2 x - \sinh^2 x} dx$

$$= \int (\cosh x - \sinh x) dx = \sinh x - \cosh x + C$$

18. $\int \frac{1}{1+\cos x}dx$ **on**

Sol. $\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx$

$= \int \left(\frac{1-\cos x}{1-\cos^2 x}\right)dx = \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$

$= \int \csc^2(x)dx - \int \csc x \cot x dx$

$= -\cot x + \csc x + C$

EXERCISE – 6(b)

I. క్రింది సమాకలనులను గణించండి

1. $\int e^{2x}dx,\ x\in R$.

Sol. $\int e^{2x}dx=\frac{e^{2x}}{2}+C$

2. $\int \sin 7x\,dx, x\in R$

Sol. $\int \sin 7x\,dx=-\frac{\cos 7x}{7}+C$

3. $\int \frac{x}{1+x^2}dx,\ \ x\in R$

Sol. $\int \frac{x}{1+x^2}dx=\frac{1}{2}\int \frac{2x\,dx}{1+x^2}=\frac{1}{2}\log(1+x^2)+C$

4. $\int 2x\sin(x^2+1)dx, x\in R$

Sol. $\int 2x\sin(x^2+1)dx$

Put $x^2+1=t\Rightarrow 2x\,dx=dt$

$\int 2x\cdot\sin(x^2+1)dx=\int \sin t\,dt=-\cot t+C$

$=-\cos(x^2+1)+C$

5). $\int \frac{(\log x)^2}{x}dx$.

Sol. $\int \frac{(\log x)^2}{x}dx$

put $\log x=t\Rightarrow dt=\frac{1}{x}dx$

$\int \frac{(\log x)^2}{x}dx=\int t^2\cdot dt=\frac{t^3}{3}+C=\frac{(\log x)^3}{3}+C$

6. $\int \frac{e^{Tan^{-1}x}}{1+x^2}dx$ **on I ⊂ (0, ∞).**

Sol. $\int \frac{e^{Tan^{-1}x}}{1+x^2}dx$

put $\tan^{-1}x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$\int \frac{e^{Tan^{-1}x}}{1+x^2}dx = \int e^t \cdot dt = e^t + C = e^{\tan^{-1}x} + C$

7. $\int \frac{\sin(Tan^{-1}x)}{1+x^2}dx, x \in R$

Sol. $\int \frac{\sin(Tan^{-1}x)}{1+x^2}dx$

put $\tan^{-1}x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$\int \frac{\sin(Tan^{-1}x)}{1+x^2}dx = \int \sin t\,dt$

$= -\cos t + t = -\cos(\tan^{-1}x) + C$

8. $\int \frac{1}{8+2x^2}dx$ **on R.**

Sol. $\int \frac{1}{8+2x^2}dx = \frac{1}{2}\int \frac{dx}{x^2+2^2}$

$= \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C = \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C$

9. $\int \frac{3x^2}{1+x^6}$ **dx, on R.**

Sol. $\int \frac{3x^2}{1+x^6}dx$

put $x^3 = t \Rightarrow 3x^2dx = dt$

$$\int \frac{3x^2dx}{1+x^6} = \int \frac{dt}{1+t^2}$$

$$= \tan^{-1}(t) + C = \tan^{-1}(x^3) + C$$

10. $\int \frac{2}{\sqrt{25+9x^2}} dx$ **on R.**

Sol. $\int \frac{2}{\sqrt{25+9x^2}} dx = \frac{2}{3}\int \frac{dx}{\sqrt{x^2+\left(\frac{5}{3}\right)^2}}$

$$= \frac{2}{3}\sinh^{-1}\left(\frac{x}{5/3}\right) + C = \frac{2}{3}\sinh^{-1}\left(\frac{3x}{5}\right) + C$$

11. $\int \frac{3}{\sqrt{9x^2-1}} dx$ on $\left(\frac{1}{3}, \infty\right)$

Sol. $\int \frac{3}{\sqrt{9x^2-1}} dx = \int \frac{dx}{\sqrt{x^2-\left(\frac{1}{3}\right)^2}}$

$$= \cosh^{-1}\left(\frac{x}{1/3}\right) + C = \cosh^{-1}(3x) + C$$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

12. $\int \sin mx \cos nx \, dx$ **on R, $m \neq n$, m , n** లు ధనాత్మక సంఖ్యలు

Sol: $\int \sin mx \cos nx \, dx = \frac{1}{2}\int 2\sin mx \cos nx \, dx$

$$= \frac{1}{2}\int [\sin(mx+nx) + \sin(mx-nx)dx$$

$$= \frac{1}{2}\int [\sin(m+n)x + \sin(m-n)x]dx$$

$$= \frac{-1}{2(m+n)}\cos(m+n)x - \frac{1}{2(m-n)}\cos(m-n)x + c$$

$$= \frac{-1}{2}\left[\frac{\cos(m+n)x}{m+n} + \cos\frac{(m-n)x}{m-n}\right] + c$$

13. $\int \sin mx \sin nx\, dx$ on R, $m \neq n$, m ,n లు ధనాత్మక సంఖ్యలు

Sol: $\int \sin mx \sin nx\, dx = \frac{1}{2}\int 2\sin mx \sin nx\, dx$

$$= \frac{1}{2}\int [\cos(mx - nx) - \sin(mx + nx)dx$$

$$= \frac{1}{2}\int [\cos(m-n)x - \cos(m+n)x]dx$$

$$= \frac{1}{2(m-n)}\sin(m-n)x - \frac{1}{2(m+n)}\sin(m+n)x + c$$

$$= \frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} - \sin\frac{(m+n)x}{m+n}\right] + c$$

14. $\int \cos mx \cos nx\, dx$ on R, $m \neq n$, m , n లు ధనాత్మక సంఖ్యలు

Sol: $\int \cos mx \cos nx\, dx = \frac{1}{2}\int 2\cos mx \cos nx\, dx$

$$= \frac{1}{2}\int [\cos(mx + nx) + \cos(mx - nx)]dx$$

$$= \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]dx$$

$$= \frac{-1}{2(m+n)}\sin(m+n)x - \frac{1}{2(m-n)}\sin(m-n)x$$

$$= -\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \sin\frac{(m-n)x}{m-n}\right] + c.$$

15. $\int \sin x \sin 2x \sin 3x \ dx$ **on R.**

Sol: Consider sin x sin 2x sin 3x

$$= \frac{1}{2}(2\sin x \sin 2x \sin 3x)$$

$$= \frac{1}{2}[\cos(3x-2x)-\cos(3x+2x)]\sin x$$

$$= \frac{1}{2}[\sin x \cos x - \sin x \cos 5x]$$

$$= \frac{1}{4}[2\sin x \cos x - 2\sin x \cos 5x]$$

$$= \frac{1}{4}[\sin 2x - [\sin(5x+x)+\sin(x-5x)]$$

$$= \frac{1}{4}[\sin 2x - [\sin 6x - \sin 4x)]$$

$$= \frac{1}{4}[\sin 2x - \sin 6x + \sin 4x]$$

$$\therefore \int \sin x \sin 2x \sin 3x \ dx = \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int \sin 6x \, dx$$

$$= -\frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x + c$$

$$= \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + c$$

16. $\int \frac{\sin x}{\sin(a+x)} dx$ on $I \subset R - \{n\pi - a : n \in Z\}$.

Sol: $\int \frac{\sin x}{\sin(a+x)} dx = \int \frac{\sin(x+a-a)}{\sin(x+a)} dx$

$$= \int \left[\frac{\sin(x+a)\cos a - \cos(x+a)\sin a}{\sin(x+a)}\right] dx$$

$$= \cos a \int dx - \sin a \int \frac{\cos(x+a)}{\sin(x+a)} dx$$

$$= x\cos a - \sin a - \log|\sin(x+a)| + c.$$

II. క్రింది సమాకలనులను గణించండి

1. $\int (3x-2)^{1/2}\ dx$

Sol. Given integral $= \int (3x-2)^{1/2}\ dx$

Put $3x - 2 = t \Rightarrow \quad 3\ dx = dt$

$$\int (3x-2)^{1/2} dx = \frac{1}{3}\int t^{1/2} dt$$

$$= \frac{1}{3}\frac{t^{3/2}}{3/2} + C = \frac{2}{9}(3x-2)^{3/2} + C$$

2. $\int \frac{1}{7x+3} dx$ **on** $I \subset R \setminus \left\{-\frac{3}{7}\right\}$

Sol. $\int \frac{1}{7x+3} dx$

Put $7x + 3 = t \Rightarrow \quad 7\ dx = dt$

$$\int \frac{1}{7x+3} dx = \frac{1}{7}\int \frac{dt}{t}$$

$$= \frac{1}{7}\log|t| + C = \frac{1}{7}\log|7x+3| + C$$

3. $\int \frac{\log(1+x)}{1+x} dx$ **on $(-1, \infty)$.**

Sol. $\int \frac{\log(1+x)}{1+x} dx$

Put $1 + x = t \Rightarrow \quad dx = dt$

$$\int \frac{\log(1+x)}{1+x} dx = \int \frac{\log t}{t} \cdot dt$$

$$= \frac{(\log t)^2}{2} + C = \frac{1}{2}\left[\log(1+x)^2\right] + C$$

4). $\int (3x^2 - 4)x\,dx$ on R **.**

Sol. $\int (3x^2 - 4)x\,dx$

put $3x^2 - 4 = t \Rightarrow 6x\,dx = dt$

$$\int (3x^2 - 4)x\,dx = \frac{1}{6}\int t\,dt$$

$$= \frac{1}{6}\cdot\frac{t^2}{2} + C = \frac{(3x^2 - 4)^2}{12} + C$$

5. $\int \frac{dx}{\sqrt{1+5x}}dx$ on $\left(-\frac{1}{5}, \infty\right)$

Sol. $\int \frac{dx}{\sqrt{1+5x}}$

Put 1 + 5x = t^2 ; 5dx = 2t dt, dx = $\frac{2}{5}t\,dt$

$$\int \frac{dx}{\sqrt{1+5x}} = \frac{2}{5}\int \frac{t\,dt}{t} = \frac{2}{5}\int dt$$

$$= \frac{2}{5}t + C = \frac{2}{5}\sqrt{1+5x} + C$$

6. $\int (1-2x^3)x^2dx$ **on R.**

Sol. $\int (1-2x^3)x^2dx$

put $1 - 2x^3 = t \Rightarrow -6x^2dx = dt$

$$\int (1-2x^3)x^2dx = -\frac{1}{6}\int t\,dt$$

$$= -\frac{1}{6}\cdot\frac{t^2}{2} + C = \frac{-(1-2x^3)^2}{12} + C$$

7. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$ **on** $I \subset R \setminus \left\{ n\pi - \frac{\pi}{4} : n \in Z \right\}$

Sol. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

put $1+\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\int \frac{\sec^2 x}{(1+\tan x)^3} dx = \int \frac{dt}{t^3} = \int t^{-3} dt$$

$$= \frac{t^{-2}}{(-2)} + C = -\frac{1}{2t^2} + C = -\frac{1}{2(1+\tan x)^2} + C$$

8. $\int x^3 \sin x^4 dx$ on R

Sol. $\int x^3 \sin x^4 dx$

Put $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\int x^3 \sin x^4 dx = \frac{1}{4} \int \sin t \cdot dt$$

$$= -\frac{1}{4} \cos t + C = -\frac{1}{4} \cdot \cos x^4 + C$$

9. $\int \frac{\cos x}{(1+\sin x)^2} dx$ **on** $I \subset R \setminus \left\{ 2n\pi + \frac{3\pi}{2} : n \in Z \right\}$

Sol. $\int \frac{\cos x}{(1+\sin x)^2} dx$

Put $1+\sin x = t \Rightarrow \cos x dx = dt$

$$\int \frac{\cos x}{(1+\sin x)^2} dx = \int \frac{dt}{t^2}$$

$$= -\frac{1}{t} + C = -\frac{1}{1+\sin x} + C$$

10. $\int \sqrt[3]{\sin x}\cos x\,dx$ **on [2nπ, (2n + 1)π, (n ∈ Z)].**

Sol. $\int \sqrt[3]{\sin x}\cos x\,dx$

Put sin x=t ⇒ cos x dx =dt

$\int \sqrt[3]{\sin x}\cos x\,dx = \int \sqrt[3]{t}\cdot dt$

$= \frac{t^{4/3}}{(4/3)} + C = \frac{3}{4}t^{4/3} + C = \frac{3}{4}(\sin x)^{4/3} + C$

11. $\int 2x\,e^{x^2}\,dx$ **on R.**

Sol. $\int 2x\,e^{x^2}\,dx$

Let x^2 =t⇒ 2x dx =dt

$\int 2x\,e^{x^2}\,dx = \int e^t\,dt = e^t + C = e^{x^2} + C$

12. $\int \frac{e^{\log x}}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{e^{\log x}}{x}dx$

Put log x= t ⇒ $\frac{1}{x}dx = dt$

$\int \frac{e^{\log x}}{x}dx = \int e^t \cdot dt = e^t + C$

$= e^{\log x} + C = x + C$

13. $\int \frac{x^2}{\sqrt{1-x^6}}dx$ **on I = (–1, 1).**

Sol. $\int \frac{x^2}{\sqrt{1-x^6}}dx$

Put x^3 =t ⇒ $3x^2$ dx =dt

$$\int \frac{x^2}{\sqrt{1-x^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{3}\sin^{-1} t + C = \frac{1}{3}\sin^{-1}(x^3) + C$$

14. $\int \frac{2x^3}{1+x^8}dx$ **on R.**

Sol. let $x^{4=t} \Rightarrow \quad 4x^3\,dx = dt$

$$\int \frac{2x^3}{1+x^8}dx = \frac{1}{2}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\tan^{-1} t + C = \frac{1}{2}\tan^{-1}(x^4) + C$$

15. $\int \frac{x^8}{1+x^{18}}dx$

Sol. $\int \frac{x^8}{1+x^{18}}dx = \int \frac{x^8}{1+\left(x^9\right)^2}dx$ on R.

Put $x^9 = t \Rightarrow \quad 9x^8\,dx = dt$

$$\int \frac{x^8}{1+x^{18}}dx = \int \frac{x^8}{1+(x^9)^2}dx = \frac{1}{9}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{9}\tan^{-1} t + C = \frac{1}{9}\tan^{-1}(x^9) + C$$

16. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$ **on** $I \subset R \setminus \{x \in R : \cos(xe^x) = 0\}$

Sol. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $x\,e^x = t$

$$(x \cdot e^x + e^x)dx = e^x(1+x)dx = dt$$

$$\text{G.I.} = \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt$$

$= \tan t + C = \tan(x \cdot e^x) + C$

17. $\int \frac{\csc^2 x}{(a + b\cot x)^5} dx$ **on I $\subset$ R \ {x$\in$R : a + b cotx = 0}, a, b $\in$ R, b $\neq$ 0.**

Sol.

