అవకలనము

* $\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$

* $\frac{d}{dx}(e^x) = e^x,\ \frac{d}{dx}(a^x) = a^x \log a$

* $\frac{d}{dx}(x^x) = x^x\left(1+\log x\right) = x^x \log ex$

* $\frac{d}{dx}(\log_e x) = \frac{1}{x},\ \frac{d}{dx}\{\log_a x\} = \frac{1}{x\log a}$

* $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x},\ \frac{d}{dx}\left\{\log_a |x|\right\} = \frac{1}{x\log a}$

* $\frac{d}{dx}(x) = 1,\ \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}},\ \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$

* $\frac{d}{dx}(\sin x) = \cos x,\ \frac{d}{dx}\left(\sin ax\right) = \cos ax.\, a$

* $\frac{d}{dx}(\cos x) = -\sin x,\ \frac{d}{dx}(\tan x) = \sec^2 x.$

* $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x,$

* $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$

* $\frac{d}{dx}(\sec x) = \sec x.\tan x$

* $\frac{d}{dx}\{\sinh x\} = \cosh x;\ \frac{d}{dx}\{\cosh x\} = \sinh x;$

  $\frac{d}{dx}\{\tanh x\} = \operatorname{sech}^2 x$

  $\frac{d}{dx}\{\coth x\} = -\operatorname{cosech}^2 x;$

* $\frac{d}{dx}\{\operatorname{sech} x\} = -\operatorname{sech} x \tanh x;$

  $\frac{d}{dx}\{\operatorname{cosech} x\} = -\operatorname{cosech} x \cdot \coth x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}\ \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2},\ \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}},$$

$$\frac{d}{dx}\left(\cos ec^{-1}x\right)=-\frac{1}{|x|\sqrt{x^2-1}}$$

* $\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{1+x^2}},\frac{d}{dx}\left(\cosh^{-1}x\right)=\frac{1}{\sqrt{x^2-1}}$

* $\frac{d}{dx}\left(\tan h^{-1}x\right)=\frac{1}{1-x^2}.$

$$\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{1}{1-x^2},\ \frac{d}{dx}\left(\sec h^{-1}x\right)=\frac{1}{x\sqrt{1-x^2}},$$

$$\frac{d}{dx}\left(\cos ech^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2+1}}$$

అతిస్వల్ప సమాధాన ప్రశ్నలు

1. క్రింది ప్రమే యములకు అవకలనము కనుగొనుము

i) $y=\left(4+x^2\right).e^{2x}$

Sol : $y=\left(4+x^2\right).e^{2x}$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx}=\left(4+x^2\right)\frac{d}{dx}\left(e^{2x}\right)+e^{2x}\frac{d}{dx}\left(4+x^2\right)$$

$$=\left(4+x^2\right).2e^{2x}+e^{2x}(0+2x)$$

$$=2e^{2x}\left[4+x^2+x\right]=2e^{2x}\left(x^2+x+4\right)$$

2. $y=\frac{ax+b}{cx+d}[|c|+|d|\neq 0]$

sol : $y=\frac{ax+b}{cx+d}[|c|+|d|\neq 0]$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx}=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}=\frac{(cx+d)a-(ax+b).c}{(cx+d)^2}$$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}=\frac{ad-bc}{(cx+d)^2}$$

3. $y=e^{2x}.\log(3x+4)\left(x>\frac{-4}{3}\right)$

sol : $y=e^{2x}.\log(3x+4)\left(x>\frac{-4}{3}\right)$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx}=e^{2x}\frac{d}{dx}\left[\log(3x+4)+\log(3x+4)\left(e^{2x}\right)\right]=e^{2x}.\frac{1}{3x+4}.3+\log(3x+4).e^{2x}.2$$

$$=e^{2x}\left(\frac{3}{3x+4}+2\log(3x+4)\right)$$

4. $y=e^{x}+\sin x.\cos x$

sol : $y=e^{x}+\sin x.\cos x$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}(\sin x.\cos x)$$

$$=e^{x}+\sin\times\frac{d}{dx}(\cos x)+\cos\times\frac{d}{dx}(\sin x)$$

$$=e^{x}-\sin^2 x+\cos^2 x=e^{x}+\cos 2x$$

5. $y=5\sin x+e^{x}.\log x$

sol : $y=5\sin x+e^{x}.\log x$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx}=5\cos x+e^{x}.\frac{d}{dx}(\log x)+\log x\frac{d}{dx}\left(e^{x}\right)$$

$$=5\cos x+e^{x}.\frac{1}{x}+(\log x)\left(e^{x}\right)$$

6. $y = 5^x + \log x + x^3 e^x$

sol : $y = 5^x + \log x + x^3 e^x$

$$\frac{dy}{dx} = 5^x + \log 5 + \frac{1}{x} + x^3.e^x + e^x.3x^2$$

$$= 5^x.\log 5 + \frac{1}{x} + x^3 e^x + 3x^2 e^x$$

7. $y = \frac{1}{ax^2 + bx + c}(|a| + |b| + |c| \neq 0)$

sol : $y = \frac{1}{ax^2 + bx + c}(|a| + |b| + |c| \neq 0)$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{(-1)}{\left(ax^2 + bx + c\right)} \frac{d}{dx}\left(ax^2 + bx + c\right)$$

$$= \frac{-(2ax + b)}{\left(ax^2 + bx + c\right)^2}$$

8. $y = \log_7 (\log x)(x > 0)$

sol : $y = \log_7 (\log x)(x > 0)$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{1}{\log_7}.\frac{1}{\log x}.\frac{1}{x}$$

$$= \frac{1}{x(\log x)\left(\log_e^7\right)} = \frac{\log_7^e}{x \log_e^x}$$

9. $f(x) = 1 + x + x^2 + ... + x^{100}$, అయితే $f'(1)$. కనుగొనుము

Sol : $f(x) = 1 + x + x^2 + ... + x^{100}$,

$$\Rightarrow f'(x) = 1 + 2x + 3x^2 .... + 100x^{99}$$

$\Rightarrow f'(1) = 1 + 2 + 3.......... + 100$

$= \frac{100 \times 101}{2} = 5050\left(\Sigma x = \frac{x(x+1)}{2}\right)$

10.i) $y = \cot^n x$ అయితే $\frac{dy}{dx}$ కనుగొనుము .

Sol :

$y = \cot^n x$

x దృష్ట్యా అవకలనము చేయగ

$\frac{dy}{dx} = \frac{d\left(\cot^n x\right)}{dx} = n.\cot^{n-1} x.\frac{d}{dx}(\cot x)$

$= n.\cot^{n-1} x.\left(-\cos ec^2 x\right)$

$= -n.\cot^{n-1} x.\cos ec^2 x$

Ii) $y = \tan\left(e^x\right)$ అయితే $\frac{dy}{dx}$ కనుగొనుము .

sol : $\frac{dy}{dx} = \sec^2\left(e^x\right).\left(e^x\right)^1 = e^x.\sec^2\left(e^x\right)$

iii) $y = \frac{1-\cos 2x}{1+\cos 2x} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\tan^2 x\right) = 2\tan x.\sec^2 x$

iv) $y = \sin^m x.\cos^n x$, అయితే $\frac{dy}{dx}$ కనుగొనుము .

sol :

$\frac{dy}{dx} = \frac{d}{dx}\sin^m x\cos^n x = \sin^m x.\frac{d}{dx}\left(\cos^n x\right) + \left(\cos^n x\right)\frac{d}{dx}\left(\sin^m x\right)$

$= \sin^m xn\cos^{n-1} x(-\sin x) + \cos^n x.m\sin^{m-1} x.\cos x$

$= m.\cos^{n+1} x.\sin^{m-1} x - n.\sin^{m+1} x.\cos^{n-1} x.$

v) $y = x\tan^{-1}x$ అయితే $\frac{dy}{dx}$ కనుగొనుము .

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(x\tan^{-1}x\right)$

$= x.\frac{d}{dx}\left(\tan^{-1}x\right) + \left(\tan^{-1}x\right)\frac{d}{dx}(x)$

$= \frac{x}{1+x^2} + \tan^{-1}x. = \frac{x}{1+x^2} + \tan^{-1}x.$

vii) $y = \sin^{-1}(\cos x)$, find $\frac{dy}{dx}$.

Sol : $y = \sin^{-1}(\cos x) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)\right]$

$= \frac{\pi}{2} - x$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - x\right) = 0 - 1 = -1$

viii) $y = \log(\tan 5x)$ అయితే $\frac{dy}{dx}$ కనుగొనుము .

sol :

$\frac{dy}{dx} = \frac{d}{dx}(\log\tan 5x) = \frac{1}{\tan 5x}\frac{d}{dx}(\tan 5x)$

$= \frac{5\sec^2 5x}{\tan 5x} = 5.\frac{1}{\cos^2 5x.\frac{\sin 5x}{\cos 5x}}$

$= \frac{10}{2\sin 5x.\cos 5x}$

$= \frac{10}{\sin 10x} = 10.\cos ec10x$

ix) $y = \sin h^{-1}\left(\frac{3x}{4}\right)$, అయితే $\frac{dy}{dx}$ కనుగొనుము ..

sol :

$\frac{dy}{dx} = \frac{d}{dx}\left(\sinh^{-1}\frac{3x}{4}\right) = \frac{1}{\sqrt{1+\left(\frac{3x}{4}\right)^2}}\frac{d}{dx}\left(\frac{3x}{4}\right)$

$$= \frac{3}{4} \frac{1}{\sqrt{1+\frac{9x^2}{16}}} = \frac{3}{4.\sqrt{\frac{16+9x^2}{16}}}$$

$$= \frac{3}{\sqrt{9x^2+16}}$$

x) $y = \tan^{-1}(\log x)$ అయితే $\frac{dy}{dx}$ కనుగొనుము .