G.I. $= \int \frac{\csc^2 x}{(a + b\cot x)^5} dx$

Put a + b cot x =t $\Rightarrow$ $-b \csc^2 x\, dx = dt$

$$\int \frac{\csc^2 x}{(a + b\cot x)^5} dx = -\frac{1}{b}\int \frac{dt}{t^5} = -\frac{1}{b}\int t^{-5} dt$$

$$= -\frac{1}{b}\frac{t^{-4}}{-4} + C = \frac{1}{4bt^4} + C = \frac{1}{4b(a + b\cot x)^4} + C$$

18. $\int e^x \sin e^x dx$ **on R.**

Sol. $e^x = t \Rightarrow e^x\, dx = dt$

$$\int e^x \sin e^x dx = \int \sin t\, dt$$

$$= -\cot + C = -\cos(e^x) + C$$

19. $\int \frac{\sin(\log x)}{x} dx$ **on (0, ∞)**

Sol. $\int \frac{\sin(\log x)}{x} dx$

put log x =t$\Rightarrow$ dt = $\frac{1}{x}dx = dt$

$$\int \frac{\sin(\log x)}{x} dx = \int \sin t\, dt$$

$$= -\cot t + C = -\cos(e^x) + C$$

20. $\int \frac{1}{x\log x}dx$ **on (0, ∞)**

Sol. $\int \frac{1}{x\log x}dx$

Put log x =t⇒ dt = $\frac{1}{x}dx = dt$

$$\int \frac{1}{x\log x}dx = \int \frac{1}{t}dt = \log t + C = \log(\log x + C)$$

21. $\int \frac{(1+\log x)^n}{x}dx$ **on (0, ∞), n ≠ –1.**

Sol. $\int \frac{(1+\log x)^n}{x}dx$

Put 1 + log x= t , ⇒ $\frac{1}{x}dx = dt$

$$\int \frac{(1+\log x)^n}{x}dx = \int t^n dt = \frac{t^{n+1}}{n+1} + C$$

$$= \frac{(1+\log x)^{n+1}}{n+1} + C$$

22. $\int \frac{\cos(\log x)}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{\cos(\log x)}{x}dx$

Put log x =t⇒ dt = $\frac{1}{x}dx = dt$

$$\int \frac{\cos(\log x)}{x}dx = \int \cos t\, dt$$

$$= \sin t + C = \sin(\log x) + C$$

23. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ **on (0, ∞)**

Sol. let $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow 2dt = \frac{dx}{\sqrt{x}}$

$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t dx$

$= 2\sin t + C = 2\sin\sqrt{x} + C$

24. $\int \frac{2x+1}{x^2+x+1}dx$ **on R.**

Sol. $\int \frac{2x+1}{x^2+x+1}dx$

put $x^2 + x + 1 = t \Rightarrow (2x + 1)dx = dt$

$\int \frac{2x+1}{x^2+x+1}dx = \int \frac{dt}{t}$

$= \log|t| + C = \log|x^2+x+1| + C$

25. $\int \frac{ax^{n-1}}{bx^n+C}dx$ **, where n ∈ N, a, b, c are real numbers, b ≠ 0 and** $x \in I \subset \left\{x \in R : x^n \neq -\frac{c}{b}\right\}$

Sol. $\int \frac{ax^{n-1}}{bx^n+C}dx$

let $bx^n + C = t \Rightarrow nbx^{n-1}dx = dt,\ x^{n-1}dx = \frac{1}{nb}dt$

$\int \frac{ax^{n-1}}{bx^n+C}dx = \frac{a}{nb}\int \frac{dt}{t} = \frac{a}{nb}\log|t| + dt$

$= \frac{a}{nb}\log|bx^n + c| + k$

26. $\int \frac{1}{x\log x[\log(\log x)]}dx$ **on (1, ∞)**

Sol. G.I. $\int \frac{1}{x\log x[\log(\log x)]}dx$

Put log(log x)=t, $\frac{1}{\log x}\cdot\frac{1}{x}dx = dt$

$$\int \frac{1}{x\log x[\log(\log x)]}dx = \int \frac{dt}{t}$$

$$= \log|t| + C = \log|\log(\log x)| + C$$

27. $\int \coth x dx$ **on R.**

Sol. sinh x=t ⇒ cosh x dx= dt

$$\int \coth x dx = \int \frac{dt}{t} = \log|t| + C$$

$$= \log|\sinh x| + C$$

28. $\int \frac{1}{\sqrt{1-4x^2}}dx$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$

Sol. $\int \frac{1}{\sqrt{1-4x^2}}dx = \frac{1}{2}\int \frac{dx}{\sqrt{(1/2)^2 - x^2}}$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{1/2}\right) + C = \frac{1}{2}\sin^{-1}(2x) + C$$

29. $\int \frac{dx}{\sqrt{25+x^2}}$ **on R**

Sol. $\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{dx}{\sqrt{x^2+5^2}} = \sinh^{-1}\left(\frac{x}{5}\right) + C$

30. $\int \frac{1}{(x+3)\sqrt{x+2}}dx$ **on I ⊂ (–2, ∞)**

Sol. put $x + 2 = t^2$, $dx = 2t\,dt$

$$\int \frac{1}{(x+3)\sqrt{x+2}}dx = \int \frac{2t\,dt}{t(t^2+1)} = 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1}(t) + C = 2\tan^{-1}(\sqrt{x+2}) + C$$

31. $\int \frac{1}{1+\sin 2x}dx$ **on** $I \subset R \setminus \left\{\frac{n\pi}{2} + (-1)^n \frac{\pi}{4} : n \in Z\right\}$

Sol. $\int \frac{1}{1+\sin 2x}dx = \int \frac{dx}{1+\frac{2\tan x}{1+\tan^2 x}}$

$$= \int \frac{(1+\tan^2 x)dx}{1+\tan^2 x + 2\tan x} = \int \frac{\sec^2 x dx}{(1+\tan x)^2}$$

put $1 + \tan x = t \Rightarrow \sec^2 x dx = dt$

$$\int \frac{1}{1+\sin 2x}dx = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{1+\tan x} + C$$

32. $\int \frac{x^2+1}{x^4+1}dx$ on R.

Sol: $\int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

$$= \int \frac{\left[1+\frac{1}{x^2}\right]}{\left[x-\frac{1}{x}\right]^2+2}.dx$$

$(\because a^2 + b^2 = (a+b)^2 - 2ab)$

Take $x - \frac{1}{x} = t$ then $\left(1+\frac{1}{x^2}\right)dx = dt$

$$\therefore \int \frac{x^2+1}{x^4+1}dx = \int \frac{dt}{t^2+2} = \int \frac{dt}{t^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) + c.$$

33. $\int \frac{dx}{\cos^2 x + \sin 2x}$ on $I \subset R / \left[\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\} \cup \left\{2n\pi + \mathrm{Tan}^{-1}\frac{1}{2} : n \in Z\right\}\right]$

Sol: $\int \frac{dx}{\cos^2 x + \sin 2x} = \int \frac{dx}{\cos^2 x + 2\sin x \cos x}$

$$= \int \frac{(\sin^2 x + \cos^2 x)}{\cos^2 x + 2\sin x \cos x} dx$$

$$= \int \frac{1 + \tan^2 x}{1 + 2\tan x} dx = \int \frac{\sec^2 x dx}{1 + 2\tan x}$$

Let $1 + 2\tan x = t$ then $2\sec^2 x\, dx = dt$

$$\Rightarrow \sec^2 x dx = \frac{1}{2} dt$$

$$\therefore \int \frac{dx}{\cos^2 x + \sin 2x} = \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}\log|t| + c$$

$$= \frac{1}{2}\log|1 + 2\tan x| + c.$$

34. $\int \sqrt{1-\sin 2x}\, dx$ on $I \subset \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right], n \in Z$.

Sol: $\int \sqrt{1-\sin 2x}\, dx$

$= \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x}\, dx$

$= \int \sqrt{(\sin x - \cos x)^2} dx$

$= \int \sqrt{(\cos x - \sin x)^2} dx$

$= \int (\cos x - \sin x) dx$

$= \int \cos x\, dx - \int \sin x\, dx$

$= \sin x + \cos x + c$

For $x \in \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right]$

$\int \sqrt{1-\sin 2x}\, dx = (\sin x + \cos x) + c$.

35. $\int \sqrt{1+\cos 2x}\, dx$ on $I \subset \left\{2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right\}, n \in Z$.

Sol: $\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{1+2\cos^2 x - 1}\, dx$

$= \int \sqrt{2\cos^2 x}\, dx$

$= \sqrt{2} \int \cos x\, dx + c$

$= \sqrt{2} \sin x + c$

For $x \in \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right]$

36). $\int \frac{\cos x + \sin x}{\sqrt{1+\sin 2x}} dx$ on $I \subset \left[2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right], n \in Z$.

Sol: $\int \frac{\cos x + \sin x}{\sqrt{1+\sin 2x}} dx$

$= \int \frac{(\cos x + \sin x)}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}}$

$$= \int \frac{\cos x + \sin x}{\sqrt{(\cos x + \sin x)^2}} dx$$

$$= \int \left(\frac{\cos x + \sin x}{\cos x + \sin x} \right) dx$$

$$\int dx + c = x + c$$

For $x \in \left[2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4} \right]$

37. $\int \frac{\sin 2x}{(a + b\cos x)^2} dx$ on $\begin{cases} R, \text{if } |a| > |b| \\ I \subset \{x \in R : a + b\cos x \neq 0\}, \text{if } |a| < |b|. \end{cases}$

Sol: $\int \frac{\sin 2x}{(a + b\cos x)^2} dx = \int \frac{2\sin x \cos x}{(a + b\cos x)^2} dx$

Let a + b cos x = t, then –b sin x dx = dt

$$\Rightarrow \sin x \, dx = -\frac{1}{b} dt$$

Also b cos x = t – a

$$\Rightarrow \cos x = \frac{t - a}{b}$$

$$\therefore \int \frac{\sin 2x}{(a + b\cos x)^2} dx = -\frac{2}{b^2} \int \left(\frac{t - a}{t^2} \right) dt$$

$$= -\frac{2}{b^2} \left[\int \frac{1}{t} dt - a \int \frac{1}{t^2} dt \right]$$

$$= -\frac{2}{b^2} \left[\log(|t|) + \frac{a}{t} \right] + c$$

$$= -\frac{2}{b^2} \left[\log|a + b\cos x| + \frac{a}{a + b\cos x} \right] + c.$$

38). $\int \frac{\sec x}{(\sec x+\tan x)^2}dx$ on $I \subset R-\left\{(2n+1)\frac{\pi}{2}, n\in Z\right\}$.

Sol: $\int \frac{\sec x}{(\sec x+\tan x)^2}$

$$=\int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^3}dx$$

Let sec x + tan x = t

Then (sec x tan x + $\sec^2$ x) dx = dt

$\Rightarrow \sec x(\sec x+\tan x)dx = dt$

$$\therefore \int \frac{\sec x}{(\sec x+\tan x)^2}dx$$

$$=\int \frac{dt}{t^3}=\int t^{-3}dt=\frac{t^{-2}}{-2}$$

$$=-\frac{1}{2t^2}=-\frac{1}{2(\sec x+\tan x)^2}+c$$

39. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ on R, $a\neq 0$, $b\neq 0$.

Sol: $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$

లవ హారాలను $\cos^2 x$ చే గుణించగా

$$=\int \frac{\sec^2 x dx}{a^2.\tan^2 x+b^2}$$

Let tan x = t, then $\sec^2$ x dt = dt

$$\therefore \int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}=\int \frac{dt}{a^2t^2+b^2}$$

$$=\frac{1}{a^2}\int \frac{dt}{t^2+\left(\frac{b}{a}\right)^2}$$

$$=\frac{1}{a^2}\frac{1}{\left(\frac{b}{a}\right)}\tan^{-1}\frac{t}{\left(\frac{b}{a}\right)}=\frac{1}{ab}\mathrm{Tan}^{-1}\frac{at}{b}$$

$$= \frac{1}{ab} \mathrm{Tan}^{-1}\left(\frac{a\tan x}{b}\right) + c.$$

40. $\int \frac{\sec^2 x}{\sqrt{a + b\tan x}}\,dx$**, a, b** లు ధనాత్మక సంఖ్యలు

$\mathbf{I \subset R} \setminus \left(\left\{x \in R : \tan x < -\frac{a}{b}\right\} \cup \left\{\frac{(2n+1)\pi}{2} : n \in Z\right\}\right)$

$a + b\tan x = t$ అనుకోండి

Ans. $\frac{2}{b}\sqrt{a + b\tan x} + C$

41. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ **on** $\mathbf{I \subset R} \setminus (\{a + n\pi : n \in Z\} \cup \{b + n\pi : n \in Z\})$**.**

Sol. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(b-a)}$$

$$\int \left\{\frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}\right\} dx$$

$$= \frac{1}{\sin(b-a)} \int \{\cot(x-b) - \cot(x-a)\}\,dx$$

$$= \frac{1}{\sin(b-a)} \left[\log|\sin(x-b)| - \log|\sin(x-a)|\right] + C$$

$$= \frac{1}{\sin(b-a)} \log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$$

42. $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$ **on** $I \subset R \setminus \left(\left\{a+\frac{(2n+1)\pi}{2} : n \in Z\right\} \cup \left\{b+(2n+1)\frac{\pi}{2} : n \in Z\right\}\right)$

Sol. $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(\overline{x-b}-\overline{x-a})}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}$$

$$\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int \{\tan(x-b)-\tan(x-a)\}dx$$

$$= \frac{1}{\sin(a-b)}[\log|\sec(x-b)|-\log|\sec(x-a)|]+C$$

$$= \frac{1}{\sin(a-b)}\log\left|\frac{\sec(x-b)}{\sec(x-a)}\right|+C$$

43. $\int \sqrt{1+\sec x}\,dx$ on $\left[\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right], n \in Z.$

Sol. $\int \sqrt{1+\sec x}\,dx = \sqrt{\frac{\sec^2 x-1}{\sec x-1}}dx$

$$= \int \frac{\tan x}{\sqrt{\sec x-1}}dx = \int \frac{\frac{\sin x}{\cos x}}{\sqrt{\frac{1-\cos x}{\cos x}}}dx$$

$$= \int \frac{\sin x}{\sqrt{\cos x}\sqrt{1-\cos x}}dx$$

Put $\cos x = t \Rightarrow \sin x\,dx = -dt$

$$= \int \frac{-dt}{\sqrt{t}\sqrt{1-t}} = -\int \frac{1}{\sqrt{t-t^2}}dt$$

$$= -\int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(t - \frac{1}{2}\right)^2}}$$

$$= -\sin^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right) + C = -\left[t^2 - t + \frac{1}{4} - \frac{1}{4}\right]$$

$$= -\sin^{-1}(2t-1) + C = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2$$

$$= -\sin^{-1}[2\cos x - 1] + C$$

III.

1. $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x}dx$ **on** $I \subset R \setminus (x \in R \mid a\cos^2 x + b\sin^2 x = 0\}$

Sol. put $a\cos^2 x + b\sin^2 x = t$

$$\left(a(2\cos x)(-\sin x) + b(2\sin x\cos x)\right)dx = dt$$
$$= \sin 2x(b-a)dx$$

$$\sin 2x \cdot dx = \frac{1}{(b-a)}dt$$

$$\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x}dx = \frac{1}{(b-a)}\int \frac{dt}{t}$$

$$= \frac{1}{(b-a)}\log|t| + C$$

$$= \frac{1}{(b-a)}\log|a\cos^2 x + b\sin^2 x| + C$$

2. $\int \frac{1-\tan x}{1+\tan x}dx$ **for** $x \in I \subset R \setminus \left\{n\pi - \frac{\pi}{4} : n \in Z\right\}$

Sol. $\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}dx$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x}dx$$

$\cos x + \sin x = t$

$\Rightarrow dt = -\sin x + \cos x\, dx$

$$\int \frac{1-\tan x}{1+\tan x} dx = \int \frac{dt}{t} = \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

3. $\int \frac{\cot(\log x)}{x} dx$**,** $x \in I \subset (0,\infty) \setminus \{e^{n\pi} : n \in Z\}$**.**

Sol. Put log x =t$\Rightarrow$ dt = $\frac{1}{x}$dx =dt

$$\int \frac{\cot(\log x)}{x} dx = \int \cot t\, dt = \log(\sin t) + C$$

$$= \log(\sin(\log x)) + C$$

4. $\int e^x \cdot \cot e^x dx, x \in I \subset R \setminus \{\log n\pi : n \in Z\}$

Sol. Put e^x =t$\Rightarrow$ e^x dx =dt

$$\int e^x \cdot \cot e^x dx = \int \cot t\, dt = \log|\sin t| + C$$

$$= \log(\sin e^x) + C$$

5. $\int \sec x(\tan x)\sec^2 x\, dx$**, on**

$I \subset \left\{x \in E : \tan x \neq \frac{(2k+1)\pi}{2} \text{for any } k \in Z\right\}$ **where** $E = R / \left\{\frac{(2n+1)\pi}{2} : n \in Z\right\}$

Sol. tan x=t $\Rightarrow$ $\sec^2$ x dx=dt

$$\int \sec x(\tan x)\sec^2 x\, dx = \int \sec t \cdot dt$$

$$= \log\tan\left(\frac{\pi}{4} + \frac{t}{2}\right) + C$$

$$= \log\left(\tan\left(\frac{\pi}{4} + \frac{\tan x}{2}\right)\right) + C$$

6. $\int \sqrt{\sin x}\cos x\,dx$ **on** $[2n\pi,(2n+1)\pi]$**, n ∈ Z.**

Sol. t = sin x ⇒ dt = cosx dx

$$\int \sqrt{\sin x}\cdot\cos x\,dx = \int \sqrt{t}\ dt = \frac{2}{3}t^{3/2} + C$$

$$= \frac{2}{3}(\sin x)^{3/2} + C$$

7. $\int \tan^4 x \sec^2 x\,dx, x \in I \subset R \setminus \left\{\frac{(2n+1)\pi}{2} : n \in Z\right\}$

Sol. $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\int \tan^4 x \sec^2 x\,dx = \int t^4 dt$$

$$= \frac{t^5}{5} + C = \frac{(\tan x)^5}{t} + C$$

8. $\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx$**,** $x \in I \subset R\setminus[-4,1]$**.**

Sol. Let $x^2 + 3x - 4 = t \Rightarrow (2x + 3)dx = dt$

$$\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C$$

$$= 2\sqrt{x^2+3x-4} + C$$

9. $\int \csc^2 x\sqrt{\cot x}\,dx$ on $\left(0,\frac{\pi}{2}\right)$

Sol. put $\cot x = t \Rightarrow -\csc^2 x\,dx = dt$

$$\int \csc^2 x\sqrt{\cot x}\,dx = -\int \sqrt{t}\,dt$$

$$= -\frac{2}{3}t\sqrt{t} + C = -\frac{2}{3}\cot(x)^{3/2} + C$$

10. $\int \sec x \log(\sec x + \tan x)dx$ on $\left(0, \frac{\pi}{2}\right)$

Sol. $\log(\sec x + \tan x) = t$

$$\Rightarrow \frac{(\sec x \cdot \tan x + \sec^2 x)dx}{(\sec x + \tan x)} = dt = \sec x\, dx$$

$$\int \sec x \cdot \log\left(\sec x + \tan x\right)dx = \int t\, dt$$

$$= \frac{t^2}{2} + C = \frac{(\log(\sec x + \tan x))^2}{2} + C$$

11. $\int \sin^3 x\, dx$ **on R.**

Sol. since $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\int \sin^3 x\, dx = \frac{3}{4}\int \sin x - \frac{1}{4}\int \sin 3x\, dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

$$= \frac{1}{12}(\cos 3x - 9\cos x) + C$$

12. $\int \cos^3 x dx$ **on R.**

Sol. since $\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\int \cos^3 x\, dx = \frac{3}{4}\int \cos x\, dx + \frac{1}{4}\int \cos 3x\, dx$$

$$= \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

$$= \frac{1}{12}(9\sin x + \sin 3x) + C$$

13. $\int \cos x \cos 2x\, dx$ **on R.**

Sol. $\cos 2x \cos x = \frac{1}{2}(2\cos 2x \cdot \cos x)$

$$\int \cos x = \cos 2x\, dx = \frac{1}{2}\int (\cos 3x + \cos x)dx$$

$$= \frac{1}{2}\int \cos 3x dx + \frac{1}{2}\int \cos x dx$$

$$= \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C = \frac{\sin 3x + 3\sin x}{6} + C$$