sol : $\frac{dy}{dx} = \frac{1}{1+(\log x)^2} . \frac{d}{dx}(\log x)$

$$= \frac{1}{x\left(1+(\log x)^2\right)}$$

11. $y = ae^{nx} + be^{-nx}$ అయితే $y'' = n^2 y$ అని చూపుము

Sol : $y = ae^{nx} + be^{-nx}$

$y_1 = na\,e^{nx} - nb\,e^{-nx}$

$y_2 = n^2.ae^{nx} + n^2\ be^{-nx}$

$y'' = n^2\left(ae^{nx} + b.e^{-nx}\right) = n^2 y$

12) $y = \log\left(\sin^{-1}\left(e^x\right)\right)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\log \sin^{-1} e^x\right)$

$$= \frac{1}{\sin^{-1}\left(e^x\right)} \frac{d}{dx}\left(\sin^{-1}\left(e^x\right)\right)$$

$$= \frac{1}{\sin^{-1}\left(e^x\right)} . \frac{1}{\sqrt{1-\left(e^x\right)^2}}\left(e^x\right)$$

$$= \frac{e^x}{\sin^{-1}\left(e^x\right)\sqrt{1-\left(e^2\right)^x}}$$

13) $y = (\sin x)^2 \left(\sin^{-1} x\right)^2$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $\frac{dy}{dx} = \frac{d}{dx}(\sin x)^2 \left(\sin^{-1} x\right)^2$

$$\frac{dy}{dx} = (\sin x)^2 \frac{d}{dx}\left(\sin^{-1} x\right)^2 + \left(\sin^{-1} x\right)^2 \frac{d}{dx}(\sin x)^2$$

$$= \sin^2 x.\frac{2\sin^{-1} x}{\sqrt{1-x^2}} + \left(\sin^{-1} x\right)^2 \ (2\sin x.\cos x)$$

$$= \left(\sin^{-1} x\right)^2 (\sin 2x) + 2\sin^2 \times \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

15) $y = \cos\left(\log x + e^x\right)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $\frac{dy}{dx} = \frac{d}{dx}\cos\left(\log x + e^x\right)$

$$= -\sin\left(\log x + e^x\right)\frac{d}{dx}\left(\log x + e^x\right)$$

$$= -\sin\left(\log x + e^x\right)\left(\frac{1}{x} + e^x\right)$$

16) $y = \frac{\sin(x+a)}{\cos x}$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $\frac{dy}{dx} = \frac{d}{dx}\frac{\sin(x+a)}{\cos x} = \frac{\cos x.\frac{d}{dx}[\sin(x+a)] - \sin(x+a)\frac{d}{dx}(\cos x)}{\cos^2 x}$

$$= \frac{\cos x.\cos(x+a) + \sin(x+a).\sin x}{\cos^2 x} = \frac{\cos(x+a-x)}{\cos^2 x} = \frac{\cos a}{\cos^2 x}$$

17) $y = \cot^{-1}(\cos ec\ 3x)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}(\cos ec\ 3x) = -\frac{1}{1+\cos ec^2}.\frac{d}{dx}(\cos ec\ 3x)$

$$= -\frac{1}{1+\cos ec^2 3x}(-\cos ec\ 3x.\cot\ 3x)\frac{d}{dx}(3x) = \frac{3.\cos ec\ 3x.\ \cot 3x}{1+\cos ec^2\ 3x}$$

18. ఈ క్రింది ప్రమేయాలకు $\frac{dy}{dx}$ కనుగొనుముi) $x = \sin h^2 y$

sol : y దృష్ట్యా అవకలనము చేయగ

$$\frac{dx}{dy} = \frac{d}{dy}\sin h^2 y = 2\sin h\, y.\cos h\, y$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{2\sin h\, y \cosh y} = \frac{1}{2\sin h\, y\sqrt{1+\sin h^2 y}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x+x^2}}$$

ii) $x = \tan h^2\, y$

sol : y దృష్ట్యా అవకలనము చేయగ

$$\frac{dx}{dy} = \frac{d}{dy}\tan h^2\, y = 2\tan h.\sec h^2\, y$$

$$\frac{dy}{dx} = \frac{1}{2\tan h\, y.\sec h^2 y}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

iii) $x = e^{\sin h\, y}$

Sol : **x** దృష్ట్యా అవకలనము చేయగ

$$\frac{dx}{dy} = \frac{d}{dy}e^{\sin h\, y} = e^{\sin h\, y}\frac{d}{dx}(\sin h\, y) = e^{\sin h\, y}.\cos h\, y \;\; = x.\cos h\, y$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{dx}{dy}}} = \frac{1}{x.\cos h\, y}$$

iv) $x = \log\left(1+\sin^2 y\right)$

sol : y దృష్ట్యా అవకలనము చేయగ

$$\frac{dx}{dy} = \frac{d}{dy}\log\left(1+\sin^2 y\right) = \frac{1}{1+\sin^2 y}\left(\sin^2 y\right)^1$$

$$= \frac{2\sin y - \cos y}{1+\sin^2 y} = \frac{\sin 2y}{1+\sin^2 y} = \frac{\sin 2y}{e^x}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{e^x}{\sin 2y}$$

v) $x = \log\left(1+\sqrt{y}\right)$ ans. $2\left(y+\sqrt{y}\right)$

19. క్రింది ప్రమే యములకు అవకలజాన్ని కనుగొనుము

i) $\sin^{-1}\left(3x-4x^3\right)$

sol : Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$now\ y = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}(\sin 3\theta) = 3\,\theta = 3\sin^{-1} x.$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

ii) $\cos^{-1}\left(4x^3-3x\right)$

sol : Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$ and $y = \cos^{-1}\left(4\cos^3\theta - 3\cos\ \theta\right)$

$= \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1} x$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = -\frac{3}{\sqrt{1-x^2}}$$

iii) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

sol : Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

and $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\tan^{-1} x;$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

iv) $\tan^{-1}\left(\frac{a-x}{1+ax}\right)$

sol : Put $a = \tan\alpha, x = \tan\theta$ then $\theta = \tan^{-1}x$ and $\alpha = \tan^{-1}a$

$y = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}\right)$

$= \tan^{-1}(\tan(\alpha-\theta)) = \alpha - \theta$

$= \tan^{-1}a - \tan^{-1}x;$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = 0 - \frac{1}{1+x^2} = -\frac{1}{1+x^2}$$

v) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

sol : $\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2}$

$y = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{1}{2}$$

vi) $\sin\left[\cos\left(x^2\right)\right]$

sol : **x** దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left[\cos\left(x^2\right)\right]\frac{d}{dx}\left[\cos\left(x^2\right)\right]$$

$$= \cos\left[\cos\left(x^2\right)\right].\left[-\sin\left(x^2\right)\right]\frac{d}{dx}\left(x^2\right)$$

$$= \cos\left[\cos\left(x^2\right)\right]\left[-\sin\left(x^2\right)\right].2x$$

$$= -2x.\sin\left(x^2\right).\cos\left[\cos\left(x^2\right)\right]$$

vii) $\sec^{-1}\left(\frac{1}{2x^2-1}\right)\left(0 < x < \frac{1}{\sqrt{2}}\right)$

sol : $x = \cos\theta$ అప్పుడు $\theta=\cos^{-1}x$ మరియు

$$2x^2 - 1 = 2\cos^2\theta - 1 = \cos 2\theta$$

$$y = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta) = 2\theta$$

$$= 2\cos^{-1} x$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = 2\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-2}{\sqrt{1-x^2}}$$

viii) $\sin\left[\tan^{-1}\left(e^{-x}\right)\right]$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{d}{dx}\sin\left[\tan^{-1}\left(e^{-x}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(e^{-x}\right)\right].\left[\tan^{-1}\left(e^{-x}\right)\right]^1$$

$$= \cos\left(\tan^{-1}\left(e^{-x}\right)\right) - \frac{1}{1+\left(e^{-x}\right)^2}\left(e^{-x}\right)^1 \quad = \frac{-e^{-x}}{1+e^{-2x}}.\cos\left[\tan^{-1}\left(e^{-x}\right)\right]$$

20. $f(x)$ ను $g(x)$ దృష్ట్యా అవకలనము చేయండి

i) $f(x)=e^x, g(x)=\sqrt{x}$

sol : $y=e^x$ *and* $z=\sqrt{x}$ అనుకోండి .

అప్పుడు $\frac{dy}{dx}=e^x$ *and* $\frac{dz}{dx}=\frac{1}{2\sqrt{x}}$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)}=\frac{e^x}{\left(\frac{1}{2\sqrt{x}}\right)}=2\sqrt{x}.e^x$$

ii) $f(x)=e^{\sin x}, g(x)=\sin x.$

sol : Let $y=e^{\sin x}$ *and* $z=\sin x.$ then $\frac{dy}{dx}=e^{\sin x}.\cos x$ and $\frac{dz}{dx}=\cos x.$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)}=\frac{e^{\sin x}.\cos x}{\cos x}=e^{\sin x}.$$

iii) $f(x)=\tan^{-1}\left(\frac{2x}{1-x^2}\right), g(x)=\sin^{-1}\left[\frac{2x}{1+x^2}\right]$

sol : $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *and* $z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$x=\tan\theta$ అనుకోండి

అప్పుడు $y=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)=\tan^{-1}(\tan 2\theta)=2\theta$

$$z=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta$$

$$y=z \Rightarrow \frac{dy}{dz}=1$$

21. $y = e^{a\sin^{-1}x}$ అయితే $\dfrac{dy}{dx} = \dfrac{ay}{\sqrt{1-x^2}}$ అని చూపుము

Sol : $y = e^{a\sin^{-1}x}$ **x** దృష్ట్యా అవకలనము చేయగ

$$\frac{dy}{dx} = \frac{d}{dx}e^{a\sin^{-1}x} = e^{a\sin^{-1}x}\frac{d}{dx}\left(a\sin^{-1}x\right)$$

$$= e^{a\sin^{-1}}.a\frac{1}{\sqrt{1-x^2}} \quad = \frac{ay}{\sqrt{1-x^2}}$$

22. $x^4 + y^4 - a^2\, xy = 0$ అయితే $\dfrac{dy}{dx}$ కనుగొనుము

sol : **x** దృష్ట్యా అవకలనము చేయగ

$$\frac{d}{dx}\left(x^4 + y^4 - a^2\, xy\right) = 0$$

$$4x^3 + 4y^3.\frac{dy}{dx} - a^2\left(x.\frac{dy}{dx} + y.1\right) = 0$$

$$4x^3 + 4y^3.\frac{dy}{dx} - a^2x\frac{dy}{dx} - a^2\, y = 0$$

$$\left(4y^3 - a^2x\right)\frac{dy}{dx} = a^2y - 4x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2y - 4x^3}{4y^3 - a^2x}$$

క్రింది ప్రమే యములకు అవకలజాన్ని కనుగొనుము

1. $f(x) = 7^{x^3+3x}\,(x > 0)$ అయితే $f'(x)$ కనుగొనుము

2. $f(x) = x.e^{x}\sin x,$ అయితే $f'(x)$ కనుగొనుము

3. $f(x) = \sin(\log x)(x > 0),$ అయితే $f'(x)$ కనుగొనుము

4. $y = \operatorname{cosec}^{-1}\left(e^{2x+1}\right)$ అయితే $\dfrac{dy}{dx}$ కనుగొనుము

5. $y = \log(\cosh 2x),$ అయితే $\dfrac{dy}{dx}$ కనుగొనుము

6. $y = \log(\sin(\log x)),$ అయితే $\dfrac{dy}{dx}$ కనుగొనుము

స్వల్ప సమాధాన ప్రశ్నలు

THEOREM: **f** మరియు **g** లు **x** వద్ద రెండు అవకలనీయ ప్రమేయులు ఐతే **f.g** కూడా **x** వద్ద అవకలనీయం మరియు **(fg)' (x) = f '(x). g(x) + f(x) . g'(x)** అని చూపుము

Proof:

f మరియు **g** లు **x** వద్ద రెండు అవకలనీయ ప్రమేయులు

కావున **f '(x) =** $\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

and $\underset{h\to 0}{Lt}\frac{g(x+h)-g(x)}{h}=g'(x)$

$\underset{h\to 0}{Lt}\frac{(fg)(x+h)-(fg)(x)}{h}=$

$=\underset{h\to 0}{Lt}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$

$=\underset{h\to 0}{Lt}\frac{1}{h}[f(x+h).g(x+h)-f(x)g(x+h)]+\underset{h\to 0}{Lt}\frac{1}{h}[f(x)g(x+h)-f(x)g(x)]$

$=\underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}\right]\underset{h\to 0}{Lt}g(x+h)+f(x).\underset{h\to 0}{Lt}\left[\frac{g(x+h)-g(h)}{h}\right]$

$\therefore$ **(fg)'(x)= f '(x) g(x) + f(x) g'(x)**

Note : u మరియు v లు **x** వద్ద రెండు అవకలనీయ ప్రమేయులు ఐతే uv కూడా **x** వద్ద అవకలనీయం మరియు

$\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{du}{dx}$

Theorem: **f** మరియు **g** లు **x** వద్ద రెండు అవకలనీయ ప్రమేయములుమరియు **g(x)≠0** ఐతే $\frac{f}{g}$ కూడా **x** వద్ద అవకలనీయం మరియు

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x).f'(x) - f(x)g'(x)}{[g(x)]^2}$$ అని చూపుము

Proof:

f మరియు **g** లు **x** వద్ద రెండు అవకలనీయ ప్రమేయులు కావున

f '(x) = $\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

and $\underset{h\to 0}{Lt}\frac{g(x+h)-g(x)}{h} = g'(x)$

$$\left(\frac{f}{g}\right)^1 = \underset{h\to 0}{Lt}\frac{(f/g)(x+h)-(f/g)(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{f(x+h)g(x) - g(x+h)f(x)}{g(x)g(x+h)}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{g(x).g(x+h)}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}\right]$$

$$= \frac{1}{[g(x)]^2}\cdot\left\{g(x)\underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\underset{h\to 0}{Lt}\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \frac{1}{[g(x)]^2}\left\{g(x)\underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\underset{h\to 0}{Lt}\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \frac{1}{[g(x)]^2}\{g(x)f'(x) - f(x)g'(x)\}$$

$$\therefore \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

u మరియు v లు x వద్ద రెండు అవకలనీయ ప్రమేయములుమరియు v ≠0 ఐతే $\frac{u}{v}$ కూడా x వద్ద అవకలనీయం మరియు $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

1. ప్రధమ సూత్రం ద్వారా క్రింది ప్రమే యములకు అవకలనము కనుగొనుము

i) $f(x) = x^4 + 4$

sol : $f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

$= \underset{h\to 0}{Lt} \frac{\left((x+h)^4+4\right)-\left(x^4+4\right)}{h}$

$= \underset{h\to 0}{Lt} \frac{(x+h)^4+4-x^4-4}{h}$

$= \underset{h\to 0}{Lt} \frac{h\left[4x^3+6x^2h+4xh^2+h^3\right]}{h} = \underset{h\to 0}{Lt}\ 4x^3+6x^2h+4xh^2+h^3$

$= 4x^3+0+0 = 4x^3$

II) $f(x) = \sqrt{x+1}$

sol : $f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$

$= \underset{h\to 0}{Lt} \frac{\left(\sqrt{x+h+1}-\sqrt{x+1}\right)\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{\sqrt{x+h+1}+\sqrt{x+1}} = \underset{h\to 0}{Lt} \frac{x+h+1-x-1}{\sqrt{x+h+1}+\sqrt{x+1}}$

$= \underset{h\to 0}{Lt} \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$

$= \underset{h\to 0}{Lt} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$

$= \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$

iii) $f(x) = \cos ax$

sol : $f^1(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\dfrac{\cos a(x+h)-\cos ax}{h}$

$$= \underset{h\to 0}{Lt}\frac{-2\sin\dfrac{ax+ah+ax}{2}.\sin\dfrac{ax+ah-ax}{2}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{-2\sin\left(ax+\dfrac{ah}{2}\right).\sin\left(\dfrac{ah}{2}\right)}{h}$$

$$\underset{h\to 0}{Lt}-2\sin\left(ax+\frac{ah}{2}\right)\underset{h\to 0}{Lt}\frac{\sin\left(\dfrac{ah}{2}\right)}{h}$$

$$= -2\sin ax.\frac{a}{2} \quad = -a.\sin ax$$

iv) $f(x) = \tan 2x$

sol : $f'(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h} \qquad = \underset{h\to 0}{Lt}\dfrac{\tan 2(x+h)-\tan 2x}{h}$

$$= \underset{h\to 0}{Lt}\frac{\dfrac{\sin(2x+2h)}{\cos(2x+2h)}-\dfrac{\sin 2x}{\cos 2x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\dfrac{\sin(2x+2h).\cos 2x-\cos(2x+2h).\sin 2x}{\cos(2x+2h).\cos 2x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{\sin(2x+2j-2x)}{\cos(2x+2h)\cos 2x}$$

$$= \underset{h\to 0}{Lt}\frac{\sin 2h}{h}.\underset{h\to 0}{Lt}\frac{1}{\cos(2x+2h)\cos 2x}$$

$$= 2.\frac{1}{\cos 2x.\cos 2x} = 2\sec^2 2x.$$

v. $f(x) = \cot x$

sol : $f^1(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\dfrac{\cot(x+h)-\cot x}{h}$

$$= \underset{h\to 0}{Lt}\frac{\dfrac{\cos(x+h)}{\sin(x+h)}-\dfrac{\cos x}{\sin x}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\dfrac{\cos(x+h).\sin x - \sin(x+h).\cos x}{\sin(x+h).\sin x}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-\sin(x+h-x)}{\sin(x+h).\sin x}\frac{1}{h}$$

$$\underset{h\to 0}{Lt} \frac{-\sin h}{h} \underset{h\to 0}{Lt} \frac{1}{\sin(x+h).\sin x}$$

$$= -\frac{1}{\sin x.\sin x} = -\cos ec^2\ x$$

vi. $f(x) = \sec 3x$

sol : $f'(x) = \underset{h\to 0}{Lt} \dfrac{f(x+h) - f(x)}{h}$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left(\frac{1}{\cos(3x+3h)} - \frac{1}{\cos 3x}\right)$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\frac{\cos 3x - \cos(3x+3h)}{\cos(3x+3h).\cos 3x}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\frac{2.\sin\dfrac{3x+3h+3x}{2}.\sin\dfrac{3x+3h-3x}{2}}{\cos(3x+3h).\cos 3x}$$

$$f'(x) = \frac{\underset{h\to 0}{Lt}\ 2\sin\left(3x+\dfrac{3h}{2}\right).\ \underset{h\to 0}{Lt}\dfrac{\sin\dfrac{3h}{2}}{h}}{\underset{h\to 0}{Lt}\ \cos(3x+3h).\cos\ 3x}$$

$$= \frac{2\sin 3x.\dfrac{3}{2}}{(\cos 3x)(\cos\ 3x)} = 3.\frac{\sin 3x}{\cos 3x}.\frac{1}{\cos 3x}$$

$$= 3.\tan 3x.\ \sec 3x$$

vii $f(x) = x\sin x$

sol : $f^1(x) = \underset{h\to 0}{Lt} \dfrac{f(x+h) - f(x)}{h}$

$$= \underset{h\to 0}{Lt} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{x(\sin(x+h) - \sin x) + h.\sin(x+h)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{x\left(2\cos\dfrac{x+h+x}{2}.\sin\dfrac{x+h-x}{2}\right) + h.\sin(x+h)}{h}$$

$$= 2x.\underset{h\to 0}{Lt}\cos\left(x+\frac{h}{2}\right).\underset{h\to 0}{Lt}\frac{\sin\frac{h}{2}}{h}+\underset{h\to 0}{Lt}h.\frac{\sin(x+h)}{h}$$

$$= 2x.\cos x.\frac{1}{2}+\sin x$$

viii $f(x)=\cos^2 x$

sol : $f^1(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$$f^1(x)=\underset{h\to 0}{Lt}\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$=\underset{h\to 0}{Lt}\frac{-\left(\cos^2 x-\cos^2(x+h)\right)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{-\sin(x+h+x)\sin(x+h-x)}{h}$$

$$f'(x)=\underset{h\to 0}{Lt}-\sin(2x+h)\underset{h\to 0}{Lt}.\frac{\sin h}{h}$$

$$=-\sin 2x.1=-\sin 2x$$

4. $f(x)=\begin{cases}x, & if \quad 0\le x\le 2\\ 2, & if \quad x\ge 2\end{cases}$ ప్రమేయము 2 వద్ద అవకలనీయమా?