14. $\int \cos x \cos 3x\, dx$ **on R.**

Sol. $\cos 3x \cos x = \frac{1}{2}(2\cos 3x \cdot \cos x)$

$$\frac{1}{2}(\cos 4x + \cos 2x)$$

$$\int \cos x \cos 3x\, dx = \frac{1}{2}\int \cos 4x\, dx + \frac{1}{2}\cos 2x\, dx$$

$$= \frac{1}{2}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right) + C$$

$$= \frac{1}{8}(\sin 4x + 2\sin 2x) + C$$

15. $\int \cos^4 x\, dx$ **on R.**

Sol. $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}(2 + 4\cos 2x + 1 + \cos 4x)$$

$$= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$

$$=\frac{1}{8}\left(3\int dx+4\int \cos 2x\,dx+\int \cos 4x\,dx\right)$$

$$=\frac{1}{8}\left(3x+4\frac{\sin 2x}{2}+\frac{\sin 4x}{4}\right)+C$$

$$=\frac{1}{32}(12x+8\sin 2x+\sin 4x)+C$$

16. $\int x\sqrt{4x+3}\,dx$ **on** $\left(-\frac{3}{4},\infty\right)$**.**

Sol. put $4x+3=t^2 \Rightarrow 4dx=2t\,dt$

$$dx=\frac{1}{2}t\,dt \Rightarrow x=\frac{t^2-3}{4}$$

$$\int x\sqrt{4x+3}\,dx=\int\frac{t^2-3}{4}\cdot t\cdot\frac{1}{2}t\,dt$$

$$=\frac{1}{8}\int(t^4-3t^2)dt=\frac{1}{8}\left(\frac{t^5}{5}-t^3\right)+C$$

$$=\frac{(4x+3)^{5/2}}{40}-\frac{1}{8}(4x+3)^{3/2}+C$$

17. $\int\frac{dx}{\sqrt{a^2-(b+cx)^2}}$ **on** $\{x\in R:|b+cx|<a\}$**, where a, b, c are real numbers** $c\neq 0$ **and** $a>0$**.**

Sol. $\int\frac{dx}{\sqrt{a^2-(b+cx)^2}}=\int\frac{dx}{c\sqrt{\left(\frac{a}{c}\right)^2-\left(\frac{b}{c}-x\right)^2}}$

$$=\frac{1}{c}\sin^{-1}\left(\frac{\left(\frac{b}{c}+x\right)}{\left(\frac{a}{c}\right)}\right)+K=\frac{1}{c}\sin^{-1}\left(\frac{b+cx}{a}\right)+K$$

18. $\int \frac{dx}{a^2+(b+cx)^2}$

Sol. $\int \frac{dx}{a^2+(b+cx)^2} = \frac{1}{c^2}\int \frac{dx}{\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}+x\right)^2}$

$$= \frac{1}{a^2 \cdot \frac{a}{c}} \tan^{-1}\left(\frac{\frac{b}{c}+x}{\frac{a}{c}}\right)+C$$

$$= \frac{1}{ac}\tan^{-1}\left(\frac{b+cx}{a}\right)+C$$

19. $\int \frac{dx}{1+e^x}, x \in R$

Sol. $\int \frac{dx}{1+e^x} = \int\left(\frac{1+e^x-e^x}{1+e^x}\right)dx$

$$= \int\left(1-\frac{e^x}{1+e^x}\right)dx = x-\log(1+e^x)+C$$

20. $\int \frac{x^2}{(1+bx)^2}dx$, $x \in I \subset R \setminus \left\{-\frac{a}{b}\right\}$

Sol. Put $a + bx = t$, $\Rightarrow b\,dx = dt \Rightarrow dx = \frac{1}{b}\cdot dt$

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b}\int \frac{\left(\frac{t-a}{b}\right)^2}{t^2}dt$$

$$= \frac{1}{b^3}\int \frac{t^2-2at+a^2}{t^2}dt$$

$$= \frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)+C$$

$$= \frac{1}{b^3}\left(t-2a\log|t|-\frac{a^2}{t}\right)+C$$

$$= \frac{1}{b^3}\left[(a+bx) - 2a\log|a+bx| - \frac{a^2}{(a+bx)}\right] + C$$

21. $\int \frac{x^2}{\sqrt{1-x}}dx, x \in (-\infty, 1)$

Sol. Put $1-x = t^2, -dx = 2t\,dt$

$$\int \frac{x^2}{\sqrt{1-x}}dx = \int (1-t^2)^2 \cdot \frac{-2t}{t}dt$$

$$= 2\int (1-2t^2+t^4)dt = -2\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) + C$$

$$= -2\left(\sqrt{1-x} - \frac{2}{3}(1-x)^{3/2} + \frac{1}{5}(1-x)^{5/2}\right) + C$$

EXERCISE – 6(C)

I. క్రింది సమాకలనులను గణించండి

1. $\int x\sec^2 x\,dx$ **on** $I \subset R \setminus \left\{\frac{(2n+1)\pi}{2} :\ \text{n is an integer}\right\}$

Sol. $\int x\sec^2 x\,dx = x(\tan x) - \int \tan x\,dx$

$= x\tan x - \log|\sec x| + C$

2. $\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$, **x ∈ R.**

Sol.

Let $f(x) = \tan^{-1}x$ so that $f'(x) = \frac{1}{1+x^2}$

$\therefore\ \int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx = e^x\tan^{-1}x + C\ \left(\because \int e^x[f(x)+f'(x)]dx = e^x\cdot f(x)+C\right)$

3. $\int \frac{\log x}{x^2}dx$ **on (0, ∞).**

Sol. $\int \frac{\log x}{x^2}dx = (\log x)\left(-\frac{1}{x}\right) + \int \frac{1}{x}\cdot\frac{1}{x}dx$

$= -\frac{1}{x}\log x - \frac{1}{x} + C$

4. $\int (\log x)^2 dx$ **on (0, ∞).**

Sol. $\int (\log x)^2 dx = (\log x)^2 x - \int x\cdot 2\log x\cdot\frac{1}{x}dx$

$= x(\log x)^2 - 2\int \log x\,dx$

$= x(\log x)^2 - 2\left(x\log x - \int x\frac{1}{x}dx\right)$

$= x(\log x)^2 - 2x\cdot\log x + x + c$

5. $\int e^x(\sec x + \sec x \tan x)dx$ **on** $I \subset R \setminus \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Sol. $\int e^x(\sec x + \sec x \tan x)dx = e^x \cdot \sec x + C$

$\left(\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + C\right)$

6. $\int e^x \cos x\, dx$ **on R.**

Sol. $I = \int e^x \cos x\, dx = e^x \sin x - \int \sin x \cdot e^x dx$

$= e^x \cdot \sin x + e^x \cdot \cos x - \int e^x \cdot \cos x\, dx$

$= e^x(\sin x + \cos x) - I$

$2I = e^x(\sin x + \cos x)$

$I = \frac{e^x}{2}(\sin x + \cos x) + C$

7. $\int e^x(\sin x + \cos x)dx$ **on R.**

Sol. $\int e^x(\sin x + \cos x)dx$

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$\therefore \int e^x(\sin x + \cos x)dx = e^x \cdot \sin x + C$

8. $\int(\tan x + \log \sec x)e^x dx$ **on** $\left(\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right) n \in Z$

Sol. $f = \log|\sec x| \Rightarrow f' = \frac{1}{\sec x} \cdot \sec x \cdot \tan x \cdot$

$= \tan x$

$\int(\tan x + \log \sec x)e^x dx = e^x \cdot \log|\sec x| + C \left(\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + C\right)$

II. Evaluate the following integrals.

1. $\int x^n \log x\, dx$ **on (0, ∞), n is a real number and n ≠ –1.**

Sol. $\int x^n \log x\, dx = (\log x)\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx$

$$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}(\log x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$$

$$= \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\log x - 1\right] + C$$

2. $\int \log(1+x^2)dx$ **on R.**

Sol. $\int \log(1+x^2)dx = \int 1.\log(1+x^2)dx =$

$$= \log(1+x^2)\cdot x - \int x\frac{1}{1+x^2}2x\, dx$$

$$= x\log(1+x^2) - 2\int\frac{1+x^2-1}{1+x^2}dx$$

$$= x\log(1+x^2) - 2\int dx + 2\int\frac{dx}{1+x^2}$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C$$

3. $\int \sqrt{x}\log x\, dx$ **on (0, ∞).**

Sol. $\int \sqrt{x}\log x\, dx =$

$$= \log x\cdot\frac{2}{3}x^{3/2} - \frac{2}{3}\int x^{3/2}\cdot\frac{1}{x}dx$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{2}{3}\frac{x^{3/2}}{3/2} + C$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{4}{9}x^{3/2} + C$$

4. $\int e^{\sqrt{x}}dx$ **on (0, ∞).**

Sol. let $\sqrt{x} = t \Rightarrow x = t^2$, $dx = 2t\,dt$

$$\int e^{\sqrt{x}}dx = 2\int t\,e^t dt = 2\left[te^t - \int e^t dt\right]$$
$$= 2(te^t - e^t) + C$$
$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

5. $\int x^2 \cos x\,dx$ **on R.**

Sol. $\int x^2 \cos x\,dx = x^2(\sin x) - \int \sin x(2x\,dx)$

$$= x^2 \sin x + 2\int x(-\sin x)dx$$
$$= x^2 \cdot \sin x + 2[x\cos x - \int \cos x\,dx]$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

6. $\int x\sin^2 x\,dx$ **on R.**

Sol. $\int x\sin^2 x\,dx = \frac{1}{2}\int x(1-\cos x)dx$

$$= \frac{1}{2}\left[\int xdx - \int x\cos 2x\,dx\right]$$
$$= \frac{1}{2}\left[\frac{x^2}{2} - \left\{x\cdot\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right\}\right]$$
$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{4}\int \sin 2x\,dx$$
$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$$

7. $\int x\cos^2 x\,dx$ **on R.**

Sol. $\int x\cos^2 x\,dx = \frac{1}{2}\int x(1+\cos 2x)dx$

$$= \frac{1}{2}\left[\int x\,dx + \int x\cos 2x\,dx\right]$$
$$= \frac{1}{2}\left[\frac{x^2}{2} + \left\{x\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right\}\right]$$

$$= \frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{4}\int \sin 2x\, dx$$

$$= \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + C$$

8. $\int \cos\sqrt{x}\, dx$ **on R.**

Sol. $x = t^2 \Rightarrow dx = 2t\, dt$

$$I = 2\int t\cdot \cos t\, dt = 2(t\sin t - \int \sin t\, dt)$$

$$= 2(t\sin t + \cos t) + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

9. $\int x\sec^2 2x\, dx$ **on** $I \subset R \setminus \left\{(2n\pi+1)\frac{\pi}{4} : n \in Z\right\}$

Sol. $\int x\sec^2 2x\, dx = x\frac{\tan 2x}{2} - \frac{1}{2}\int \tan 2x\, dx$

$$= x\frac{\tan 2x}{2} - \frac{1}{2}\cdot\frac{1}{2}\log|\sec 2x| + C$$

$$= x\frac{\tan 2x}{2} - \frac{1}{4}\log|\sec 2x| + C$$

10. $\int x\cot^2 x\, dx$ **on** $I \subset R \setminus \{n\pi : n \in Z\}$.

Sol. $\int x\cot^2 x\, dx = \int x(\csc^2 x - 1)dx$

$$= \int x\csc^2 x\, dx - \int x\, dx$$

$$= x(-\cot x) + \int \cot x\, dx - \frac{x^2}{2}$$

$$= -x\cot x + \log|\sin x| - \frac{x^2}{2} + C$$

11. $\int e^x(\tan x + \sec^2 x)dx$ on $I \subset R \setminus \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Sol. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x\, dx$

$$I = \int e^x[f(x) + f'(x)]dx = e^x f(x) + C$$

$$= e^x \tan x + C$$

12. $\int e^x\left(\frac{1+x\log x}{x}\right)dx$ **on (0, ∞).**

Sol. $\int e^x\left(\frac{1+x\log x}{x}\right)dx=\int e^x\left(\log x+\frac{1}{x}\right)dx$

$$=e^x\log x+C$$

13. $\int\frac{dx}{(x^2+a^2)^2},(a>0)$ on R.

Sol: $x = a\tan\theta$ అనుకోండి. అప్పుడు $dx = a\sec^2\theta\, d\theta$

$$\therefore\int\frac{dx}{(x^2+a^2)^2}=\int\frac{a\sec^2\theta d\theta}{(a^2\tan^2\theta+a^2)^2}$$

$$=\int\frac{a\sec^2\theta d\theta}{a^4(1+\tan^2\theta)^2}=\frac{1}{a^3}\int\frac{\sec^2\theta d\theta}{\sec^4\theta}$$

$$=\frac{1}{a^3}\int\cos^2\theta d\theta$$

$$=\frac{1}{a^3}\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{2a^3}\left[\int 1\cdot d\theta+\int\cos 2\theta\, d\theta\right]$$

$$=\frac{1}{2a^3}\left[\theta+\frac{1}{2}\sin 2\theta\right]$$

$$=\frac{1}{2a^3}\left[\tan^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}\sin\left[2\tan^{-1}\left(\frac{x}{a}\right)\right]\right]+c$$

$$=\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)+\frac{1}{4a^3}\sin\left[2\tan^{-1}\left(\frac{x}{a}\right)\right]+c.$$

14. $\int e^x \log(e^{2x}+5e^x+6)dx$ on r.

Sol: $e^{2x}+5e^x+6=(e^x)^2+5e^x+6$

$=(e^x)^2+3e^x+2e^x+6$

$=e^x(e^x+3)+2(e^x+3)$

$=(e^x+3)(e^x+2)$

$\int e^x \log(e^{2x}+5e^x+6)dx$

$=\int e^x \log[(e^x+2)(e^x+3)]dx$

$=\int e^x \log(e^x+2)dx+\int e^x \log(e^x+3)dx$

$(\because \log ab=\log a+\log b)$

$e^x = t$ అనుకోండి. అప్పుడు $e^x\,dx = dt$

$\therefore \int e^x \log(e^{2x}+5e^x+6)dx$

$=\int \log(t+2)dt+\int \log(t+3)dt$

$=\log(t+2)t-\int \frac{t}{t+2}dt+\log(t+3)\cdot t-\int \frac{t}{t+3}dt$

$=t\cdot\log(t+2)-\int\left(\frac{(t+2)-2}{t+2}\right)dt+t\cdot\log(t+3)-\int\left(\frac{(t+3)-3}{t+3}\right)dt$

$=t\cdot\log(t+2)-\int dt+2\int\frac{dt}{t+2}+t\log(t+3)-\int dt+3\int\frac{dt}{t+3}$

$=t\log(t+2)-t+2\log(t+2)+t\log(t+3)-t+3\log(t+3)$

$=2\log|(t+2)|+3\log|(t+3)|-2t+t[\log(t+2)(t+3)]$

$=t[\log(t^2+5t+6)]-2t+2\log|t+2|+3\log|t+3|+c$

$=e^x[\log(e^{2x}+5e^x+6)]-2e^x+2\log|e^x+2|+3\log|e^x+3|+c.$

15. $\int \cos(\log x)dx$ on $(0,\infty)$.

Sol: $I=\int \cos(\log x)dx=\int \cos(\log x)1\cdot dx$

ఇక్కడ $u=\cos(\log x)$, $v=1$ and using integration by parts successively.