Sol : $f'\left(2^-\right)=\underset{x\to 2^-}{Lt}\frac{f(x)-f(2)}{x-2}$

$$=\underset{x\to 2-}{Lt}\frac{x-2}{x-2}=1$$

$$f'\left(2^+\right)=\underset{x\to 2^+}{Lt}\frac{f(x)-f(2)}{x-2}$$

$$=\underset{x\to 2+}{Lt}\frac{2-2}{x-2}=0$$

$f'(2-)\ne f'\left(2^+\right)$; $f(x)$ ప్రమేయము 2 వద్ద అవకలనీయంకాదు.

5) $y = \frac{\cos x}{\sin x + \cos x}$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $\frac{dy}{dx} = \frac{d}{dx}\frac{\cos x}{\sin x + \cos x}$

$$\frac{dy}{dx} = \frac{(\sin x + \cos x)\frac{d}{dx}(\cos x) - \cos \times \frac{d}{dx}(\sin x + \cos x)}{(\sin x + \cos x)^2} = \frac{(\sin x + \cos x)(-\sin x) - \cos \times (\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= -\frac{\sin^2 x - \sin x \cos x - \cos^2 x + \sin x \cos x}{(\sin x + \cos x)^2} = -\frac{1}{\sin^2 x + \cos^2 x + 2\sin x \cos x} = -\frac{1}{1 + \sin 2x}$$

6. క్రింది ప్రమే యములకు $\frac{dy}{dx}$ కనుగొనుము

i) $y = \cos[\log(\cot x)]$

sol : $\frac{dy}{dx} = \frac{d}{du}\cos[\log(\cot x)]$

$$= -\sin[\log(\cot x)].\frac{1}{\cot x}\left(-\cos ec^2 x\right)$$

$$= \frac{1}{\sin^2 x}.\frac{1}{\cos x}.\sin[\log(\cot\ x)]$$

$$= \frac{\sin[\log(\cot x)].\cos ecx}{\cos x}$$

Ii) $y = \log\left(\cot\left(1 - x^2\right)\right)$

sol :

$$\frac{dy}{dx} = \frac{d}{dx}\log\left(\cot\left(1 - x^2\right)\right)$$

$$= \frac{1}{\cot\left(1 - x^2\right)}\frac{d}{dx}\left(\cot\left(1 - x^2\right)\right)$$

$$= \frac{-\cos ec^2\left(1 - x^2\right)\frac{d}{dx}\left(1 - x^2\right)}{\cot\left(1 - x^2\right)}$$

$$= \frac{-\cos ec^2\left(1 - x^2\right)(-2x)}{\cot\left(1 - x^2\right)}$$

$$= \frac{2x.\cos ec^2\left(1 - x^2\right)}{\cot\left(1 - x^2\right)}$$

$$= 2x.\frac{1}{\sin^2\left(1-x^2\right).\frac{\cos\left(1-x^2\right)}{\sin\left(1-x^2\right)}}$$

$$= \frac{4x}{2\sin\left(1-x^2\right).\cos\left(1-x^2\right)}$$

$$= \frac{4x}{\sin 2\left(1-x^2\right)} \quad = 4x.\cos ec\left(2\left(1-x^2\right)\right)$$

iii) $y = \sin\left[\cos\left(x^2\right)\right]$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin\left[\cos\left(x^2\right)\right]$

$$= \cos\left[co\left(x^2\right)\right].\frac{d}{dx}\left[\cos\left(x^2\right)\right]$$

$$= \cos\left[\cos\left(x^2\right)\right]\left(-\sin\left(x^2\right)\right).\frac{d}{dx}\left(x^2\right)$$

$$= -2x.\sin\left(x^2\right).\cos\left[\cos\left(x^2\right)\right]$$

iv) $y = \sin\left[\tan^{-1}\left(e^x\right)\right]$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin\left[\tan^{-1}\left(e^x\right)\right]$

$$= \cos\left[\tan^{-1}\left(e^x\right)\right].\frac{d}{dx}\left[\tan^{-1}\left(e^x\right)\right]$$

$$= \cos\left(\tan^{-1}\left(e^x\right)\right)\left[\frac{1}{1+\left(e^x\right)^2}\right]\left(e^x\right)$$

$$= \frac{e^x}{1+e^{2x}}.\cos\left[\tan^{-1}\left(e^x\right)\right]$$

V) $y = \tan^{-1}\left(\tan h\frac{x}{2}\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left(\tan h\frac{x}{2}\right)$

$$= \frac{1}{1+\tan h^2\frac{x}{2}}.\frac{d}{dx}\left(\tan h\frac{x}{2}\right)$$

$$= \frac{\sec h^2 \frac{x}{2} - \frac{1}{2}}{1 + \tan h^2 \frac{x}{2}} = \frac{\sec h^2 \frac{x}{2}}{2\left(1 + \tan h^2 \frac{x}{2}\right)}$$

Vi) $y = \sin x.\left(Tan^{-1}x\right)^2$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin x.\left(Tan^{-1}x\right)^2$

$$= \sin x.\frac{d}{dx}\left(Tan^{-1}x\right)^2 + \left(Tan^{-1}x\right)^2.\frac{d}{dx}(\sin x)$$

$$= \sin x.\ 2Tan^{-1}x.\frac{1}{1+x^2} + \left(Tan^{-1}x\right)^2.\cos\ x$$

$$= \frac{2\sin xTan^{-1}x}{1+x^2} + \cos\ x.\left(Tan^{-1}x\right)^2$$

xii) $y = \log\left(\frac{x^2 + x + 2}{x^2 - x + 2}\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\log\left(x^2 + x + 2\right) - \log\left(x^2 - x + 2\right)\right)$

$$= \frac{1}{x^2 + x + 2} - \frac{d}{dx}\left(x^2 + x + 2\right)$$

$$- \frac{1}{x^2 - x + 2}\frac{d}{dx}\left(x^2 - x + 2\right)$$

$$= \frac{2x+1}{x^2 + x + 2} = \frac{2x-1}{x^2 - x + 2}$$

$$= \frac{(2x+1)\left(x^2 - x + 2\right) - (2x-1)\left(x^2 + x + 2\right)}{\left(x^2 + x + 2\right)\left(x^2 - x + 2\right)}\ 2x^3 - 2x^2 + 4x + x^2 - x + 2$$

$$= \frac{-2x^3 - 2x^2 - 4x + x^2 + x + 2}{\left(x^2 + 2\right)^2 - x^2}$$

$$= \frac{4 - 2x^2}{x^4 + 4x^2 + 4 - x^2} = \frac{4 - 2x^2}{x^4 + 3x^2 + 4}$$

7. క్రింది ప్రమేయములకు $\frac{dy}{dx}$ కనుగొనుము

i) y = $\tan^{-1}\left(\frac{3a^2x - x^3}{a\left(a^2 - 3x^2\right)}\right)$

sol : Put $x = a\tan\theta \Rightarrow$ θ = $\tan^{-1}$ (x/a)

$$y = \tan^{-1}\frac{3a^2x - x^3}{a^3 - 3ax^2} = \tan^{-1}\frac{3a^2(a\tan\theta) - (a\tan\theta)^3}{a^3 - 3a(a\tan\theta)^2}$$

$$= \tan^{-1}\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a\left(a^2\tan^2\theta\right)} = \tan^{-1}\frac{a^3\left(3\tan\theta - \tan^3\theta\right)}{a^3\left(1 - 3\tan^2\theta\right)}$$

$$= \tan^{-1}\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan^{-1}\tan 3\theta$$

$$\Rightarrow y = 3\theta = 3.Tan^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}3.Tan^{-1}\left(\frac{x}{a}\right) = 3.\frac{1}{1 + \frac{x^2}{a^2}}.\left(\frac{1}{a}\right) = \frac{3a}{x^2 + a^2}$$

ii) y= $\tan^{-1}(\sec x + \tan x)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol. $\sec x + \tan x = \frac{1 + \sin x}{\cos x}$

$$= \frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2};$$

x దృష్ట్యా అవకలనము చేయగ $\frac{dy}{dx} = 0 + \frac{1}{2} = \frac{1}{2}$

iii) y= $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : Put $x = \tan\theta$ $\Rightarrow \theta = \tan^{-1}x$ and

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sqrt{1+\tan^2\theta-1}}{\tan\theta} = \frac{\sec\theta-1}{\tan\theta}$$

$$= \frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}$$

$$y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x\right) = \frac{1}{2\left(1+x^2\right)}$$

iv) y= $(\log x)^{\tan x}$ **find** $\frac{dy}{dx}$

sol : $let\ \ y = (\log x)^{\tan x}$

ఇరుపైపులా సంవర్గమానములను తీసుకొనగా

$$\log y = \log(\log x)^{\tan x}$$

$$= (\tan x).\log(\log x)$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{1}{y}.\frac{dy}{dx} = \tan x.\frac{dy}{dx}(\log(\log x)) + \log(\log x)\frac{d}{dx}(\tan x)$$

$$= \tan x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).\sec^2 x$$

$$\frac{dy}{dx} = y\left(\frac{\tan x}{x\log x} + \log(\log x).\sec^2 x\right) = (\log x)^{\tan x}$$

$$\left(\frac{\tan x}{x\log x} + \log(\log x).\sec^2 x\right)$$

v) $\left(x^{x}\right)^{x} = x^{x^{2}}$

$let \quad y = x^{x^{2}}$

ఇరువైపులా సంవర్గమానములను తీసుకొనగా

$\log y = \log x^{x^{2}} = x^{2}.\log x$

x దృష్ట్యా అవకలనము చేయగ

$\frac{1}{y}.\frac{dy}{dx} = x^{2}.\left(\frac{\log}{x}x\right) + (\log x)\frac{d}{dx}\left(x^{2}\right)$