$I=\cos(\log x)x-\int -\sin(\log x)\frac{1}{x}x\cdot dx$

$= x\cos(\log x) + \int \sin(\log x)dx$

$= x\cos(\log x) + \sin(\log x)\cdot x - \int \cos(\log x)\frac{1}{x}\cdot x \cdot dx$

$= x\cos(\log x) + x\cdot\sin(\log x) - \int \cos(\log x)dx$

$= x[\cos(\log x) + \sin(\log x)] - 1$

$\therefore 2I = x[\cos(\log x) + \sin(\log x)]$

$\Rightarrow I = \frac{x}{2}[\cos(\log x) + \sin(\log x)] + c$

$\therefore \int \cos(\log x)dx = \frac{x}{2}[\cos(\log x) + \sin(\log x)] + c$

16. $\int e^x \frac{x+2}{(x+3)^2}dx$

Sol. $\int e^x \frac{x+2}{(x+3)^2}dx$

$= \int e^x \left\{\frac{x+3-1}{(x+3)^2}\right\}dx$

$= \int e^x \left\{\frac{1}{x+3} + \frac{(-1)}{(x+3)^2}\right\}dx \;= e^x\left(\frac{1}{x+3}\right) + C \quad \left(\because \int e^x [f(x) + f'(x)]dx = e^x.f(x) + C\right)$

17. $\int \frac{xe^x}{(x+1)^2}dx$ **on $I \subset R \setminus \{-1\}$**

Sol. $\int \frac{xe^x}{(x+1)^2}dx = \int\left[\frac{x+1-1}{(x+1)^2}\right]e^x dx$

$= \int\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]e^x dx$

$= \int\left[\left(\frac{1}{x+1}\right) + \frac{(-1)}{(x+1)^2}\right]e^x dx$

$\left(\because \int e^x [f(x) + f'(x)]dx = e^x.f(x) + C\right)$

$= \left(\frac{1}{x+1}\right)e^x + C = \frac{e^x}{x+1} + C$

III.

1. $\int x\tan^{-1}x\,dx,\ x\in R$

Sol. $\int x\tan^{-1}x\,dx = (\tan^{-1}x)\frac{x^2}{2}-\frac{1}{2}\int x^2\cdot\frac{1}{1+x^2}dx$

$$=\frac{x^2(\tan^{-1}x)}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2(\tan^{-1}x)}{2}-\frac{1}{2}(x-\tan^{-1}x)+C$$

$$=\frac{x^2(\tan^{-1}x)}{2}-\frac{x}{2}+\frac{\tan^{-1}x}{2}+C$$

$$=\frac{(x^2+1)}{2}\tan^{-1}x-\frac{x}{2}+C$$

2. $\int x^2\tan^{-1}x\,dx,\ x\in R$.

Sol. $\int x^2\tan^{-1}x\,dx = (\tan^{-1}x)\frac{x^3}{3}-\frac{1}{3}\int x^3\frac{1}{1+x^2}dx$

$$=\frac{x^3(\tan^{-1}x)}{3}-\frac{1}{3}\int\frac{x(x^2+1)-x}{1+x^2}dx$$

$$=\frac{x^3(\tan^{-1}x)}{3}-\frac{1}{3}\int xdx+\frac{1}{3}\int\frac{xdx}{1+x^2}$$

$$=\frac{x^3(\tan^{-1}x)}{3}-\frac{x^2}{6}+\frac{1}{6}\log|1+x^2|+C$$

3. $\int\frac{\tan^{-1}x}{x^2}dx, x\in I\subset R\setminus\{0\}$

Sol. $\int\frac{\tan^{-1}x}{x^2}dx=\int\tan^{-1}x\frac{1}{x^2}=(\tan^{-1}x)\left(-\frac{1}{x}\right)+\int\frac{1}{x}\frac{1}{1+x^2}dx$

$$=-\frac{\tan^{-1}x}{x}+\frac{1}{2}\int\frac{2x\,dx}{x^2(1+x^2)}$$

$$=-\frac{\tan^{-1}x}{x}+\frac{1}{2}\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)(2x\,dx)$$

$$= \frac{\tan^{-1} x}{x} + \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$$

$$= -\frac{\tan^{-1} x}{x} + \log|x| - \frac{1}{2}\log|1+x^2| + C$$

4. $\int x\cos^{-1} x\,dx, x \in (-1,1)$

Sol. $\int x\cos^{-1} x$

$$= \cos^{-1}\int x\,dx - \int\left[\frac{d}{dx}[\cos^{-1} x]\int x\,dx\right]dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \int \frac{-1}{\sqrt{1-x^2}}\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\cos^{-1} x + \frac{1}{2}\int \frac{x^2}{1-x^2}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int \sqrt{1-x^2}dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] + \frac{1}{2}\sin^{-1} x + C$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x + C$$

5. $\int x^2\sin^{-1} x\,dx, x \in (-1,1)$

Sol. $\int x^2\sin^{-1} x\,dx$

$$= (\sin^{-1} x)\frac{x^3}{3} - \frac{1}{3}\int x^3\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{3}\int \frac{x[1-(1-x^2)]}{\sqrt{1-x^2}}dx$$

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{3}\int \frac{xdx}{\sqrt{1-x^2}} + \frac{1}{3}\int x\sqrt{1-x^2}dx$$

$$=\frac{x^3}{3}\sin^{-1}x+\frac{1}{3}\sqrt{1-x^2}+\frac{1}{3}\frac{(1-x^2)^{3/2}}{(3/2)(-2)}+C$$

$$=\frac{x^3}{3}\sin^{-1}x+\frac{\sqrt{1-x^2}}{3}-\frac{1}{9}(1-x^2)^{3/2}+C$$

6. $\int x\log(1+x)dx, x\in(-1,\infty)$

Sol. $\int x\log(1+x)dx$

$$=\log(1+x)\left(\frac{x^2}{2}\right)-\frac{1}{2}\int\frac{x^2}{1+x}dx$$

$$=\frac{x^2}{2}\log(1+x)-\frac{1}{2}\int\frac{1-(1-x^2)}{1+x}dx$$

$$=\frac{x^2}{2}\log(1+x)-\frac{1}{2}\int\frac{dx}{1+x}+\frac{1}{2}\int(1-x)dx$$

$$=\frac{x^2}{2}\log(1+x)-\frac{1}{2}\log(1+x)+\frac{1}{2}\left(x-\frac{x^2}{2}\right)+C$$

$$=\frac{(x^2-1)}{2}\log(1+x)+\frac{x}{2}-\frac{x^2}{4}+C$$

7. $\int\sin\sqrt{x}\,dx$ **on (0, ∞).**

Sol. Put $x=t^2\Rightarrow dx=2t\,dt$

$$=2\left[t(-\cos t)+\int\cos t\,dt\right]$$

$$=-2t\cos t+2\sin t$$

$$=-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$$

8). $\int e^{ax}\sin(bx+c)dx$ **, (a, b, c ∈ R, b ≠ 0) on R.**

Sol.

Let $I=\int e^{ax}\sin(bx+c)dx$

$$=e^{ax}\left(-\frac{\cos(bx+c)}{b}\right)+\frac{1}{b}\int\cos(bx+c)e^{ax}a\,dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b}\int e^{ax}\cos(bx+c)dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b}\left(e^{ax}\cdot\sin\frac{bx+c}{b}\right) - \frac{1}{b}\int \sin(bx+c)e^{ax}\cdot a\cdot dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}I$$

$$\left(1+\frac{a^2}{b^2}\right)I = -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c) \quad \frac{a^2+b^2}{b^2}I = \frac{e^{ax}}{b^2}[a\sin(bx+c) - b(\cos(bx+c)]$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b(\cos(bx+c)] + C_1$$

9. $\int a^x \cos 2x\, dx$

Sol. $\int a^x \cos 2x\, dx$

$$= a^x\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\cdot a^x \log a\, dx$$

$$= \frac{a^x\cdot\sin 2x}{2} + \frac{\log a}{2}\int a^x(-\sin 2x)dx$$

$$= \frac{a^x\sin 2x}{2} + \frac{\log a}{2}\left(a^x\cdot\cos\frac{2x}{2} - \frac{1}{2}\int \cos 2x\cdot a^x \log a\, dx\right)$$

$$= \frac{a^x\sin 2x}{2} + \frac{a^x\log a\cos 2x}{4} - \frac{(\log a)^2}{4}I$$

$$\left(1+\frac{(\log a)^2}{4}\right)I = \frac{a^x[2\sin 2x + (\log a)\cos 2x]}{4}$$

$$\frac{4+(\log a)^2}{4}I = \frac{a^x[2\sin 2x + (\cos 2x)\log a]}{4}$$

$$\therefore I = \frac{2\cdot a^x\cdot\sin 2x + (a^x\cdot\log a)\cos 2x}{(\log a)^2+4} + c$$

10. $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$

Sol. Put $x = \tan t \Rightarrow dx = \sec^2 t\, dt$

Then $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$

$$= \int \tan^{-1}\left(\frac{3\tan t - \tan^3 t}{1 - 3\tan^2 t}\right)\sec^2 t\, dt$$

$$= \int \tan^{-1}(\tan 3t)\sec^2 t\, dt = 3\int t\sec^2 t\, dt$$

$$= 3\left[t\int \sec^2 t\, dt - \int\left\{\frac{d}{dt}(t)\int \sec^2 t\, dt\right\}dt\right]$$

$$= 3[t(\tan t) - \int (1)\tan t\, dt]$$

$$= 3(t\tan t - \log|\sec t|) + C$$

$$= 3\left(x\cdot\tan^{-1}x - \log\sqrt{1+x^2}\right) + C$$

$$= 3x\left[\tan^{-1}x - \frac{3}{2}\log(1+x^2)\right] + C$$

$$= 3x\tan^{-1}(x) - \frac{3}{2}\log(1+x^2) + C$$

11. $\int \sinh^{-1}x\, dx$ **on R.**

Sol. $\int \sinh^{-1}x\, dx = \int 1.\sinh^{-1}x\, dx$

$$= x.\sinh^{-1}x - \int \frac{1}{\sqrt{1+x^2}}.xdx$$

$$= x.\sinh^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx$$

$$= x.\sinh^{-1}x - \frac{1}{2}2.\sqrt{1+x^2} + c$$

$$= x.\sinh^{-1}x - \sqrt{1+x^2} + c$$

12. $\int \cosh^{-1} x dx$ **on [1, ∞].**

Sol. $\int \cosh^{-1} x dx = \int 1.\cosh^{-1} x dx$

Ans: $x\cosh^{-1} x - \sqrt{x^2 - 1} + C$

13. $\int \tanh^{-1} x\, dx$ **on (–1, 1).**

Sol. $\int \tanh^{-1} x\, dx = \int 1.\tanh^{-1} x\, dx$

$$= \int 1.\tanh^{-1} x\, dx$$

$$= x.\tanh^{-1} x - \int \frac{1}{1-x^2} x dx$$

$$= x.\tanh^{-1} x + \frac{1}{2}\int \frac{-2x}{1-x^2} dx$$

$$= x.\tanh^{-1} x + \frac{1}{2}\log\left(1-x^2\right) + c$$

EXERCISE-6(D)

I. క్రింది సమాకలనులను గణించండి

1. $\int \frac{dx}{\sqrt{2x-3x^2+1}}$

Sol. $\int \frac{dx}{\sqrt{2x-3x^2+1}}$

$$= \int \frac{dx}{\sqrt{3\left(\frac{2x}{3}-x^2+\frac{1}{3}\right)}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{x-\frac{1}{3}}{\frac{2}{3}}\right)+C = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$$

2. $\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}}d\theta$

Sol. $\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}}d\theta$

put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$

$$= \int -\frac{dt}{\sqrt{2-t^2}} = -\int \frac{dt}{\sqrt{(\sqrt{2})^2-t^2}}$$

$$= -\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)+C = -\sin^{-1}\left(\frac{\cos\theta}{\sqrt{2}}\right)+C$$

3. $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$

Sol. $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$

put $\sin x = t \Rightarrow \cos x dx = dt$

$$= \int \frac{dt}{t^2+4t+5} = \int \frac{dt}{(t+2)^2+1}$$

$$= \tan^{-1}(t+2)+C = \tan^{-1}(\sin x+2)+C$$

4. $\int \frac{dx}{1+\cos^2 x}$

Sol. $\int \frac{dx}{1+\cos^2 x} = \int \frac{\sec^2 dx}{\sec^2 x+1} = \int \frac{\sec^2 xdx}{\tan^2 x+2}$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

5. $\int \frac{dx}{2\sin^2 x+3\cos^2 x}$

Sol. $\int \frac{dx}{2\sin^2 x+3\cos^2 x} = \int \frac{\sec^2 xdx}{2\tan^2 x+3}$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{dt}{2t^2+3} = \frac{1}{2}\int \frac{dt}{t^2+\left(\sqrt{\frac{3}{2}}\right)^2}$$

$$= \frac{2}{2\sqrt{\frac{3}{2}}} \tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right) + C$$

6. $\int \frac{1}{1+\tan x}dx$

Sol. $\int \frac{1}{1+\tan x}dx$

$$= \int \frac{1}{1+\frac{\sin x}{\cos x}}dx = \int \frac{\cos x\, dx}{\sin x+\cos x} = \frac{1}{2}\int \frac{2\cos x\, dx}{\sin x+\cos x}$$

$$= \frac{1}{2}\int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{\sin x+\cos x}dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x-\sin x}{\sin x+\cos x}dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x+\cos x| + C$$

7. $\int \frac{1}{1-\cot x}dx$

Sol. $\int \frac{1}{1-\cot x}dx = \int \frac{1}{1-\frac{\cos x}{\sin x}}dx = \int \frac{\sin xdx}{\sin x-\cos x}$

$$= \frac{1}{2}\int \frac{(\sin x-\cos x)+(\cos x+\sin x)}{\sin x-\cos x}dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x+\sin x}{\sin x-\cos x}dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x-\cos x| + C$$

II. క్రింది సమాకలనులను గణించండి

1. $\int \sqrt{1+3x-x^2}dx$

Sol.
$$\int \sqrt{1+3x-x^2}dx = \int \sqrt{1-(x^2-3x)}dx$$
$$= \int \sqrt{1-(x-\frac{3}{2})^2 - \frac{9}{2}}\, dx$$
$$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right) - \left(x-\frac{3}{2}\right)^2}$$
$$= \frac{\left(x-\frac{3}{2}\right)\sqrt{1+3x-x^2}}{2} + \frac{13}{8}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$
$$= \frac{(2x-3)\sqrt{1+3x-x^2}}{2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

2. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx$

Sol. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx$

9cosx-sinx $= A\frac{d}{dx}(4\sin x + 5\cos x) + B(4\sin x + 5\cos x)$

9cosx-sinx $= A(4\cos x - 5\sin x) + B(4\sin x + 5\cos x)$

ఇరువైపులా సరి పద గుణకాలను పోల్చగా

9 = 4A+5B and -5 = -5A+4B

సమీకరణాలను సాధించగా

A =1,B=1.

$\therefore$ 9cosx-sinx $=1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)$

$=1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)$

$$\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx = \int \frac{(4\sin x + 5\cos x) + (4\cos x - 5\sin x)}{4\sin x + 5\cos x}dx$$

$$= \int dx + \int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x} dx$$

$$= x + \log|4\sin x + 5\cos x| + C$$

3. $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$

Sol. Let 2 cos x + 3 sin x = A(4 cosx + 5 sin x) + B(–4 sin x + 5 cos x)

ఇరువైపులా sinx , cosx, పద గుణకాలను పోల్చగా

4A + 5B = 2, 5A – 4B = 3.

A B 1

+5 –2 4 +5

–4 –3 5 –4

$$\frac{A}{-15-8} = \frac{B}{-10+12} = \frac{1}{-16-25}$$

$$A = \frac{23}{41}, B = -\frac{2}{41}$$

$$\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx =$$

$$= \frac{23}{41}\int dx - \frac{2}{41}\int \frac{-4\sin x + 5\cos x}{4\cos x + 5\sin x} dx$$

$$= \frac{23}{41}x - \frac{2}{41}\log|4\cos x + 5\sin x| + C$$

4. $\int \frac{dx}{1+\sin x + \cos x}$

Sol. $\int \frac{dx}{1+\sin x + \cos x}$

$$= \int \frac{dx}{\left[1 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$$

$$= \int \frac{\sec^2\frac{x}{2}dx}{1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$= \int \frac{\sec^2 \frac{x}{2}}{2+2\tan\frac{x}{2}} \quad \text{put } \tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$= 2\int \frac{dt}{2+2t} = \int \frac{dt}{1+t} \log|1+t| + C$$

$$= \log\left|1+\tan\frac{x}{2}\right| + C$$

5. $\int \frac{dx}{3x^2+x+1}$

Sol. $\int \frac{dx}{3x^2+x+1} = \int \frac{dx}{3\left(x^2+\frac{1}{3}x+\frac{1}{3}\right)}$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{6}\right)^2 + \frac{1}{3} - \frac{1}{36}} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2}$$

$$= \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{11}}{6}}\tan^{-1}\left(\frac{x+(1/6)}{(\sqrt{11}/6)}\right) + C$$

$$= \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right) + C$$

6. $\int \frac{dx}{\sqrt{5-2x^2+4x}}$

Sol. $\int \frac{dx}{\sqrt{5-2x^2+4x}}$

$5 - 2x^2 + 4x$

$$= -2\left[x^2 - 2x - \frac{5}{2}\right] = -2\left[(x-1)^2 - 1 - \frac{5}{2}\right]$$

$$= -2\left[(x-1)^2 - \frac{7}{2}\right] = 2\left[\frac{7}{2} - (x-1)^2\right]$$

$$\int \frac{1}{\sqrt{5-2x^2+4x}}dx$$

$$= \int \frac{1}{\sqrt{2}\left\{\frac{7}{2}-(x-1)^2\right\}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{7}{2}\right)^2 - (x-1)^2}} dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{(x-1)}{\sqrt{7/2}} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \sqrt{\frac{2}{7}}(x-1) + C$$

III.