$= x^{2}.\frac{1}{x} + 2x.\log x$

$= x + 2x\log x = x(1 + 2\log x) = x\left(\log e + \log x^{2}\right)$

$= x.\log(e)x^{2} \quad \frac{dy}{dx} = y.x.\log\left(ex^{2}\right) \quad = x^{x^{2}}.x.\log\left(ex^{2}\right) = x^{x^{2}+1} + \log\left(ex^{2}\right)$

vi) **y=** $20^{\log(\tan x)}$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $let \quad y = 20^{\log(\tan x)}$

ఇరువైపులా సంవర్గమానములను తీసుకొనగా

$\log y = \log(20)^{\log(\tan x)}$

$= \log(\tan x)\log 20$

x దృష్ట్యా అవకలనము చేయగ

$\frac{1}{y}.\frac{dy}{dx} = \log 20\frac{1}{\tan x}\sec^{2}x \quad = \log 20\frac{\cos x}{\sin x}.\frac{1}{\cos^{2}x}$

$= \frac{2\log 20.}{2\sin x.\cos x} = \frac{2\log 20}{\sin 2x} = (2\log 20).\operatorname{cosec} 2x$

$\frac{dy}{dx} = y.(2\log 20).\operatorname{cosec} 2x$

$= 20^{\log\tan x}(2\log 20).\operatorname{cosec} 2x$

vii) y = $x^x + e^{e^x}$, **find** $\frac{dy}{dx}$

sol : Let $u = x^x and\ v = e^{e^x}$ so that $y = u + v$

$u = x^x \Rightarrow \log u_1 = \log x^x = x \log x$

x దృష్ట్యా అవకలనము చేయగ

$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{x} + \log x = 1 + \log x$

$\frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$

$v = e^{e^x} \Rightarrow \log v = \log e^{e^x} = e^x.\log e = e^x$

x దృష్ట్యా అవకలనము చేయగ

$\frac{1}{v}.\frac{dv}{dx} = e^x \Rightarrow \frac{dv}{dx} = v.e^x = e^{e^x}.e^x$

$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= x^x(1 + \log x) + e^{e^x}.e^x$

viii) y= $x.\log.\log(\log x)$, **find** $\frac{dy}{dx}$

sol : $y = x.\log.\log(\log x)$

x దృష్ట్యా అవకలనము చేయగ

$\frac{dy}{dx} = \frac{d}{dx}x.\log.\log(\log x)$

$= x.\log x\frac{d}{dx}(\log.(\log x)) + \log(\log x)\log x.1 + x.\log(\log x)\frac{1}{x}$

$= x\log x.\frac{1}{\log x}.\frac{1}{x} + \log x.\log(\log x) + \log(\log x)$

$= 1 + \log(\log x)(1 + \log x) = 1 + \log(\log x) + \log x \log(\log x)$

$= \log e + \log(\log x) + \log x.\log(\log x)$

$= \log(e\log x) + \log x.\log(\log x)$

ix) $y = e^{-ax^2}.\sin(x\log x)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : let $y = e^{-ax^2}.\sin(x\log x)$

x దృష్ట్యా అవకలనము చేయగ $\frac{dy}{dx} = \frac{d}{dx}e^{-ax^2}.\sin(x\log x)$

$$= e^{-ax^2}.\frac{d}{dx}(\sin(x\log x)) + \sin(x\log x)\frac{d}{dx}\left(e^{-ax^2}\right)$$

$$= e^{-ax^2}\cos(x\log x).\left(x.\frac{1}{x} + \log x\right) + \sin(x\log x)e^{-ax^2}(-2ax)$$

$$= e^{-ax^2}(\cos(x\log x)(1+\log x)) - 2ax.\sin(x\log x) = e^{-ax^2}(\cos(x\log x)(\log ex) - 2ax.\sin(x\log x))$$

x) $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

sol : $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

$2^x = \tan\theta$, then θ = $\tan^{-1}2^x$

$$y = \sin^{-1}\left(\frac{2-2^x}{1+(2^x)^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}2^x$$

$$\frac{dy}{dx} = \frac{d}{dx}2\tan^{-1}2^x = 2\frac{1}{1+(2^x)^2}\frac{d}{dx}2^x$$

$$= \frac{2}{1+4^x}2^x\log 2$$

$$= \frac{2^{x+1}.\log 2}{(1+4^x)}$$

8. క్రింది ప్రమేయములకు $\frac{dy}{dx}$ కనుగొనుము.

i) $x = 3\cos t - 2\cos^3 t,$

$y = 3\sin t - 2\sin^3 t$

Sol : $\frac{dx}{dt} = \frac{d}{dt} 3\cos t - 2\cos^3 t$

$= -3\sin t + 6\cos^2 t(\sin t)$

$= -3\sin t + 6\cos^2 t(\sin t)$

$= 3\sin t\left(2\cos^2 t - 1\right)$

$= 3\sin.\cos 2t \quad y = 3\sin t - 2\sin^3 t$

$\frac{dy}{dt} = \frac{d}{dt}\left(3\sin t - 2\sin^3 t\right)$

$= 3\cos t - 2\left(3\sin^2 t\right)(-\cos t)$

$= 3\cos t\left(1 - 2\sin^2 t\right) = 3\cos t.\cos 2t$

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3\cos t - \cos 2t}{3\sin t - \cos 2t} = \cot t$

ii) $x = \frac{3at}{1+t^3}.y = \frac{3at^2}{1+t^2}$

sol : $\frac{dx}{dt} = \frac{d}{dt}\frac{3at}{1+t^3} = \frac{\left(1-t^3\right)3a - 3at\left(3t^2\right)}{\left(1+t^3\right)^2}$

$= \frac{3a\left(1+t^3-3t^3\right)}{\left(1+t^3\right)^2} = \frac{3a\left(1-2t^3\right)}{\left(1+t^3\right)^2}$

$y = \frac{3at^2}{\left(1+t^3\right)}$

$\frac{dy}{dt} = \frac{d}{dt}\frac{3at^2}{\left(1+t^3\right)} = \frac{\left(1+t^3\right)(6at) - 3at\left(3t^2\right)}{\left(1+t^3\right)^2}$

$$= \frac{3at\left(2+2t^3-3t^3\right)}{\left(1+t^3\right)^2} = \frac{3at\left(2-t^3\right)}{\left(1-t^3\right)^2}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2} \cdot \frac{\left(1-t^3\right)^2}{3a\left(1-2t^3\right)}$$

$$= \frac{t\left(2-t^3\right)}{1-2t^3}$$

iii) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$

sol : ans = $\tan t$

iv) **If** $x = a\left(\cos t + \log\tan\left(\frac{t}{2}\right)\right), y = a\sin t,$ **find** $\frac{dy}{dx}$

9. $f(x)$ ను $g(x)$ దృష్ట్యా అవకలనము చేయండి

i) $f(x) = \log_a^x,\ g(x) = a^x.$

sol : $f = \log_a^x = \frac{\log x}{\log_e^a}$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{df}{dx} = \frac{1}{x\log_e^a}$$

$$g = a^x \Rightarrow \frac{dg}{dx} = a^x.\log_e^a$$

$$\frac{df}{dg} = \frac{\left(\frac{df}{dx}\right)}{\left(\frac{dg}{dx}\right)} = \frac{\frac{1}{x.\log_e^a}}{a^x\log_e^a} = \frac{1}{x.a^x\left(\log_e^a\right)^2}$$

ii) $f(x) = \sec^{-1}\left(\frac{1}{2x^2-1}\right), g(x) = \sqrt{1-x^2}$

sol : $f = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ *and* $g = \sqrt{1-x^2}$

$x = \cos\theta$ అయితే అప్పుడు **$\theta = \cos^{-1}x$**

$$f=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)=\sec^{-1}(\sec 2\theta)=2\theta=2\cos^{-1}x$$

$$\frac{df}{dx}=\frac{d}{dx}2\cos^{-1}x=2.\frac{-1}{\sqrt{1-x^2}}$$

$$g=\sqrt{1-x^2}\Rightarrow\frac{dg}{dx}=\frac{d}{dx}\sqrt{1-x^2}$$

$$=\frac{1}{2\sqrt{1-x^2}}.\frac{d}{dx}\left(1-x^2\right)=\frac{1}{2\sqrt{1-x^2}}(-2x)\quad=\frac{1}{\sqrt{1-x^2}}(-x)$$

$$\therefore\frac{df}{dg}=\frac{\left(\frac{df}{dx}\right)}{\left(\frac{dg}{dx}\right)}=\frac{2}{x}$$

iii) $f(x)=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), g(x)=\tan^{-1}x.$

sol : $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *and* $z=\tan^{-1}x$

$x=\tan z$ అయితే అప్పుడు

z = tan ⁻¹x and $y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2 z}-1}{\tan z}\right)$

$$=\tan^{-1}\left(\frac{\sec z-1}{\tan z}\right)=\tan^{-1}\left[\frac{\frac{1}{\cos z}}{\frac{\sin z}{\cos z}}\right]=\tan^{-1}\left(\frac{1-\cos z}{\sin z}\right)=\tan^{-1}\left(\frac{2\sin^2\frac{z}{2}}{2\sin\frac{z}{2}.\cos\frac{z}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{z}{2}\right)=\frac{z}{2}=\frac{1}{2}\tan^{-1}x$$

$$\therefore\frac{dy}{dz}=\frac{d\left(\frac{1}{2}\tan^{-1}x\right)}{d\left(\tan^{-1}x\right)}=\frac{1}{2}$$