1. $\int \frac{x+1}{\sqrt{x^2-x+1}} dx$

Sol.

take $x+1 = A\frac{d}{dx}\left(x^2-x+1\right)+B$

$x+1 = A\left(2x-1\right)+B$

ఇరువైపులా సరి పద గుణకాలను పోల్చగా

$2A = 1$ and $B-A = 1$

$\Rightarrow A=\frac{1}{2}$ and $b = \frac{3}{2}$

$\therefore x+1 = \frac{1}{2}\left(2x-1\right)+\frac{3}{2}$

$$\int \frac{x+1}{\sqrt{x^2-x+1}} dx = \int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{\sqrt{x^2-x+1}} dx$$

$$= \frac{1}{2}\int \frac{(2x-1)dx}{\sqrt{x^2-x+1}} + \frac{3}{2}\int \frac{dx}{\sqrt{x^2-x+1}}$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2 - x + 1} + \frac{3}{2}\sinh^{-1}\left(\frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \sqrt{x^2 - x + 1} + \frac{3}{2}\sinh^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) + C$$

2. $\int (6x+5)\sqrt{6 - 2x^2 + x}\,dx$

Sol.

let $6x+5 = A\,\frac{d}{dx}\left(6 - 2x^2 + x\right) + B$

$\Rightarrow 6x + 5 = A(1 - 4x) + B$

ఇరుపైపులా సరి పద గుణకాలను పోల్చగా

$6 = -4A \Rightarrow A = \frac{-3}{2}$

$A + B = 5$

$B = 5 - A = 5 + \frac{3}{2} = \frac{13}{2}$

$\int (6x+5)\sqrt{6 - 2x^2 + x}\,dx$

$$= -\frac{3}{2}\int (1 - 4x)\sqrt{6 - 2x^2 + x}\,dx + \frac{13}{2}\sqrt{6 - 2x^2 + x}\,dx$$

$$= -\frac{3}{2}\frac{(6 - 2x^2 + x)^{3/2}}{3/2} + \frac{13}{2}\sqrt{2}\int\sqrt{3 - x^2 + \frac{x}{2}}dx$$

$$= -(6 - 2x^2 + x)^{3/2} + \frac{13}{\sqrt{2}}\int\sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\,dx$$

$$= -(6 - 2x^2 + x)^{3/2} + \frac{13}{\sqrt{2}}$$

$$\left(\frac{\left(x - \frac{1}{4}\right)\sqrt{3 - x^2 + \frac{x}{2}}}{2} + \frac{49}{32}\sin^{-1}\frac{\left(x - \frac{1}{4}\right)}{\left(\frac{7}{4}\right)}\right) + C$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{\sqrt{2}}$$

$$\left[\frac{(4x-1)\sqrt{6-2x^2+x}}{16\times 2} + \frac{49}{32}\sin^{-1}\left(\frac{4x-1}{7}\right)\right] + C$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{16}(4x-1)$$

$$\sqrt{6-2x^2+x} + \frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + C$$

3. $\int \frac{dx}{4+5\sin x}$

Sol. $\int \frac{dx}{4+5\sin x} = \int \frac{dx}{4+5\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$$\text{put } \tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$$

$$\Rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+t^2}$$

$$\text{G.I.} = 2\int \frac{\frac{dt}{1+t^2}}{4+5\frac{2t}{1+t^2}} = 2\int \frac{dt}{4+4t^2+10t}$$

$$= \frac{1}{2}\int \frac{dt}{t^2+\frac{5t}{2}+1} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right| + C = \frac{1}{3}\log\left|\frac{2t+1}{2t+4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\left(\tan\frac{x}{2}\right)+2}\right|+C$$

4. $\int \frac{1}{2-3\cos 2x}dx$

Sol. $\int \frac{1}{2-3\cos 2x}dx = \int \frac{dx}{2-3\frac{1-\tan^2 x}{1+\tan^2 x}}$

put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$dx = \frac{dt}{1+t^2}$$

$$GI = \int \frac{\frac{dt}{1+t^2}}{2-3\frac{1-t^2}{1+t^2}} = \int \frac{dt}{2+2t^2-3+3t^2}$$

$$= \int \frac{dt}{5t^2-1} = \frac{1}{5}\int \frac{dt}{t^2-\left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\frac{(1/2)}{\sqrt{5}}\log\left|\frac{t-\frac{1}{\sqrt{5}}}{t+\frac{1}{\sqrt{5}}}\right|+C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right|+C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right|+C$$

5. $\int x\sqrt{1+x-x^2}dx$

Sol. Let x = A(1 – 2x) + B

ఇరువైపులా సరి పద గుణకాలను పోల్చగా

$1 = -2A \Rightarrow A = -1/2, \qquad 0 = A + B \Rightarrow B = -A = 1/2$

$\int x\sqrt{1+x-x^2}\,dx =$

$-\frac{1}{2}\int(1-2x)\sqrt{1+x-x^2}dx + \frac{1}{2}\int\sqrt{1+x-x^2}dx$

$= -\frac{1}{2}\frac{(1+x-x^2)^{3/2}}{3/2} + \frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}dx$

$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left(\frac{\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2}}{2} + \frac{25}{8}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)\right)$

$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{(2x-1)\sqrt{1+x-x^2}}{8} + \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C$

6. $\int\frac{dx}{(1+x)\sqrt{3+2x-x^2}}$

Sol. $\int\frac{dx}{(1+x)\sqrt{3+2x-x^2}} = \int\frac{dx}{(1+x)\sqrt{(3-x)(1+x)}}$

Put $1+x = t^2 \Rightarrow dx = 2t\,dt$

$G.I. = \int\frac{2t\,dt}{t^2\sqrt{t^2(4-t^2)}} = \int\frac{2dt}{t^2\sqrt{4-t^2}} = \int\frac{2}{t^3}\frac{dt}{\sqrt{\frac{4}{t^2}-1}}$

Put $\frac{4}{t^2} - 1 = y^2 \Rightarrow -\frac{8}{t^3}dt = 2y\,dy$

$\Rightarrow \frac{2}{t^3}dt = -\frac{y}{4}dy$

$G.I. = 2\int -\frac{y}{4}\frac{dy}{\sqrt{y^2}} = -\frac{1}{2}\int dy = -\frac{1}{2}y + C$

$$= -\frac{1}{2}\sqrt{\frac{4}{t^2} - 1} + C$$

$$= -\frac{1}{2}\sqrt{\frac{4}{1+x} - 1} + C - \frac{1}{2}\sqrt{\frac{3-x}{3+x}} + C$$

7. $\int \frac{dx}{4\cos x + 3\sin x}$

Sol. $\int \frac{dx}{4\cos x + 3\sin x} = \int \frac{dx}{4\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 3\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}$

$$I = \int \frac{\frac{2dt}{1+t^2}}{4\frac{(1-t^2)}{1+t^2} + \frac{3\cdot 2t}{1+t^2}} = 2\int \frac{dt}{4 - 4t^2 + 6t}$$

$$= -\frac{1}{2}\int \frac{dt}{t^2 - \frac{3}{2}t - 1} = -\frac{1}{2}\int \frac{dt}{\left(t - \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2}$$

$$= -\frac{1}{2}\frac{1}{2\cdot\frac{5}{4}}\log\left|\frac{t - \frac{3}{4} - \frac{5}{4}}{t - \frac{3}{4} + \frac{5}{4}}\right| + C$$

$$= -\frac{1}{5}\log\left|\frac{t-2}{t+(1/2)}\right| + C = -\frac{1}{5}\log\left|\frac{2t-4}{2t+1}\right| + C$$

$$= -\frac{1}{5}\log\left|\frac{2\left(\tan\frac{x}{2} - 2\right)}{2\tan\frac{x}{2} + 1}\right| + C$$

8. $\int \frac{1}{\sin x + \sqrt{3}\cos x} dx$

Sol.Let $t = \tan\frac{x}{2}$ so that $dx = \frac{2dt}{1+t^2}$

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{2\frac{dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{\sqrt{3}(1-t^2)}{1+t^2}} = 2\int \frac{dt}{\sqrt{3}(1-t^2)+2t}$$

$$= \frac{2}{\sqrt{3}}\int \frac{dt}{1-t^2+\frac{2}{\sqrt{3}}t} = \frac{2}{\sqrt{3}}\int \frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2 - \left(t-\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\frac{1/4}{\sqrt{3}}\log\left|\frac{\frac{2}{\sqrt{3}}+t-\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}-t+\frac{1}{\sqrt{3}}}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{t+\frac{1}{\sqrt{3}}}{\sqrt{3}-t}\right| + C = \frac{1}{2}\log\left|\frac{\sqrt{3}t+1}{\sqrt{3}(\sqrt{3}-t)}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{3}\tan\frac{x}{2}+1}{\sqrt{3}\left(\sqrt{3}-\tan\frac{x}{2}\right)}\right| + c$$

9. $\int \frac{dx}{5+4\cos 2x}$

Sol. $t = \tan x \Rightarrow dt = \sec^2 x\, dx,\ dx = \frac{dt}{1+t^2}$

$$I = \int \frac{\frac{dt}{1+t^2}}{\frac{5+4(1-t^2)}{1+t^2}} = \int \frac{dt}{5+5t^2+4-4t^2}$$

$$= \int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

$$=\frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right)+C$$

10. $\int \frac{2\sin x+3\cos x+4}{3\sin x+4\cos x+5}dx$

Sol.

$$\int \frac{2\sin x+3\cos x+4}{3\sin x+4\cos x+5}dx$$

Let $2\sin x + 3\cos x + 4 = A(3\sin x + 4\cos x + 5) + B\frac{d}{dx}(3\sin x + 4\cos x + 5) + C$

$2\sin x + 3\cos x + 4$

$= A(3\sin x + 4\cos x + 5) + 3(3\cos x - 4\sin x) + C$

ఇరుపైపులా సరి పద గుణకాలను పోల్చగా

$3A - 4B = 2$ & $4A + 3B = 3$

$$A=\frac{18}{25}, B=\frac{1}{25}$$

$4 = 5A + C$

$$C = 4 - 5A = 4 - 5\cdot\frac{18}{25}=\frac{2}{5}$$

$$2\sin x + 3\cos x + 4 = \frac{18}{25}(3\sin x + 4\cos x + 5) + \frac{1}{25}(3\cos x - 4\sin x) + \frac{2}{5}$$

$$\therefore \int \frac{2\sin x+3\cos x+4}{3\sin x+4\cos x+5}dx$$

$$=\frac{18}{25}\int dx+\frac{1}{25}\int\frac{3\cos x-4\sin x}{3\sin x+4\cos x+5}$$

$$+\frac{2}{5}\int\frac{dx}{3\sin x+4\cos x+5}$$

$$=\frac{18}{25}x+\frac{1}{25}\log|3\sin x+4\cos x+5|$$

$$+\frac{2}{5}\int\frac{dx}{3\sin x+4\cos x+5} \qquad ...(1)$$

Let $I=\int\frac{dx}{3\sin x+4\cos x+5}$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}$ $I = \int \frac{\frac{2dt}{1+t^2}}{\frac{3-2t}{1+t^2} + \frac{4(1+t^2)}{1+t^2} + 5}$

$$= 2\int \frac{dt}{6t+4-4t^2+5+5t^2} = 2\int \frac{dt}{t^2+6t+9}$$

$$= 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} = -\frac{2}{3+\tan\frac{x}{2}}$$

(1) లో ప్రతిక్షేపించగా

$$I = \frac{18}{25}\cdot x + \frac{1}{25}\log|3\sin x + 4\cos x + 5| - \frac{4}{5\left(3+\tan\frac{x}{2}\right)} + C$$

11. $\int \sqrt{\frac{5-x}{x-2}}\,dx$ on (2, 5).

Sol: $\int \sqrt{\frac{5-x}{x-2}}\,dx = \int \sqrt{\frac{(5-x)^2}{(x-2)(5-x)}}\,dx$

$$= \int \frac{5-x}{\sqrt{(x-2)(5-x)}}\,dx$$

$$= \int \frac{5-x}{\sqrt{5x - x^2 - 10 + 2x}}\,dx$$

$$= \int \frac{5-x}{\sqrt{7x - x^2 - 10}}\,dx$$

Let $5 - x = A\frac{d}{dx}(7x - x^2 - 10) + B$

$$= A(7-2x) + B$$

ఇరువైపులా సరి పద గుణకాలను పోల్చగా

$-2A = -1 \Rightarrow A = 1/2$ and $7A + B = 5 \Rightarrow B = 5 - 7A = 5 - \frac{7}{2} = \frac{3}{2}$.

$\therefore 5-x=\frac{1}{2}(7-2x)+\frac{3}{2}$

$\therefore \int \frac{5-x}{\sqrt{7x-x^2-10}}dx$

$=\frac{1}{2}\int \frac{(7-2x)}{\sqrt{7x-x^2-10}}dx+\frac{3}{2}\int \frac{dx}{\sqrt{7x-x^2-10}}\ \ldots(1)$

Consider the first integral on RHS and suppose $7x - x^2 - 10 = t \Rightarrow (7 - 2x)d = dt$

$\therefore \frac{1}{2}\int \frac{(7-2x)}{\sqrt{7x-x^2-10}}dx=\frac{1}{2}\int \frac{dt}{\sqrt{t}}=\frac{1}{2}\int t^{-1/2}dt$

$=\frac{1}{2}\frac{t^{1/2}}{1/2}=\sqrt{t}=\sqrt{7x-x^2-10}\quad \ldots(2)$

$7x-x^2-10=-(x^2-7x+10)$

$=-\left(x^2-2\left(\frac{7}{2}\right)x+\frac{49}{4}-\frac{49}{4}+10\right)=-\left[\left(x-\frac{7}{2}\right)^2-\frac{9}{4}\right]$

$=-\left[\left(x-\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]=\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2$

$\therefore \int \frac{dx}{\sqrt{7x-x^2-10}}=\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}}$

$=\sin^{-1}\left(\frac{x-\frac{7}{2}}{\frac{3}{2}}\right)=\sin^{-1}\left(\frac{2x-7}{3}\right)\ldots(3)$

$\therefore$ From (1), (2) and (3)

$\int \frac{5-x}{\sqrt{7x-x^2-10}}dx=\sqrt{7x-x^2-10}+\frac{3}{2}\sin^{-1}\left(\frac{2x-7}{3}\right)+c$

12. $\int \sqrt{\frac{1+x}{1-x}}dx \text{ on } (-1,1)$

Sol: $\int \sqrt{\frac{1+x}{1-x}}dx = \int \sqrt{\frac{(1+x)^2}{1-x}}dx$

$= \int \sqrt{\frac{1+x}{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{xdx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \frac{1}{2}\int \frac{(-2x)dx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \sqrt{1-x^2} + c$

13. $\int \frac{dx}{(1-x)\sqrt{3-2x-x^2}}$ on (–1, 3).

Sol: Put 1 – x = 1/t ⇒ x = $1-\frac{1}{t}$

$\therefore dx = +\frac{1}{t^2}dt$

Also $3-2x-x^2 = 3-\left(1-\frac{1}{t}\right)-\left(1-\frac{1}{t}\right)^2$

$= 3+\frac{2}{t}-2-\left(\frac{1}{t^2}-\frac{2}{t}+1\right)$

$= \frac{4}{t}-\frac{1}{t^2} = \frac{4t-1}{t^2}$

$\therefore \int \frac{dx}{(1-x)\sqrt{3-2x-x^2}} = \int \frac{\left(\frac{1}{t^2}\right)dt}{\left(\frac{1}{t}\right)\sqrt{\frac{4t-1}{t^2}}}$

$= \int \frac{dt}{\sqrt{4t-1}} = \frac{2\sqrt{4t-1}}{4}+c$

$= \frac{1}{2}\sqrt{4t-1}+c$

$= \frac{1}{2}\sqrt{4\left(\frac{1}{1-x}\right)-1}+c$

$= \frac{1}{2}\sqrt{\frac{4-1+x}{1-x}}+c = \frac{1}{2}\sqrt{\frac{3+x}{1-x}}.$

14. $\int \frac{dx}{(x+2)\sqrt{x+1}}$

Sol. $\int \frac{dx}{(x+2)\sqrt{x+1}}$

x+1 =t^2 అనుకొనుము

$dx = 2tdt$,x+2=1+t^2

$G.I. = \int \frac{2t}{(1+t^2)t} dt$

$= 2.\tan^{-1} t + c = 2\tan^{-1}\sqrt{x+1} + c$

15. $\int \frac{dx}{(2x+3)\sqrt{x+2}}$

Sol. x+2 =t^2 అనుకొనుము

$dx = 2tdt$ and 2x+3=2t^2 -1

$G.I. = \int \frac{2t}{(2t^2-1)t} dt = 2\int \frac{1}{(\sqrt{2}\ t)^2 - 1} dt$

$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}\ t - 1}{\sqrt{2}\ t + 1} + c$

$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2x+4}\ -1}{\sqrt{2x+4}\ +1} + c$

16. $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$

Sol. $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}} = \int \frac{dx}{(1+\sqrt{x})\sqrt{x}\sqrt{1-x}}$

x=$\sin^2 t \Rightarrow dx = 2\sin t.\cos t.dt$

$= \int \frac{2\sin t.\cos t.dt}{(1+\sin t)\sin t.\cos t} = 2\int \frac{1}{1+\sin t} dt = 2.(\tan t - \sec t) + c$

$= 2\int \frac{1}{1+\sin t} dt = 2.(\tan t - \sec t) + c$

17. $\int \frac{dx}{(x+1)\sqrt{2x^2+3x+1}}$

Sol. $\int \frac{dx}{(x+1)\sqrt{2x^2+3x+1}}$

$x+1=\frac{1}{t} \Rightarrow x=\frac{1-t}{t}$ *and* $dx=\frac{-1}{t^2}dt$

$$g.i.=\int \frac{1}{\frac{1}{t}.\sqrt{2\left(\frac{1-t}{t}\right)^2+3\left(\frac{1-t}{t}\right)+1}}.\frac{-1}{t^2}dt$$

$$=\int \frac{-1}{\sqrt{2+2t^2-4t+3t-3t^2+t^2}}dt$$

$$=-\int \frac{1}{\sqrt{2-t}}dt=2\sqrt{2-t}+c$$

$$=2\sqrt{2-\frac{1}{x+1}}+c=2\sqrt{\frac{2x+1}{x+1}}+c$$

18. $\int \sqrt{e^x-4}dx$

Sol. $\int \sqrt{e^x-4}dx=\int \frac{e^x-4}{\sqrt{e^x-4}}dx$

$$=\int \frac{e^x}{\sqrt{e^x-4}}dx-\int \frac{4}{\sqrt{e^x-4}}dx$$

$$=2\sqrt{e^x-4}-4\int \frac{e^{-x/2}}{\sqrt{1-4e^{-x}}}dx$$

$$=2\sqrt{e^x-4}+4\int \frac{e^{-x/2}}{\sqrt{1-4\left(e^{-x/2}\right)^2}}dx$$

$$=2\sqrt{e^x-4}+4.\int \frac{e^{-x/2}}{\sqrt{1-\left(2e^{-x/2}\right)^2}}dx$$

$$=2\sqrt{e^x-4}+4\sin^{-1}2e^{-x/2}+c$$

$$=2\sqrt{e^x-4}+4\tan^{-1}\frac{2e^{-x/2}}{\sqrt{1-4e^{-x}}}+c$$

$$=2\sqrt{e^x-4}+4\tan^{-1}\frac{2}{\sqrt{e^x-4}}+c$$

19. $\int \sqrt{1+\sec x}\,dx$

$$\int \sqrt{1+\sec x}\,dx = \int \sqrt{\frac{1+\cos x}{\cos x}}\,dx = \int \sqrt{\frac{2\cos^2\frac{x}{2}}{1-2\sin^2\frac{x}{2}}}\,dx$$

$$= \int \frac{\sqrt{2}.\cos\frac{x}{2}}{\sqrt{1-\left(\sqrt{2}\sin\frac{x}{2}\right)^2}}\,dx = 2\int \frac{\frac{1}{2}\sqrt{2}.\cos\frac{x}{2}}{\sqrt{1-\left(\sqrt{2}\sin\frac{x}{2}\right)^2}}\,dx$$

$$= 2\sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right)+c$$

20. $\int \frac{1}{1+x^4}\,dx$

Sol. $\int \frac{1}{1+x^4}\,dx$

$$= \frac{1}{2}\int \frac{2}{1+x^4}\,dx = \frac{1}{2}\int \frac{1+x^2+1-x^2}{1+x^4}\,dx$$

$$= \frac{1}{2}\int \left(\frac{1+x^2}{1+x^4}+\frac{1-x^2}{1+x^4}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}+\frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}}\right)dx$$

$$= \frac{1}{2}\left(\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{2}\right)^2}\,dx - \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-\left(\sqrt{2}\right)^2}\,dx\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\log\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+c$$

$$= \frac{1}{2\sqrt{2}}\left(\tan^{-1}\frac{x^2-1}{x\sqrt{2}} - \frac{1}{2}\log\frac{x^2+1-\sqrt{2}}{x^2+1+\sqrt{2}}\right)+c$$

EXERCISE – 6(E)

I. Evaluate the following integrals.

1. $\int \frac{(x-1)dx}{(x-2)(x-3)}$

Sol.

$$\frac{(x-1)}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} \quad x-1 = A(x-3)+B(x-2)$$

సరి పద గుణకాలను పోల్చగా

$$\therefore \frac{(x-1)}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}$$

$$\therefore \int \frac{(x-1)dx}{(x-2)(x-3)} = \int \left(\frac{-1}{x-2} + \frac{2}{x-3} \right) dx$$

$$= = 2\log(x-3) - \log(x-2) + c$$

2. $\int \frac{x^2}{(x+1)(x+2)^2} dx$

Sol. $\frac{x^2}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$$\Rightarrow x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

Put x = –2 in (1)

$(-2)^2 = A(0) + B(0) + C(-2+1) \Rightarrow C = -4$

(1) లో $x = -1$ ప్రతిక్షేపించగా

$(-1)^2 = A(-1+2)^2 + B(0) + C(0) \Rightarrow A = 1$

(1) లో x^2 పద గుణకాలను పోల్చగా

$1 = A + B \Rightarrow B = 1 - A = 1 - 1 = 0$

$$\therefore \frac{x^2}{(x+1)(x+2)^2} = \frac{1}{x+1} + \frac{0}{x+2} + \frac{-4}{(x+2)^2}$$

$$\therefore \int \frac{x^2}{(x+1)(x+2)^2}dx$$

$$= \int \frac{1}{x+1}dx - 4\int \frac{1}{(x+2)^2}dx$$

$$= \log|x+1| - 4\left(\frac{-1}{x+2}\right)$$

$$= \log|x+1| + \frac{4}{x+2} + C$$

3. $\int \frac{x+3}{(x-1)(x^2+1)}dx$

Sol. Let $\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\Rightarrow (x+3) = A(x^2+1) + (Bx+C)(x-1) \ldots(1)$

(1) లో $x = 1$ ప్రతిక్షేపించగా

Then $4 = A(1 + 1) + 0 \Rightarrow A = 2$

(1) లో $x = 0$ ప్రతిక్షేపించగా

$3 = A(1) + C(-1)$

$\Rightarrow A - C = 3 \Rightarrow C = A - 3 = 2 - 3 = 1$

(1) లో x^2 పద గుణకాలను పోల్చగా

$0 = A + B \Rightarrow B = -A = -2$

$$\therefore \frac{x+3}{(x-1)(x^2+1)} = \frac{+2}{(x-1)} + \frac{-2x-1}{x^2+1}$$

$$\int \frac{x+3}{(x-1)(x^2+1)}dx$$

$$= 2\int \frac{1}{x-1}dx - \int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$

$$= 2\log|x-1| - \log|x^2+1| - \tan^{-1}(x) + C$$

4. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Sol. పాక్షిక భిన్నాలనుండి

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(b^2-a^2)}\left(\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right)$$

$$\therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{(b^2-a^2)}\left[\int \frac{1}{(x^2+a^2)}dx - \int \frac{1}{(x^2+b^2)}dx\right]$$

$$= \frac{1}{(b^2-a^2)}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right] + C$$

5. $\int \frac{dx}{e^x+e^{2x}}$

Sol. $\frac{1}{e^x+e^{2x}} = \frac{1}{e^x(1+e^x)} = \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)$

$$\therefore \int \frac{1}{e^x+e^{2x}}dx = \int \frac{1}{e^x}dx - \int \frac{1}{1+e^x}dx$$

$$= \int e^{-x}dx - \int \frac{e^x}{e^x(1+e^x)}dx$$

$$= \int e^{-x}dx + \int e^x\left\{\frac{1}{e^x} - \frac{1}{1+e^x}\right\}dx$$

$$= \int e^{-x}dx - \int 1dx + \int \frac{e^x}{1+e^x}dx$$

$$= \frac{e^x}{(-1)} - x + \log|1+e^x| + C$$

$$= -e^x + \log(1+e^x) - \log(e^x) + C \quad [\because x = \log e^x]$$

$$= -e^{-x} + \log\left(\frac{1+e^x}{e^x}\right) + C$$

6. $\int \frac{dx}{(x+1)(x+2)}$

Sol. $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

7. $\int \frac{1}{e^x - 1} dx$

Sol. $\int \frac{1}{e^x - 1} dx = \frac{e^x - (e^x - 1)}{e^x - 1} dx = \int \frac{e^x dx}{e^x - 1} - \int dx$

$$= \log(e^x - 1) - x = \log(e^x - 1) - \log e^x + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

8. $\int \frac{1}{(1-x)(4+x^2)} dx$

Sol. Let $\frac{1}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$

$\Rightarrow 1 = A(4 + x^2) + (Bx + C)(1 - x) \quad \ldots(1)$

Put x = 1 in (1)

$1 = A(4 + 1) \Rightarrow A = 1/5$

x = 0 in (1)

$1 = A(4) + C(1) \Rightarrow C = 1 - 4A = 1/5$

సరి పద గుణకాలను పోల్చగా

$0 = A - B \Rightarrow B = A = 1/5$

$$\therefore \frac{1}{(1-x)(4+x^2)} = \left(\frac{1}{5}\right)\frac{1}{1-x} + \frac{\left(\frac{1}{5}x+\frac{1}{5}\right)}{4+x^2}$$

$$= \frac{1}{5}\left(\frac{1}{1-x}\right) + \frac{1}{5}\frac{x}{4+x^2} + \frac{1}{5}\frac{1}{4+x^2}$$

$$\therefore \int \frac{1}{(1-x)(4+x^2)}dx = \frac{1}{5}\int\frac{1}{1-x}dx + \frac{1}{5}\int\frac{x}{4+x^2}dx + \frac{1}{5}\int\frac{1}{4+x^2}dx$$

$$= -\frac{1}{5}\int\frac{1}{1-x}dx + \frac{1}{10}\int\frac{2x}{4+x^2}dx + \frac{1}{5}\int\frac{1}{4+x^2}dx$$

$$= -\frac{1}{5}\log|1-x| + \frac{1}{10}\log|4+x^2| + \frac{1}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{5}\log|1-x| + \frac{1}{10}\log|4+x^2| + \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right) + C$$

9. $\int \frac{2x+3}{x^3+x^2-2x}dx$

Sol. $\frac{2x+3}{x(x^2+x-2)} = \frac{2x+3}{x(x+2)(x-1)}$

Let $\frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$

$\Rightarrow 2x+3 = A(x+2)(x-1) + Bx(x-1) + C(x)(x+2)$ ...(1)

$x = 0$ ప్రతిక్షేపించగా (1)

$3 = A(2)(-1) + B(0) + C(0) \Rightarrow A = -3/2$

$x = 1$ ప్రతిక్షేపించగా

$2 + 3 = A(0) + B(0) + C(1)(3) \Rightarrow C = 5/3$

$x = -2$ ప్రతిక్షేపించగా

$2(-2) + 3 = A(0) + B(-2)(-2-1) + C(0)$

$\Rightarrow -1 = 6B \Rightarrow B = -1/6$

$$\therefore \frac{2x+3}{x^3+x^2-2x} = \frac{2x+3}{x(x+2)(x-1)}$$

$$= \frac{-\frac{3}{2}}{x} + \frac{-\frac{1}{6}}{x+2} + \frac{\frac{5}{3}}{x-1}$$

$$\int \frac{2x+3}{x^3+x^2-2x}dx = = -\frac{3}{2}\int \frac{1}{x}dx - \frac{1}{6}\int \frac{1}{x+2}dx + \frac{5}{3}\int \frac{1}{x-1}dx$$

$$= -\frac{3}{2}\log|x| - \frac{1}{6}\log|x+2| + \frac{5}{3}\log|x-1| + C$$

క్రింది సమాకలనులను కనుగొనండి

1. $\int \frac{dx}{6x^2-5x+1}$

Sol.

$6x^2 - 5x + 1 = (3x - 1)(2x - 1)$

Let $\frac{1}{6x^2-5x+1} = \frac{A}{3x-1} + \frac{B}{2x-1}$

$\Rightarrow 1 = A(2x-1) + B(3x-1)$

Put x = 1/3, $1 = A\left(\frac{2}{3}-1\right) \Rightarrow A = -3$

$x = \frac{1}{2} \Rightarrow 1 = B\left(\frac{3}{2}-1\right) \Rightarrow B = 2$

$\therefore \frac{1}{6x^2-5x+1} = \frac{-3}{3x-1} + \frac{2}{2x-1}$

$\int \frac{1}{6x^2-5x+1}dx = -3\int \frac{dx}{3x-1} + 2\int \frac{dx}{2x-1}$

$= -3\frac{\log|3x-1|}{3} + 2\frac{\log|2x-1|}{2}$

$= \log\left|\frac{2x-1}{3x-1}\right| + C$

2. $\int \frac{dx}{x(x+1)(x+2)}$

Sol. $\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

$\Rightarrow 1 = A(x+1)(x+2) + B(x)(x+2) + C(x)(x+1)$

$x = 0$ ప్రతిక్షేపించగా

$1 = A(1)(2) + B(0) + C(0) \Rightarrow A = 1/2$

$x = -1$ప్రతిక్షేపించగా

$1 = A(0) + B(-1)(-1+2) + C(0) \Rightarrow B = -1$

$x = -2$ప్రతిక్షేపించగా

$1 = A(0) + B(0) + C(-2)(-2+1) \Rightarrow C = 1/2$

$\frac{1}{x(x+1)(x+2)} = \frac{1/2}{x} - \frac{1}{x+1} + \frac{1/2}{x+2}$

$\int \frac{1}{x(x+1)(x+2)} dx =$

$\frac{1}{2}\int \frac{1}{x} dx - \int \frac{1}{x+1} dx + \frac{1}{2}\int \frac{1}{x+2} dx$

$= \frac{1}{2}\log|x| - \log|x+1| + \frac{1}{2}\log|x+2| + C$

3. $\int \frac{3x-2}{(x-1)(x+2)(x-3)} dx$

Sol. $\frac{3x-2}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$3x - 2 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$

Put $x = 1$

$3(1) - 2 = A(1+2)(1-3) + B(0) + C(0) \Rightarrow A = -\frac{1}{6}$

$x = 3$ప్రతిక్షేపించగా

$3(3) - 2 = A(0) + B(0) + C(3-1)(3+2) \Rightarrow C = \frac{7}{10}$

$x = -2$ ప్రతిక్షేపించగా

$3(-2) - 2 = A(0) + B(-2-1)(-2-3) + C(0) - 8$

$= 15B \Rightarrow B = \frac{-8}{15}$

$$\therefore \frac{3x-2}{(x-1)(x+2)(x-3)}$$

$$= \frac{-1}{6}\cdot\frac{1}{x-1} - \frac{8}{15}\cdot\frac{1}{x+2} + \frac{7}{10}\cdot\frac{1}{x-3}$$

$$\int \frac{3x-2}{(x-1)(x+2)(x-3)}dx = -\frac{1}{6}\log|x-1|$$

$$-\frac{8}{15}\log|x+2| + \frac{7}{10}\log|x-3| + C$$

4. $\int \frac{7x-4}{(x-1)^2(x+2)}dx$

Sol. $\frac{7x-4}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$\Rightarrow 7x - 4 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \ldots(1)$

Put $x = 1$ in (1)

$7 - 4 = A(0) + B(1+2) + C(0) \Rightarrow B = 1$

$x = -2$ in (1)

$7(-2) - 4 = A(0) + B(0) + C(-2-1)^2 \Rightarrow C = -2$

(1)లో x^2 గుణకాలను పోల్చగా

$0 = A + C \Rightarrow A = -C \Rightarrow A = 2$

$$\therefore \frac{7x-4}{(x-1)^2(x+2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{2}{x+2}$$

$$\therefore \int \frac{7x-4}{(x-1)^2(x+2)}dx = 2\int\frac{dx}{x-1} + \int\frac{dx}{(x-1)^2} - 2\int\frac{dx}{x+2}$$

$$= 2\log|x-1| - \left(\frac{1}{x-1}\right) - \log|x+2| + C$$

III. క్రింది సమాకలనులను కనుగొనండి

1. $\int \frac{1}{(x-a)(x-b)(x-c)} dx$

Sol. $\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

$= \frac{A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$

$\Rightarrow 1 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$ ...(1)

Put x = a, we get

$1 = A(a-b)(a-c) \Rightarrow A = \frac{1}{(a-b)(a-c)}$

Put x = b, we get $B = \frac{1}{(b-a)(c-b)}$

ఇదేవిధంగా $C = \frac{1}{(c-a)(c-b)}$

$\therefore \frac{1}{(x-a)(x-b)(x-c)} =$

$= \frac{\frac{1}{(a-b)(a-c)}}{x-a} + \frac{\frac{1}{(b-a)(b-c)}}{x-b} + \frac{\frac{1}{(c-a)(c-b)}}{x-c}$