10. **x** దృష్ట్యా $f(x)$ కు రెండవ ఘాత అవకలనము కనుగొనుము

i) $\cos^3 x$

sol : $y = \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$

x దృష్ట్యా అవకలనము చేయగ

$y_1 = \frac{1}{4}[-3\sin 3x - 3\sin x]$

x దృష్ట్యా అవకలనము చేయగ

$y_2 = \frac{1}{4}(-9\cos 3x - 3\cos x)$

$= -\frac{1}{4}(\cos x + 3\cos 3x)$

ii) $y = \sin^4 x$

sol : $y = \sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{(1-\cos 2x)^2}{2}\right)$

$= \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right]$

$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right]$

$= \frac{1}{8}[2 - 4\cos 2x + 1 + \cos 4x]$

$= \frac{1}{8}[3 - 4\cos 2x + \cos 4x]$

x దృష్ట్యా అవకలనము చేయగ

$y_1 = \frac{1}{8}(8\cos 2x - 4\sin 4x)$

$y_2 = \frac{1}{8}(16\cos 2x - 16\cos 4x) = 2(\cos 2x - \cos 4x)$

iii) $y = \log\left(4x^2 - 9\right)$

$= \log(2x-3)(2x+3) = \log(2x-3) + \log(2x+3)$

x దృష్ట్యా అవకలనము చేయగ

$y_1 = \frac{2}{2x-3} + \frac{2}{2x+3}$

x దృష్ట్యా అవకలనము చేయగ

$$y_2 = \frac{2(-1)^{-2}}{(2x-3)^2} + \frac{2(-1)^2}{(2x+3)^2}$$

$$= -4\frac{\left((2x+3)^2+(2x+3)^2\right)}{\left(4x^2-9\right)^2} = \frac{-4\left(2\left\{4x^2+9\right\}\right)}{\left(4x^2-9\right)^2}$$

$$= -8\frac{\left(4x^2+9\right)}{\left(4x^2-9\right)^2}$$

11. $y = ax^{n+1} + bx^{-n}$, అయితే $x^2y'' = n(n+1)y$. అని చూపుము

Sol : $y = ax^{n+1} + bx^{-1}$

x దృష్ట్యా అవకలనము చేయగ

$$y_1 = (n+1).ax^n - nbx^{-n-1}$$

x దృష్ట్యా అవకలనము చేయగ

$$y_2 = n(n+1).ax^{n-1} + n(n+1)bx^{-n-2}$$

$$\therefore x^2y_2 = n(n+1)ax^{n+1} + n(n+1)bx^{-n}$$

$$= n(n+1)\left(ax^{n+1}+bx^{-n}\right) = n(n+1)y$$

$$\therefore x^2y'' = n(n+1)y$$

12. $f(x) = \dfrac{x\cos x}{\sqrt{1+x^2}}$ యొక్క యొక్క అవకలజాన్ని కనుగొనుము

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

$$f'(x) = \frac{\sqrt{1+x^2}\dfrac{d}{dx}(x\cos x) - x\cos x\dfrac{d}{dx}\sqrt{1+x^2}}{1+x^2}$$

Sol.

$$f'(x) = \frac{1}{1+x^2}\left[\begin{array}{c}(\sqrt{1+x^2})(\cos x - x\sin x) - \\ \dfrac{x}{\sqrt{1+x^2}}(x\cos x)\end{array}\right]$$

$$= (1+x^2)^{-3/2}[\cos x - x(1+x^2)\sin x]$$

13. y = x^x (x > 0) అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol.

y = x^x

ఇరువైపులా సంవర్గమానములను తీసుకొనగా

log y = x log x.

x దృష్ట్యా అవకలనము చేయగ

$$\frac{y'}{y} = x \cdot \frac{1}{x} + \log x = 1 + \log x$$

$$\therefore \frac{dy}{dx} = y' = y(1+\log x) = x^x(1+\log x).$$

14. ప్రథమ సూత్రం ద్వారా sin 2x ప్రమేయానికి అవకలజాన్ని కనుగొనుము

Sol. f(x + h) – f(x) = sin 2(x + h) – sin 2x

$$= 2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}$$

$$= 2\cos(2x+h)\sinh$$

$$f'(x) = \underset{h\to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ 2\cos(2x+h) \cdot \underset{h\to 0}{\text{Lt}} \frac{\sinh}{h}$$

$$= 2\cos 2x - 1 = 2\cos 2x$$

15. $x^n n^x \log(nx)(x > 0, n \in N)$

Sol. $y = x^n n^x \log(nx)$

$$\frac{dy}{dx} = x^n \cdot n^x (\log nx) + n^x \cdot \log x^n (x^n) + x^n \cdot \log nx(n^x)$$

$$= x^n \cdot n^x \cdot \frac{n}{\log nx} + x^n \cdot \log nx(nx^{n-1}) + x^n \cdot \log nx \cdot (n^x \cdot \log nx)$$

$$= x^{n-1} \cdot n^x \left[\frac{nx}{\log nx} + \log nx + x^n \log nx\right]$$

16. $x = \tan(e^{-y})$ యొక్క అవకలజాన్ని కనుగొనుము

Sol. $\frac{dx}{dy} = \sec^2(e^{-y}) \cdot (e^{-y})^1 = -e^{-y} \cdot \sec^2 e^{-y}$

$= -e^{-y}(1+\tan^2(e^{-y}) = -e^{-y}(1+x^2)$

$\frac{dx}{dy} = \frac{1}{\left(\frac{dx}{dy}\right)} = -\frac{1}{e^{-y}(1+x^2)} = -\frac{e^y}{1+x^2}$

17.. $x = \log(1+\sqrt{y})$ యొక్క అవకలజాన్ని కనుగొనుము

Sol. $1+\sqrt{y} = e^x$

$\sqrt{y} = e^x - 1$

$y = (e^x - 1)^2$

$\frac{dy}{dx} = 2(e^x - 1) \cdot e^x = 2\sqrt{y} \cdot e^x$

$= 2\sqrt{y}(\sqrt{y}+1) = 2(y+\sqrt{y})$

18. $x^{\log y} = \log x$ అయితే $\frac{dy}{dx} = \frac{y}{x}\frac{1-\log x \cdot \log y}{\log_x^2}$. అని చూపుము

Sol. Given $x^{\log y} = \log x$, $\log x^{\log y} = \log\log x$

(log y) (log x) = log(log x)

x దృష్ట్యా అవకలనము చేయగ

$\log x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot \frac{1}{x} = \frac{1}{\log x} \cdot \frac{1}{x}$

$\frac{\log x}{y} . \frac{dy}{dx} = \frac{1}{x \log x} - \frac{1}{x} \cdot \log y$

$= \frac{1-\log x \cdot \log y}{x \log x}; \frac{x}{y}\frac{dy}{dx} = \frac{1-\log x \log y}{(\log x)^2}$

19.. $f(x) = \log(\sec x + \tan x)$ అయితే $f'(x)$ కనుగొనుము

Sol. $f'(x) = \frac{d}{dx}\log(\sec x + \tan x)$,

$$= \frac{1}{\sec x + \tan x}\cdot\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{\sec x + \tan x}\sec x(\sec x + \tan x)$$

$$= \sec x$$

20. $y = \sec(\sqrt{\tan x})$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$\frac{dy}{du} = \frac{d}{dx}(\sec\sqrt{\tan x})$$

$$\frac{dy}{dx} = (\sec\sqrt{\tan x})(\tan\sqrt{\tan x})\cdot\frac{1}{2\sqrt{\tan x}}\sec^2 x$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}\sec(\sqrt{\tan x})\tan(\sqrt{\tan x})$$

21. If $y = (\cot^{-1} x^3)^2$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol. $u = \cot^{-1} x^3, u = x^3, y = u^2$

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}(x^3)\right)^2$$

$$= 2\cot^{-1}(x^3)\frac{d}{dx}\left(\cot^{-1}(x^3)\right)$$

$$= -\frac{1}{1+x^6}$$

$$= 2\cot^{-1}(x^3)\left(-\frac{1}{1+x^6}\right)\frac{d}{dx}x^3$$

$$= 2\cot^{-1}(x^3)\left(-\frac{1}{1+x^6}\right)3x^2$$

$$= -\frac{6x^2}{1+x^6}\cot^{-1}(x^3)$$

22. $y=\tan^{-1}(\cos\sqrt{x})$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{du}{dx}$

$=\frac{d}{dx}\tan^{-1}(\cos\sqrt{x})$

$=\frac{1}{1+\cos^2(\sqrt{x})}.\frac{d}{dx}\cos\sqrt{x}$

$\frac{1}{1+\cos^2(\sqrt{x})}.\left(-\sin\sqrt{x}\right)\frac{d}{dx}\sqrt{x}$

$=\frac{1}{1+\cos^2(\sqrt{x})}.-\sin\sqrt{x}\,\frac{1}{2\sqrt{x}}$

$=\frac{-\sin\sqrt{x}}{2\sqrt{x}\left[1+\cos^2(\sqrt{x})\right]}$

23. f(x) = $x^2\, 2^x \log x$ (x > 0) అయితే f(x) కనుగొనుము

Sol.

$f'(x)=uv\cdot\frac{dw}{dx}+vw\cdot\frac{du}{dx}+uw\cdot\frac{dv}{dx}$

$f'(x) = \frac{d}{dx}x^2 2^x \log x$

$=x^2 2^x\frac{1}{x}+2^x\log x(2x)+x^2\cdot\log x\cdot 2^x\log 2$

$=x2^x(\log x^2+x\log x(\log 2)+1)$

24. $\sin y = x\sin(a+y)$, ఐతే $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$ అని చూపుము

Sol. $x=\frac{\sin y}{\sin(a+y)}$

y దృష్ట్యా అవకలనము చేయగ

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\sin^2(a+y)}.$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{\sin^2(a+y)}{\sin a}$$

25. $x^y = e^{x-y}$ అయితే $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ అని చూపుము

Sol. $x^y = e^{x-y}$

$$\log x^y = \log e^{x-y}$$

$$y \log x = x - y$$

$$y = \frac{x}{1+\log x}$$

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x \cdot \frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

దీర్ఘ సమాధాన ప్రశ్నలు

1. $y = \sin^{-1}\left(\frac{b + a\sin x}{a + b\sin x}\right)(a > 0, b > 0)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol : $Let\ u = \frac{b + a\sin x}{a + b\sin x}$

$$\frac{du}{dx} = \frac{d}{dx}\frac{b + a\sin x}{a + b\sin x}$$

$$= \frac{(a + b\sin x)(a\cos x) - (b + a\sin x)(b\cos x)}{(a + b\sin x)^2}$$

$$= \frac{a^2\cos x + ab\sin x\cos x - b^2\cos x - ab\sin x\cos x}{(a + b\sin x)^2}$$

$$= \frac{\left(a^2 - b^2\right)\cos x}{(a + b\sin x)^2}$$

$$now\ \ y = \sin^{-1}u.\ \frac{dy}{dx} = \frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}\ \frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{b + a\sin x}{a + b\sin x}\right)^2}}\left\{\frac{\left(a^2 - b^2\right)\cos x}{(a + b\sin x)^2}\right\}$$

$$= \frac{a + b\sin x}{\sqrt{(a + b\sin x)^2 - (b + a\sin x)^2}}\left\{\frac{\left(a^2 - b^2\right)\cos x}{(a + b\sin x)^2}\right\}$$

$$= \frac{\left(a^2 - b^2\right)\cos x}{\sqrt{a^2 + b^2\sin^2 x - b^2 - a^2\sin^2 x}}\frac{1}{(a + b\sin x)}$$

$$= \frac{\left(a^2 - b^2\right)\cos x}{\sqrt{\left(a^2 - b^2\right) - \left(a^2 - b^2\right)\sin^2 x}}\frac{1}{(a + b\sin x)} = \frac{\sqrt{a^2 - b^2}}{(a + b\sin x)}$$

2. $y = \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right)(a > 0, b > 0)$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol : Let $u = \frac{b + a\cos x}{a + b\cos x}$ $then\ y = \sin^{-1}(u)$

$$\frac{du}{dx} = \frac{(a + b\cos x)(-a\sin x) + (b + a\cos x)(-b\sin x)}{(a + b\cos x)^2}$$

$$= \frac{\sin x\left\{a^2 + ab\cos x - b^2 - ab\cos x\right\}}{(a + b\cos x)^2} = \frac{\sin x\left(a^2 - b^2\right)}{(a + b\cos x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(u) = \frac{1}{\sqrt{1-u^2}}.\frac{d}{dx}(u)$$

$$= -\frac{1}{\sqrt{1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2}}.\left\{\frac{\sin x.(a^2-b^2)}{(a+b\cos x)^2}\right\}$$

$$= \frac{\sin x.(a^2-b^2)}{\sqrt{(a+b\cos x)^2-(b+a\cos x)^2}}.\frac{1}{(a+b\cos x)} = \frac{(a^2-b^2)\sin x}{\sqrt{a^2+b^2\cos^2 x-b^2-a^2\cos^2 x}}.\frac{1}{(a+b\cos x)} = \frac{\sqrt{a^2-b^2}}{a+b\cos x}$$

3. y= $Tan^{-1}\left[\frac{\cos x}{1+\cos x}\right]$ అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol : Let $u = \frac{\cos x}{1+\cos x} \Rightarrow \frac{du}{dx} = \frac{(1+\cos x)(-\sin x)-\cos x(-\sin x)}{(1+\cos x)^2}$

$$= \frac{-\sin x - \sin x\cos x + \sin x.\cos x}{(1+\cos x)^2} = \frac{-\sin x}{(1+\cos x)^2} \quad y = \tan^{-1}u \Rightarrow \frac{dy}{du} = \frac{1}{1+u^2}\frac{du}{dx}$$

$$= \frac{1}{1+\frac{\cos^2 x}{(1+\cos x)^2}}.\frac{-\sin x}{(1+\cos x)^2} = \frac{(1+\cos x^2)}{(1+\cos x)^2+\cos^2 x}.\frac{-\sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)^2}{2\cos^2 x+2\cos x+1}.\frac{-\sin x}{(1+\cos x)^2} = \frac{(1+\cos x)^2}{2\cos^2 x+2\cos x+1}\times\frac{-\sin x}{(1+\cos x)^2}$$

$$= \frac{-\sin x}{2\cos^2 x+2\cos x+1}$$

4 క్రింది ప్రమే యములకు $\frac{dy}{dx}$ కనుగొనుము.

i. $y = x^y$

ఇరుపైపులా సంవర్గమానములను తీసుకొనగా

$$\log y = \log x^y = y\log x$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{1}{y}.\frac{dy}{dx} = \log x.\frac{dy}{dx} + y.\frac{1}{x}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} - \log x.\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}; \frac{1 - y\log x}{y}\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)} = \frac{y^2}{x(1 - \log y)} \; [by (1)]$$

ii) $y^x = x^{\sin y}$

sol : ఇరువైపులా సంవర్గమానములను తీసుకొనగా

$$\log y^x = \log x^{\sin y} \Rightarrow x.\log y = (\sin y)\log x$$

x దృష్ట్యా అవకలనము చేయగ

$$x.\frac{1}{y}.\frac{dy}{dx} + \log y = \sin y.\frac{1}{x} + \log x.\cos y\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{x}{y} - \log x.\cos y\right).\frac{dy}{dx} = \frac{\sin y}{x} - \log y \Rightarrow$$

$$\Rightarrow \frac{x - y\log x.\cos y}{y}.\frac{dy}{dx} = \frac{\sin y - x\log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(\sin y - x\log y)}{x(x - y\log x.\cos y)}$$

5 $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, అయితే $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}$ అని చూపుము

sol : $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$

Put $x = \sin\theta, y = \sin\varphi$ then $\theta = \sin^{-1} x$ and $\varphi = \sin^{-1} y$ $\therefore \sqrt{1 - \sin^2\theta} + \sqrt{1 - \sin^2\varphi} = a(\sin\theta - \sin\varphi)$

$$\cos\theta + \cos\varphi = a(\sin\theta - \sin\varphi)$$

$$2\cos\frac{\theta + \varphi}{2}.\cos\frac{\theta - \varphi}{2}$$

$$= a\left[2\cos\frac{\theta + \varphi}{2}.\sin\frac{\theta - \varphi}{2}\right]$$

$$\therefore \cos\frac{\theta - \varphi}{2} = a.\sin\frac{\theta - \varphi}{2}$$

$$\tan\frac{\theta - \varphi}{2} = \frac{1}{a} \Rightarrow \frac{\theta - \varphi}{2} = \tan^{-1}\left(\frac{1}{a}\right)$$

$\varphi = \theta - 2\tan^{-1}\left(\frac{1}{a}\right)$

$\Rightarrow \sin^{-1} y = \sin^{-1} x - 2\tan^{-1}\left(\frac{1}{a}\right)$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

6. $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{x^2+a^2}\right)$, అయితే $\frac{dy}{dx} = 2\sqrt{a^2+x^2}$ అని చూపుము

sol : $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d}{dx}y = \frac{d}{dx}\left(x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)\right)$$

$$\Rightarrow \frac{dy}{dx} = x.\frac{1}{2\sqrt{a^2+x^2}}\sqrt{2}\,x + \sqrt{a^2+x^2}.1 + \frac{a^2}{x+\sqrt{a^2+x^2}}\left(1 - \frac{1}{2\sqrt{a^2+x^2}}2x\right)$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2} + \frac{a^2}{x+\sqrt{a^2+x^2}}.\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2} + \frac{a^2}{\sqrt{a^2+x^2}}$$

$$= \frac{x^2+a^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2}$$

$$= \sqrt{a^2+x^2} + \sqrt{a^2+x^2} = 2\sqrt{a^2+x^2}$$

7. $x^{\log y} = \log x$, అయితే $\frac{x}{y}.\frac{dy}{dx} = \frac{1-\log x.\log y}{(\log x)^2}$ అని చూపుము

sol :

$x^{\log y} = \log x$ ఇరుపైపులా సంవర్గమానములను తీసుకొనగా

$\Rightarrow \log x^{\log y} = \log\log x$

$(\log y)(\log x) = \log(\log x)$.

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d}{dx}(\log y)(\log x) = \frac{d}{dx}\log(\log x)$$

$$\Rightarrow \log x.\frac{1}{y}.\frac{dy}{dx} + \log y.\frac{1}{x} = \frac{1}{\log x}.\frac{1}{x}$$

$$\Rightarrow \frac{\log x}{y}.\frac{dy}{dx} = \frac{1}{x\log x} - \frac{1}{x}.\log y$$

$$= \frac{1-\log x.\log y}{x\log x}; \frac{x}{y}\frac{dy}{dx} = \frac{1-\log x.\log y}{(\log x)^2}$$

8) $y = Tan^{-1}\left(\frac{2x}{1-x^2}\right) + Tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) - \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$ అయితే $\frac{dy}{dx} = \frac{1}{1+x^2}$ అని చూపుము

Sol : $y = Putx = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) - \tan^{-1}\left(\frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) + \tan^{-1}(\tan 3\theta) + \tan^{-1}(\tan 4\theta)$$

$$= 2\theta + 3\theta - 4\theta = \theta = \tan^{-1}x$$

$$\frac{d}{dx}y = \frac{d}{dx} = \tan^{-1}x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