$\therefore \int \frac{1}{(x-a)(x-b)(x-c)} dx$

$= \frac{1}{(a-b)(a-c)} \int \frac{1}{x-a} dx + \frac{1}{(b-a)(b-c)}$

$\int \frac{1}{x-b} dx + \frac{1}{(c-a)(c-b)} \int \frac{1}{x-c} dx$

2. $\int \frac{2x+3}{(x+3)(x^2+4)} dx$

Sol. $\frac{2x+3}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$

$2x + 3 = A(x^2 + 4) + (Bx + C)(x + 3)$

$x = -3 \Rightarrow -3 = A(9+4) = 13A \Rightarrow A = -\frac{3}{13}$

x^2 గుణకాలను పోల్చగా

$0 = A + B \Rightarrow B = -A = \frac{3}{13}$

స్థిర పదాలను పోల్చగా

$3 = 4A + 3C$

$3C = 3 - 4A = 3 + \frac{12}{13} = \frac{39+12}{13} = \frac{51}{13} \Rightarrow C = \frac{17}{13}$

$$\frac{2x+3}{(x+3)(x^2+4)} = \frac{-3}{13}\cdot\frac{1}{x+3} + \frac{3x+17}{13(x^2+4)}$$

$$\int \frac{2x+3}{(x+3)(x^2+4)}dx$$

$$= \frac{-3}{13}\int\frac{dx}{x+3} + \frac{3}{26}\int\frac{2xdx}{x^2+4} + \frac{17}{13}\int\frac{dx}{x^2+4}$$

$$= \frac{-3}{13}\log|x+3| + \frac{3}{26}\log|x^2+4| + \frac{17}{26}\tan^{-1}\left(\frac{x}{2}\right) + C$$

3. $\int \frac{2x^2+x+1}{(x+3)(x-2)^2}dx$

Sol. $\frac{2x^2+x+1}{(x+3)(x-2)^2} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$2x^2 + x + 1 = A(x-2)^2 + B(x+3)(x-2) + C(x+3)$

$x = 2 \Rightarrow 8 + 2 + 1 = C(2+3) = 5C \Rightarrow C = \frac{11}{5}$

$x = -3 \Rightarrow 18-3+1 = A(-5)^2 = 25A \Rightarrow A = \frac{16}{25}$

x^2 గుణకాలను పోల్చగా

$2 = A + B \Rightarrow B = 2 - A = 2 - \frac{16}{25} = \frac{34}{25}$

$$\int \frac{2x^2+x+1}{(x+3)(x-2)^2} = \frac{16}{25}\int\frac{dx}{x+3} + \frac{34}{25}\int\frac{dx}{x-2} + \frac{11}{5}\int\frac{1}{(x-2)^2}dx$$

$$= \frac{16}{25}\log|x+3| + \frac{34}{25}\log|x-2| - \frac{11}{5(x-2)} + C$$

4. $\int \frac{dx}{x^3+1}$

Sol. $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$

$$\frac{1}{x^3+1} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

$\Rightarrow 1 = A(x^2 - x + 1) + (Bx + C)(x + 1) \quad \ldots(1)$

$x = -1$ ప్రతిక్షేపించగా

$1 = A(1 + 1 + 1) + (-B + C)(0)$

$\Rightarrow 3A = 1 \Rightarrow A = 1/3$

$x = 0$ ప్రతిక్షేపించగా

$1 = A(1) + C(1) \Rightarrow C = 1 - A = 1 - \frac{1}{3} = \frac{2}{3}$

x^2 గుణకాలను పోల్చగా

$0 = A + B \Rightarrow B = -A = -1/3$

$$\frac{1}{x^3+1} = \frac{1}{3(x+1)} + \frac{-x+2}{3|x^2-x+1|}$$

$$\int \frac{1}{x^3+1}dx = \int\left(\frac{1}{3(x+1)} + \frac{-x+2}{3\left(x^2-x+1\right)}\right)dx \text{ --------(1)}$$

Let $-x+2 = A\frac{d}{dx}\left(x^2-x+1\right) + B = A(2x-1) + B$

సరి పద గుణకాలను పోల్చగా

2A = -1 and B-A =2

A = - 1/2, B =3/2

$$-x+2=\frac{-1}{2}(2x-1)+\frac{3}{2}$$

((1) నుండి

$$=\frac{1}{3}\log|x+1|-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx+\frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$=\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$=\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

5. $\int\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$

Sol.

$\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\int\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx=\int\frac{-t\,dt}{t^2+3t+2}$$

$$=-\int\frac{t}{t^2+3t+2}dt \qquad \ldots(1)$$

Let $\frac{t}{t^2+3t+2}=\frac{t}{(t+1)(t+2)}=\frac{A}{t+1}+\frac{B}{t+2}$

$\Rightarrow t = A(t + 2) + B(t + 1) \qquad \ldots(2)$

(2) లో $t = -1$ ప్రతిక్షేపించగా

$-1 = A(-1 + 2) \Rightarrow A = -1$

$t = -2$ ప్రతిక్షేపించగా

$-2 = B(-2 + 1) \Rightarrow B = 2$

$$\therefore \frac{t}{t^2+3t+2} = \frac{-1}{t+1} + \frac{2}{t+2} \qquad \ldots(3)$$

$\therefore$ (1) , (3) ల నుండి

$$\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx = -\left[\int \frac{-1}{t+1} dt + 2\int \frac{1}{t+2} dt\right]$$

$$= \int \frac{1}{t+1} dt - 2\int \frac{1}{t+2} dt$$

$$= \log|t+1| - 2\log|t+2| + C$$

$$= \log|1+\cos x| - 2\log|2+\cos x| + C$$

$$= \log|1+\cos x| - \log(2+\cos x)^2 + C$$

$$= \log\left|\frac{1+\cos x}{(2+\cos x)^2}\right| + C$$

EXERCISE – 6(F)

I. క్రింది సమాకలనులను కనుగొనండి

1. $\int e^x(1+x^2)dx$

Sol. $\int e^x(1+x^2)dx = \int e^x dx + \int x^2 e^x dx$

$= e^x + (x^2 \cdot e^x - 2\int x \cdot e^x dx)$

$= e^x + x^2 \cdot e^x - 2(x \cdot e^x - \int e^x dx)$

$= e^x + x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C$

$= e^x(x^2 - 2x + 3) + C$

2. $\int x^2 e^{-3x} dx$

Sol. $\int x^2 e^{-3x} dx = \frac{x^2 e^{-3x}}{-3} + \frac{1}{3}\int e^{-3x} \cdot 2x\, dx$

$= -\frac{x^2 e^{-3x}}{3} + \frac{2}{3}\left(\frac{xe^{-3x}}{-3} + \frac{1}{3}\int e^{-3x} dx\right)$

$= -\frac{x^2 e^{-3x}}{3} - \frac{2}{9} x \cdot e^{-3x} - \frac{2}{27} e^{-3x} + C$

$= \frac{-e^{-3x}}{27}\left(9x^2 + 6x + 2\right) + C$

3. $\int x^3 e^{ax} dx$

Sol. $\int x^3 e^{ax} dx = \frac{x^3 e^{ax}}{a} - \frac{1}{a}\int e^{ax}(3x^2\, dx)$

$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax} dx$

$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left(\frac{x^2 e^{ax}}{a} - \frac{1}{a}\int e^{ax} 2x\, dx\right)$

$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2} \cdot x^2 \cdot e^{ax} + \frac{6}{a^2}\int x \cdot e^{ax} dx$

$$= \frac{x^3e^{ax}}{a} - \frac{3x^2e^{ax}}{a^2} + \frac{6}{a^2}\left(\frac{xe^{ax}}{a} - \frac{1}{a}\int e^{ax}\cdot dx\right)$$

$$= \frac{x^3e^{ax}}{a} - \frac{3x^2e^{ax}}{a^2} + \frac{6x\cdot e^{ax}}{a^3} - \frac{6}{a^4}e^{ax} + C$$

$$= \frac{e^{ax}}{a^4}\left[a^3x^3 - 3a^2x^2 + 6ax - 6\right] + C$$

II.

1. $\int x^n \cdot e^{-x}dx = -x^ne^{-x} + n\int x^{n-1}\cdot e^{-x}dx$ అని చూపుము

Sol. $\int x^n \cdot e^{-x}dx = \frac{x^ne^{-x}}{(-1)} + \int e^{-x}\cdot nx^{n-1}dx$

$$= -x^ne^{-x} + n\int x^{n-1}\cdot e^{-x}dx$$

2. $I_n = \int \cos^n x\,dx$ అయితే $I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-2}$ అని చూపుము

.

Sol. $I_n = \int \cos^n x\,dx = = \int \cos^{n-1}x\cos x\,dx$

$$= \cos^{n-1}x\sin x - \int \sin x(n-1)\cos^{n-2}x(-\sin x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n(1+n-1) = \cos^{n-1}x\cdot\sin x + (n-1)I_n$$

$$I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$$

III.

1. $I_n = \int \cot^n x\, dx$ **, $n \geq 2$ కు ల ఘూకరణ సూత్రం రాబట్టండి** $\int \cot^4 x\, dx$ **ను కనుగొనుము.**

Sol. $I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cdot \cot^2 x\, dx$

$$= \int \cot^{n-2} x \cdot (\csc^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x \cdot \csc^2 x dx - I_{n-2}$$

$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$n = 4 \Rightarrow I_4 = -\frac{\cot^3 x}{3} - I_2$$

$n = 2 \Rightarrow I_2 = -\cot x - I_0$ where $I_0 = \int dx = x$

$$I_2 = -\cot x - x$$

$$I_4 = -\frac{\cot^3 x}{3} - (-\cot x - x) + C$$

$$= -\frac{\cot^3 x}{3} + \cot x + x + C$$

2. $I_n = \int \csc^n x\, dx$ **, $n \geq 2$ కు ల ఘూకరణ సూత్రం వ్రాయండి**

$\int \operatorname{cosec}^5 x\, dx$ **ను కనుగొనుము.**

Sol. $I_n = \int \csc^n x\, dx = \int \csc^{n-2} x \cdot \csc^2 x\, dx$

$$= \csc^{n-2} x(-\cot x) + \int \cot x (n-2) \csc^{n-3} x (\cot x) dx$$

$$= -\csc^{n-2} x \cot x + (n-2) \int \csc^{n-2} x (\csc^2 x - 1) dx$$

$$= -\csc^{n-2} x \cot x + (n-2) I_{n-2} - (n-2) I_n$$

$$I_n (1 + n - 2) = -\csc^{n-2} x \cdot \cot x + (n-2) I_{n-2}$$

$$I_n = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$n = 5 \Rightarrow I_5 = -\frac{\csc^3 x \cdot \cot x}{4} + \frac{3}{4} I_3$$

$$I_3 = -\frac{\csc x \cdot \cot x}{2} + \frac{1}{2} I_1$$

$$I_1 = \int \csc x\, dx = \log\left|\tan\frac{x}{2}\right|$$

$$I_3 = -\frac{\csc x \cdot \cot x}{2} + \frac{1}{2}\log\left|\tan\frac{x}{2}\right|$$

$$I_5 = -\frac{\csc^3 x \cdot \cot x}{4} - \frac{3}{8}\csc x \cot x + \frac{3}{8}\log\left|\tan\frac{x}{2}\right| + C$$

3. $I_{m,n} = \int \sin^m x \cos^n x\, dx$ **అయితే** $I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2}, n$ **అని చూపుము m ≥ 2.**

Sol. $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$= \int \sin^{m-1} x \cdot (\cos x)^n \sin x\, dx$$

$$= \int \sin^{m-1}(x)(\cos x)^n (-\sin x) dx$$

$$= -\left[\sin^{m-1}(x)\int (\cos x)^n (-\sin x) dx\right.$$

$$\left. -\int\left\{\frac{d}{dx}\sin^{m-1}(x) \cdot \int \cos^n(x)(-\sin x)dx\right\}dx\right]$$

$$= -\left[\sin^{m-1}(x)\frac{\cos^{n+1}(x)}{n+1} - \int\left\{(m-1)\sin^{m-2}(x)\cos x \frac{\cos^{n+1} x}{n+1}\right\}dx\right]$$

$$= -\sin^{m-1}(x)\frac{\cos^{n+1}(x)}{n+1} + \frac{m-1}{n+1}\int\{\sin^{m-2}(x)\cos^n x \cos^2 x\}dx$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)\int\{\sin^{m-2}(x)\cos^n x - \sin^m(x)\cos^n(x)\}dx$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)\left[\int \sin^{m-2}(x)\cos^n x dx - \int \sin^m(x)\cos^n(x)dx\right]$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$-\left(\frac{m-1}{n+1}\right)I_{m,n}$$

$$\therefore I_{m,n} + \left(\frac{m-1}{n+1}\right)I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\Rightarrow\left(1+\frac{m-1}{n+1}\right)I_{m,n}=-\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1}+\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\therefore\left(\frac{m+n}{n+1}\right)I_{m,n}=-\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1}+\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

$$\therefore I_{m,n}=\frac{1}{m+n}(\sin^{m-1}(x)\cos^{n+1}(x))+\left(\frac{m-1}{n+1}\right)I_{m-2,n}$$

4. $\int\sin^5 x\cos^4 x dx$ ను కనుగొనుము.

Sol.

$$I_{m,n}\frac{-\sin^{m-1}x\cdot\cos^{n+1}x}{m+n}+\frac{m-1}{m+n}I_{m-2,n}$$

$$I_{5,4}=-\frac{\sin^4 x\cos^5 x}{9}+\frac{4}{9}I_{3,4}$$

$$I_{3,4}=-\frac{\sin^2 x\cos^5 x}{7}+\frac{2}{7}I_{1,4}$$

$$I_{1,4}=\int\sin x\cos^4 x\,dx=-\frac{\cos^5 x}{5}$$

$$I_{3,4}=-\frac{\sin^2 x\cos^5 x}{7}-\frac{2}{35}\cos^5 x$$

$$I_{5,4}=-\frac{\sin^4 x\cos^5 x}{9}+\frac{4}{9}\left(-\frac{\sin^2 x\cos^5 x}{7}-\frac{2}{35}\cos^5 x\right)+C$$

$$=-\frac{\sin^4 x\cos^5 x}{9}-\frac{4}{63}\sin^2 x\cdot\cos^5 x-\frac{8}{315}\cos^5 x+C$$

5. $\int x^n e^{nx}dx$ కు ల ఘూకరణ సూత్రం వ్రాయండి $\int x^3 e^{5x}dx$ ను కనుగొనుము.

Sol:Let $I_n=\int x^n e^{ax}dx$

$$=x^n\frac{e^{ax}}{a}-\int n\cdot x^{n-1}\frac{e^{ax}}{a}dx$$

$$=\frac{x^n e^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}dx$$

$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$ ల ఘూకరణ సూత్రం

$I_3 = \int x^3 e^{5x} dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} I_2$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}\left[\frac{x^2 e^{5x}}{5} - \frac{2}{5} I_1\right]$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25}\int x^3 e^{5x} dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25}\left(\frac{xe^{5x}}{5} - \int \frac{e^{5x}}{5} dx\right)$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x^2 e^{5x} - \frac{6}{625} e^{5x} + c = \frac{x^3 e^{5x}}{5} - \frac{3}{5^2} x^2 e^{5x} + \frac{6}{5^3} x^2 e^{5x} - \frac{6}{5^4} e^{5x} + c.$

6. $\int \sin^n x\, dx$ **$n \geq 2$, కు ల ఘూకరణ సూత్రం రాబట్టండి**

$\int \sin^4 x\, dx$ ను కనుగొనుము.

Sol: Let $I_n = \int \sin^n x\, dx$

$= \int \sin^{n-1} x \sin x\, dx$

$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x) dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x\, dx$

$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n = \frac{1}{n}[(n-1) I_{n-2} - \sin^{n-1} x \cos x]$

$= \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$.

$\int \sin^4 x\, dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} I_2$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left[\frac{-\sin x \cos x}{2} + \frac{1}{2} I_0\right]$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x\, dx + \frac{3}{8}\int dx + c$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x\, dx + \frac{3}{8} x + c.$$

7. $\int \sin^m x \cos^n x\, dx$ కు ల ఘూకరణ సూత్రం రాబట్టండి.

Sol: $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$= \int \sin^m x \cos^{n-1} x \cos x\, dx$$

$u = \cos^{n-1}x$ and $v = \sin^m x$.

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x(-\sin x)\frac{\sin^{m+1} x}{m+1} dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \left(\frac{n-1}{m+1}\right)\int \sin^{m+2} x \cos^{n-2} x dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \left(\frac{n-1}{m+1}\right)\int \sin^{m} x \cos^{n-2} x(1-\cos^2 x) dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1}\int \sin^{m} x \cos^{n-2} x - \frac{n-1}{m+1}\int \sin^m x \cos^n x\, dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\therefore I_{m,n}\left(1 + \frac{n-1}{m+1}\right) = \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2}$$

$$\Rightarrow I_{m,n}\left(\frac{m+n}{n+1}\right) = \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2}$$

$$\Rightarrow I_{m,n} = \frac{m+n}{n+1}\left[\frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2}\right]$$

$$= \frac{1}{m+1}[\cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}]$$

is the reduction formula for

$$I_{m,n} = \int \sin^m x \cos^n x\, dx.$$

8. $\int \tan^n x\,dx$ **, $n \geq 2$** కు ల ఘూకరణ సూత్రం రాబట్టండి. $\int \tan^6 x\,dx$ **.** ను కనుగొనుము

Sol: Let $I_n = \int \tan^n x\,dx$

$= \int \tan^{n-2} x \tan^2 x\,dx$

$= \int \tan^{n-2} x(\sec^2 x - 1)dx$

$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$.

$I_6 = \int \tan^6 x\,dx$

$= \dfrac{\tan^5 x}{5} - I_4$

$= \dfrac{\tan^5 x}{5} - \left[\dfrac{\tan x^3}{3} - I_2\right]$

$= \dfrac{\tan^5 x}{5} - \dfrac{\tan x^3}{3} + \dfrac{\tan x}{1} - I_0$

$= \dfrac{\tan^5 x}{5} - \dfrac{\tan x^3}{3} + \tan x - x + c.$

9. $\int \sec^n x\,dx$ **, $n \geq 2$** కు ల ఘూకరణ సూత్రం రాబట్టండి $\int \sec^5 x\,dx$ ను కనుగొనుము.

Sol: $I_n = \int \sec^n x\,dx$

$= \int \sec^{n-2} x \sec^2 x\,dx$

$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \cdot \tan x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$

$= \sec^{n-2} x \tan x - (n-2)I_n - (n-2)I_{n-2}$

$= \sec^{n-2} x \tan x - (n-2)[I_n - I_{n-2}]$

$\Rightarrow I_n(1 + n - 2) = \sec^{n-2} \tan x + (n-2)I_{n-2}$

$\Rightarrow I_n = \dfrac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2)I_{n-2}\right]$

$= \dfrac{1}{n-1}\sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$

$$\int \sec^5 x\,dx = I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4} I_3$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left[\frac{1}{2}\sec x \tan x + \frac{1}{2} I_1\right]$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\int \sec x\,dx$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\log|\sec x + \tan x| + c$$

10. $I_n = \int (\log x)^n dx$ అయితే $I_n = x(\log x)^n - nI_{n-1}$ అని చూపుము $\int (\log x)^4 dx$ ను కనుగొనుము.

Sol. $I_n = \int (\log x)^n dx$

$$= (\log x)^n x - \int x \cdot n \cdot (\log x)^{n-1} \frac{1}{x} dx$$

$$= x(\log x)^n - n\int (\log x)^{n-1} dx$$

$$= x(\log x)^n - n \cdot I_{n-1}$$

$I_4 = x(\log x)^4 - 4 \cdot I_3$

$I_3 = (x \log x)^3 - 3 \cdot I_2$

$I_2 = (x \log x)^2 - 2 \cdot I_1$

$I_1 = x \log x - I_0$, $I_0 = \int dx = x$

$I_1 = x \log x - x$

$I_2 = (x \log x)^2 - 2x \log x + 2x$

$I_3 = (x)(\log x)^3 - 3x(\log x)^2 - 2x \log x + 2x$

$= x(\log x)^3 - 3x(\log x)^2 + 6x(\log x) - 6x$

$I_4 = x(\log x)^4 - 4(x)(\log x)^3 - 3x(\log x)^2 + 6x(\log x) - 6x] + C$

$= x[(\log x)^4 - 4(\log x)^3 + 12(\log x)^2 - 24(\log x) + 24] + C$

SOLVED PROBLEMS

క్రింద ఉన్న సమాకలనులను రాబట్టండి

1. $\int\left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx$ **hint:** $x+\frac{1}{x}=t$ అని ప్రతిక్షేపించగా **Ans.** $e^{\left(x+\frac{1}{x}\right)}+C$

11. $\int\frac{1}{\sqrt{\sin^{-1}x}\sqrt{1-x^2}}dx$ **on I = (0, 1).**

Ans. $2\sqrt{\sin^{-1}x}+C$

2. $\int\frac{1}{a\sin x+b\cos x}dx$ ను రాబట్టండి

Sol. a = cos θ, b = r sin θ అనుకోండి

Then $r=\sqrt{a^2+b^2}, \cos\theta=\frac{a}{r}$ and $\sin\theta=\frac{b}{r}$

$a\sin x+b\cos x=r\cdot\cos\theta\sin x+r\sin\theta\cos x$

$=r[\cos\theta\sin x+\sin\theta\cos x]=r\sin(x+\theta)$

$\int\frac{1}{a\sin x+b\cos x}dx=\frac{1}{r}\int\frac{1}{\sin(x+\theta)}dx$

$=\frac{1}{r}(\csc(x+\theta))dx=\frac{1}{r}\log\left|\tan\frac{1}{2}(x+\theta)\right|+C$

$=\frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\frac{1}{2}(x+\theta)\right|+C$

For all x ∈ I where I is an internal disjoint with $\{n\pi-\theta : n\in Z\}$.

3. $\int\frac{1}{a^2-x^2}dx$ for $x\in I=(-a,a)$.

Ans. $\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

4. $\int e^{ax}\cos(bx+c)dx$ ను కనుగొనుము

Sol. A = $\int e^{ax}\cos(bx+c)dx$ అనుకోండి

$$A = e^{ax}\left[\frac{\sin(bx+c)}{b}\right]-\int ae^{ax}\left[\frac{\sin(bx+c)}{b}\right]dx$$

$$=\frac{1}{b}e^{ax}\sin(bx+c)-\frac{a}{b}\int e^{ax}\sin(bx+c)dx$$

$$=\frac{1}{b}e^{ax}\sin(bx+c)-\frac{a}{b}\left[e^{ax}\left\{\frac{-\cos(bx+c)}{b}\right\}-\int ae^{ax}\left\{-\frac{\cos(bx+c)}{b}\right\}dx\right]+C_1$$

$$=\frac{1}{b}e^{ax}\sin(bx+c)+\frac{a}{b^2}e^{ax}\cos(bx+c)-\frac{a^2}{b^2}A+C_2$$

$$\left(1+\frac{a^2}{b^2}\right)A=\frac{a}{b^2}e^{ax}\cos(bx+c)+\frac{1}{b}e^{ax}\sin(bx+c)+C_2$$

$$(a^2+b^2)A = ae^{ax}\cos(bx+c)+be^{ax}\sin(bx+c)+C_3$$

$$A=\frac{e^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)]+K$$

ఇక్కడ $k=\frac{C_3}{a^2+b^2}$ స్థిర రాశి

By taking c = 0, we get

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]+K$$

5. $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$ **on (–1, 1).**

Sol. Put x = cos θ, θ ∈ (0, π)dx = –sin θ dθ

$$\frac{1-x}{1+x}=\frac{1-\cos\theta}{1+\cos\theta}=\frac{2\sin^2\theta/2}{2\cos^2\theta/2}=\tan^2\frac{\theta}{2}$$

$$\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx=\int \tan^{-1}\sqrt{\tan^2\frac{\theta}{2}}(-\sin\theta)d\theta$$

$$= -\int \tan^{-1}\left(\tan\frac{\theta}{2}\right)(\sin\theta)d\theta$$

$$= -\frac{1}{2}\int \theta \cdot \sin\theta\, d\theta$$

$$= -\frac{1}{2}\left[\theta(-\cos\theta) - \int(-\cos\theta)d\theta\right] + C$$

$$= \frac{1}{2}(\theta\cos\theta - \sin\theta) + C$$

$$= \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$$

6. $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$ **, $I \subset R \setminus \{2n\pi : n \in Z\}$.**

Sol. $\frac{1-\sin x}{1-\cos x} = \frac{1-\sin x}{2\sin^2 x/2}$

$$= \frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} = \frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\csc^2\frac{x}{2} - \cot\frac{x}{2}$$

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int e^x\left(\frac{1}{2}\csc^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$$

$$= \int e^x[f(x) + f'(x)]dx \text{ where } f(x) = -\cot\frac{x}{2}$$

$$= e^x f(x) + C = -e^x\cot\frac{x}{2} + C$$

7. $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$ **, $I \subset R \setminus (-1, 1)$.**

Sol. Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}(\tan 2\theta) = 2\theta + n\pi$$

$n = 0$ if $|x| < 1$

$= -1$ if $x > 1$

$= 1$ if $x < -1$

We have $d\theta = \frac{1}{1+x^2}dx$ and

$1+x^2 = 1 + \tan^2\theta = \sec^2\theta$

$$\therefore \int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

$$= \int\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)(1+x^2)\frac{1}{1+x^2}dx$$

$$= \int(2\theta + n\pi)\int \sec^2\theta d\theta$$

$$= 2\int \theta\sec^2\theta d\theta + n\pi\int \sec^2\theta d\theta + c$$

$$= 2\left(\theta\tan\theta - \int \tan\theta d\theta\right) + n\pi\tan\theta + c$$

$$= 2(\theta\tan\theta + \log|\cos\theta| + n\pi\tan\theta + c$$

$$= (2\theta + n\pi)\tan\theta + 2\log\cos\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\cos^2\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\sec^2\theta + c$$

$$= x\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \log(1+x^2) + c$$

8. $\int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx$ ను కనుగొనుము

Sol. $t = \sin^{-1}x$, అప్పుడు

$x = \sin t, dt = \frac{1}{\sqrt{1-x^2}}dx$, for $x \in (-1, 1)$

$$\int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx = \int e^{mt}\sin^2 t dt$$

$$= \int e^{mt}\left(\frac{1-\cos 2t}{2}\right)dt$$

$$= \frac{1}{2}\int e^{mt}dt - \frac{1}{2}\int e^{mt}\cdot\cos 2t\,dt + c \qquad \ldots(1)$$

Case (i) : m = 0

From (1) $\int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx$

$$= \frac{1}{2}\int dt - \frac{1}{2}\int \cos 2t\,dt + C$$

$$= \frac{t}{2} - \frac{\sin 2t}{4} + C$$

$$= \frac{\sin^{-1}x}{2} - \frac{1}{4}\sin(2\sin^{-1}x) + C$$

Case (ii) : $m \neq 0$

From (1) $\int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx$

$$= \frac{1}{2}\frac{e^{mt}}{m} - \frac{1}{2}\frac{e^{mt}}{m^2+4}(m\cos 2t + 2\sin 2t) + C_1$$

$$= \frac{e^{mt}}{2}\left(\frac{1}{m} - \frac{1}{m^2+4}(m\cos 2t + 2\sin 2t)\right) + C_1$$

$$= \frac{e^{m\sin^{-1}x}}{2}\left(\frac{1}{m} - \frac{1}{m^2+4}(m\cos(2\sin^{-1}x) + -2\sin(2\sin^{-1}x)\right) + C_1$$

9. $\int \frac{dx}{3\cos x + 4\sin x + 6}$ ను కనుగొనుము

Ans. $\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3\tan(x/2)+4}{\sqrt{11}}\right) + C$

10. $\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1}dx$ ను కనుగొనుము

Ans. $-\log|\cos x + \sin x + 1| + 2x + 5\log\left|1 + \tan\frac{x}{2}\right| + C$

11. $\int \frac{e^x}{e^{x/2}+1}dx$ **on R.**

Sol. $t = 1+e^{x/2} \Rightarrow dt = \frac{1}{2}e^{x/2}dx$

$$\int \frac{e^x}{e^{x/2}+1}dx = 2\int \frac{e^{x/2}\left(\frac{1}{2}e^{x/2}dx\right)}{e^{x/2}+1}$$

$$= 2\int \frac{(t-1)dt}{t} = 2\int\left(1-\frac{1}{t}\right)dt = 2(t-\log t)+C$$

$$= 2(1+e^{x/2}-\log(1+e^{x/2}))+C$$

12. $\int \frac{(x^3-x)^{1/3}}{x^4}dx$ ను కనుగొనుము

Ans. $\frac{3}{8}\left(1-\frac{1}{x^2}\right)^{4/3}+C$

13. $\int\left(x+\frac{1}{x}\right)^3 dx, x>0$. ను కనుగొనుము

Sol: $\int\left(x+\frac{1}{x}\right)^3 dx = \int\left[x^3+\frac{1}{x^3}+\left(x+\frac{1}{x}\right)\right]dx$

$$= \int x^3dx + 3\int x\,dx + 3\int\frac{dx}{x} + \int\frac{dx}{x^3}+c$$

$$= \frac{x^4}{4}+\frac{3x^2}{2}+3\log|x|-\frac{1}{2x^2}+c.$$

14. $\int \sqrt{1+\sin 2x}\,dx$ ను కనుగొనుము

Sol: $\int \sqrt{1+2\sin x\cos x}\,dx$

$= \int \sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}\,dx$

$= \int \sqrt{(\sin x+\cos x)^2}dx$

$= \int (\sin x+\cos x)dx$

If $2n\pi-\frac{\pi}{4}\le x\le 2n\pi+\frac{3\pi}{4}$ for some $n\in Z$

$= \int -(\sin x+\cos x)dx$ other wise

$\therefore \int \sqrt{1+2\sin x\cos x}\,dx = -\cos x+\sin x+c$

If $2n\pi-\frac{\pi}{4}\le x\le 2n\pi+\frac{3\pi}{4}$

$= \cos x - \sin x + c$

If $2n\pi+\frac{3\pi}{4}\le x\le 2n\pi+\frac{7\pi}{4}$.

15. $\int\left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx$. ను కనుగొనుము

Sol: Let $x+\frac{1}{x}=t$ then $\left(1-\frac{1}{x^2}\right)dx=dt$

$\therefore \int\left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx=\int e^t dt$

$= e^t+c=e^{\left(x+\frac{1}{x}\right)}+c.$

16. $\int\frac{1}{\sqrt{\sin^{-1}x}\sqrt{1-x^2}}dx$ ను కనుగొనుము

Sol: Let $\sin^{-1} x = t$ then $\frac{1}{\sqrt{1-x^2}}dx=dt$

$\therefore \int\frac{1}{\sqrt{\sin^{-1}x}\sqrt{1-x^2}}dx=\int\frac{1}{\sqrt{t}}dt$

$= \int t^{-1/2}dt$

$= 2\sqrt{t} + c = 2\sqrt{\sin^{-1} x} + c$.

17. $\int \frac{x}{\sqrt{1-x}} dx$, ను కనుగొనుము

Sol: $1 - x = t^2$ over (0, 1)

అప్పుడు $-dx = 2t\,dt \quad x = 1 - t^2$

$$\therefore \int \frac{x}{\sqrt{1-x}} dx = -\int \frac{(1-t^2)2tdt}{t}$$

$$= -2\int (1-t)^2 dt = -2\left[t - \frac{t^3}{3}\right]$$

$$= -2\left[\sqrt{1-x} - \frac{(1-x)^{3/2}}{3}\right]$$

$$= \frac{2}{3}(1-x)^{3/2} - 2\sqrt{1-x} + c$$

18. $\int \frac{dx}{(x+5)\sqrt{x+4}}$ ను కనుగొనుము

Sol: $x + 4 = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \int \frac{dx}{(x+5)\sqrt{x+4}} = \int \frac{2t\,dt}{(t^2+1)t} = 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1} t + c$$

$$= 2\tan^{-1}(\sqrt{x+4}) + c$$

19. $\int \frac{1}{a^2 - x^2} dx$ **for $x \in I = (-a, a)$.**

Sol: $\frac{1}{a^2 - x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}$

$\Rightarrow 1 = A(a + x) + B(a - x)$

Put $x = a$, $1 = 2a\,A \Rightarrow A = \frac{1}{2a}$

And if $x = -a$, then $2a\,B = 1 \Rightarrow B = \frac{1}{2a}$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\int \frac{dx}{a-x} + \frac{1}{2a}\int \frac{dx}{a+x}$$

$$= -\frac{1}{2a}\log|a-x| + \frac{1}{2a}\log|x+a| + c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a+x}\right| + c.$$

20. $\int \frac{dx}{\sqrt{x^2+2x+10}}$. ను కనుగొనుము

Sol: $\sqrt{x^2+2x+10} = \sqrt{x^2+2x+1+9}$

$$= \sqrt{(x+1)^2 + 3^2}$$

$$\therefore \int \frac{dx}{\sqrt{x^2+2x+10}} = \int \frac{dx}{\sqrt{(x+1)^2+3^2}}$$

Take x + 1 = t $\Rightarrow$ dx = dt

$$= \int \frac{dt}{\sqrt{t^2+3^2}} = \sinh^{-1}\left(\frac{t}{3}\right) + c$$

$$= \sinh^{-1}\left(\frac{x+1}{3}\right) + c$$

$$\left(\text{use} \int \frac{dx}{\sqrt{a^2 - x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)\right).$$

21. $\int \frac{dx}{5+4\cos x}$. ను కనుగొనుము

Sol: Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2\,dt}{1+t^2}$

And $\cos x = \frac{1-t^2}{1+t^2}$

$$\int \frac{dx}{5+4\cos x} = \int \frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{2dt}{9+t^2} = 2\int \frac{dt}{3^2+t^2}$$

$$= 2\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{t}{3}\right)$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + c$$

22. $\int \frac{dx}{3\cos x + 4\sin x + 6}$ ను కనుగొనుము

Sol: $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2\,dt}{1+t^2}$

$\sin x = \frac{2\,t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore \int \frac{dx}{3\cos x + 4\sin x + 6}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) + 6}$$

$$= \int \frac{2\,dt}{3 - 3t^2 + 8t + 6 + 6t^2}$$

$$= \int \frac{2dt}{3t^2 + 8t + 9}$$

$$= \frac{2}{3}\int \frac{dt}{t^2 + \frac{8}{3}t + 3}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2 + 3 - \frac{16}{9}}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2 + \frac{11}{9}}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{2}{3}\frac{3}{\sqrt{11}}\tan^{-1}\left(\frac{t+\frac{4}{3}}{\frac{\sqrt{11}}{3}}\right)$$

$$= \frac{2}{\sqrt{11}}\text{Tan}^{-1}\left(\frac{3t+4}{\sqrt{11}}\right)+c.$$