9) $x^y = y^x$, అయితే $\frac{dy}{dx} = \frac{y(x\log y - y)}{x(y\log x - x)}$ అని చూపుము

sol : $x^y = y^x \Rightarrow \log x^y = \log y^x$

$$y\log x = x\log y$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d}{dx}y\log x = \frac{d}{dx}x\log y$$

$$y.\frac{1}{x}\log x.\frac{dy}{dx} = x\frac{1}{y}.\frac{dy}{dx} + \log y$$

$$\therefore \log x.\frac{dy}{dx} - \frac{x}{y}.\frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\frac{y\log x - x}{y}.\frac{dy}{dx} = \frac{x\log y - y}{x}$$

$$\frac{dy}{dx} = \frac{y(x\log y - y)}{x(y\log x - x)}$$

10. క్రింది ప్రమే యములకు అవకలజాలను కనుగొనుము

i) $y = (\sin x)^{\log x} + x^{\sin x}$

sol : $u = (\sin x)^{\log x}, v = x^{\sin x}$ అయితే అప్పుడు $y = u + v$,

$u = (\sin x)^{\log x}$ applying logs,

$$\Rightarrow \log u = \log\{(\sin x)^{\log x}\} = \log x.\log(\sin x)$$

x దృష్ట్యా అవకలనము చేయగ

$$\Rightarrow \frac{d}{dx}\log u = \frac{d}{dx}\log x.\log(\sin x)$$

$$\frac{1}{u}.\frac{dy_1}{dx} = \log x\frac{1}{\sin x}.\cos x + \log(\sin x)\frac{1}{x}$$

$$\frac{dy_1}{dx} = u\left[\cot x.\log x + \frac{\log(\sin x)}{x}\right]$$

$$= (\sin x)^{\log x}\left[\cot x.\log x. + \frac{\log\sin x}{x}\right]$$

$$v = x^{\sin x}$$

$$\log v = (\log x)^{\sin x} = \sin x.\log x$$

$$\frac{1}{v}.\frac{dv}{dx} = \sin x.\frac{1}{x} + (\log x.\cos x)$$

$$\frac{dv}{dx} = v\left[\frac{\sin x}{x}.\cos x.\log x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x.\log x\right]$$

$$\sin ce\ \ y = u+v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\sin x)^{\log x}\left(\cot x.\log x. + \frac{\log\sin x}{x}\right) + x^{\sin x}\left(\frac{\sin x}{x} + \cos x.\log x\right)$$

2) $\left(x^x\right)^x$

$let \quad y=\left(x^x\right)^x \qquad \Rightarrow \log y=\log x^{\left(x^x\right)}=x^x.\log x$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{1}{y}.\frac{dy}{dx}=x^x.\frac{1}{x}+(\log x)x^x(1+\log x)$$

$$\left[\frac{d}{dx}\left(x^x\right)=x^x(1+\log x)\right]$$

$$=x^{x-1}[1+x\log x(\log e+\log x)]=x^{x-1}(1+x.\log x.\log ex)$$

$$\frac{dy}{dx}=y.x^{x-1}(1+x\log x.\log ex)=x^{\left(x^x\right)}.x^{x-1}(1+x\log x.\log ex)$$

$$=x^{x^x+x-1}(1+x\log x.\log ex)$$

3) $x^y+y^x=a^b$, అయితే $\frac{dy}{dx}=-\left(\frac{y.x^{y-1}+y^x.\log y}{x^y.\log x+x.y^{x-1}}\right)$ అని చూపుము

sol : **Let** $u=x^y$ and $v=y^x$. **so that** $u+v=a^b$

$$u=x^y\Rightarrow\log u=\log x^y=y\log x$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{1}{y_1}.\frac{dy_1}{dx}=y.\frac{1}{x}+\log x.\frac{dy}{dx},\frac{dy_1}{dx}=y_1\left(\frac{y}{x}+\log x.\frac{dy}{dx}\right)=\left(x^y\right)\left(\frac{y}{x}+\log x.\frac{dy}{dx}\right)\quad ....(1)$$

$$v=y^x\Rightarrow\log v=\log y^x=x\log y$$

$$\frac{1}{v}.\frac{dv}{dx}=x.\frac{1}{y}.\frac{dy}{dx}+\log y.1$$

$$\frac{dv}{dx}=v\left(\frac{x}{y}.\frac{dy}{dx}+\log y\right)$$

$$=y^2\left(\frac{x}{y}.\frac{dy}{dx}+\log y\right)\quad ......(2)$$

$$but\quad u+v=a^b\Rightarrow\frac{du}{dx}+\frac{dv}{dx}=0$$

$$\Rightarrow\left(x^y\right)\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}.\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow y.x^{y-1}+x^y.\log x\frac{dy}{dx}+x.y^{x-1}\frac{dy}{dx}+y^x.\log y=0$$

$$\Rightarrow \frac{dy}{dx}.\left(x^y.\log x + x.y^{x-1}\right) = -\left(y.x^{y-1} + y^x.\log y\right)$$

$$\therefore \frac{dy}{dx} = -\frac{\left(y.x^{y-1} + y^x.\log y\right)}{\left(x^y.\log x + x.y^{x-1}\right)}$$

11) $f(x) = \sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$, $g(x) = \tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$, అయితే $f'(x) = g'(x)(\beta < x < \alpha)$ అని చూపుము

Sol : $y = \sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$

$$\sin y = \sqrt{\frac{x-\beta}{\alpha-\beta}} \Rightarrow \sin^2 y = \frac{x-\beta}{\alpha-\beta}$$

$$\cos^2 y = 1 - \sin^2 y = 1 - \frac{x-\beta}{\alpha-\beta}$$

$$= \frac{\alpha-\beta-x+\beta}{\alpha-\beta} = \frac{\alpha-x}{\alpha-\beta} \Rightarrow \cos y = \sqrt{\frac{\alpha-x}{\alpha-\beta}}$$

$$\therefore \tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{\frac{x-\beta}{\alpha-\beta}}}{\sqrt{\frac{\alpha-x}{\alpha-\beta}}} = \sqrt{\frac{x-\beta}{\alpha-\beta}.\frac{\alpha-\beta}{\alpha-x}} = \sqrt{\frac{x-\beta}{\alpha-x}};$$

$y = \tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$ **i.e.,** $\sin^{-1}\sqrt{\frac{x-\beta}{\alpha-x}} = \tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$

$$f'(x) = g'(x)$$

12) $a > b > 0$ *and* $0 < x < \pi$; అయితే $f'(x) = (a + b\cos x)^{-1}$ అని చూపుము

Sol : **Let** $u = \cos^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right)$

$$\frac{du}{dx} = -\frac{1}{\sqrt{1 - \frac{(a\cos x + b)^2}{(a + b\cos x)^2}}}.\frac{d}{dx}\left(\frac{a\cos x + b}{a + b\cos x}\right)$$

$$\frac{(a + b\cos x)(-a\sin x) - (a\cos x + b)(-b\sin x)}{(a + b\cos x)^2}$$

$$= \frac{a+b\cos x}{\sqrt{(a+b\cos x)^2-(a\cos x+b)^2}} \frac{(a+b\cos x)(-a\sin x)-(a\cos x+b)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{\left(a^2-b^2\right)\sin x}{(a+b\cos x)\sqrt{\left(a^2-b^2\right)\left(1-\cos^2 x\right)}} = \frac{\left(a^2-b^2\right)\sin x}{(a+b\cos x)\sqrt{\left(a^2-b^2\right)}.\sin x} = \frac{\sqrt{a^2-b^2}}{a+b\cos x}$$

$$f'(x) = \left(a^2-b^2\right)^{-1/2} \frac{\sqrt{a^2-b^2}}{a+b\cos x} = \frac{1}{a+b\cos x} = (a+b\cos x)^{-1}$$

13. $y = x^{\tan x} + (\sin x)^{\cos x}$, అయితే $\frac{dy}{dx}$ కనుగొనుము

Sol : **Let** $u = x^{\tan x} and\ v = (\sin x)^{\cos x}$

$$\log u = \log x^{\tan x} = (\tan x)\log x$$

$$\frac{1}{u}.\frac{dy}{dx} = \tan x \frac{1}{x} + (\log x)\sec^2 x.$$

$$\frac{du}{dx} = u\left(\frac{\tan x}{x} + (\log x).\sec^2 x\right)$$

$$= x^{\tan x}\left(\frac{\tan x}{x} + (\log x).\sec^2 x\right)$$

$$\log v = \log(\sin x)\cos x] = \cos x.\log\sin x$$

$$\frac{1}{v}.\frac{dv}{dx} = \cos x.\frac{1}{\sin x}\cos x + (\log\sin x)(-\sin x)$$

$$= \frac{\cos^2 x}{\sin x} - \sin x\log\left(\sin^x\right)^{\cos x}$$

$$\frac{dv}{dx} = v\left(\frac{\cos^2 x}{\sin x} - \sin x\log\sin x\right)$$

$$= (\sin x)^{\cos x}\left(\frac{\cos^2}{\sin x} - \sin x\log(\sin x)\right)$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\tan x}$$

$$\left(\frac{\tan x}{x} + (\log x)\left(\sec^2 x\right)\right) + (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right)$$

15. $x^y = e^{x-y}$, అయితే $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ అని చూపుము

Sol : $x^y = e^{x-y} \Rightarrow \log x^y = \log e^{x-y}$

$\Rightarrow y\log x = x - y(\log e = 1)$

$y(1+\log x) = x$, $y = \frac{x}{1+\log x}$ c

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x.\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

16. $\sin y = x\sin(a+y)$, అయితే $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ అని చూపుము

Sol : $x = \frac{\sin y}{\sin(a+y)}$

$$\frac{dy}{dx} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}.\frac{dy}{dx} = \frac{\sin(a+y-y)}{\sin^2(a+y)}\frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{\sin^2(a+y)}{\sin a}$$

17. $y = \frac{2x+3}{4x+5}$ అయితే **y"** కనుగొనుము

Sol : $y = \frac{2x+3}{4x+5}$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d}{dx}y = \frac{d}{dx}\frac{2x+3}{4x+5}$$

$$\frac{dy}{dx} = \frac{(4x+5).2 - (2x+3).4}{(4x+5)^2}$$

$$= \frac{8x+10-8x-12}{(4x+5)^2}$$

$$= \frac{-2}{(4x+5)^2} = 2(4x+5)^{-2}$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d^2y}{dx^2} = \frac{d}{dx}2(4x+5)^{-2}$$

$$\frac{d^2y}{dx^2} = (-2)(-2)(4x+5)^{-3}.4$$

$$y'' = \frac{16}{(4x+5)^3}$$

18. క్రింది ప్రమేయములకు రెండవ ఘాత అవకలజాలను కనుగొనుము

i) y $= e^x.\sin x.\cos 2x$

sol : $y = e^x.\sin x.\cos 2x = \frac{e^x}{2}(2\cos 2x.\sin x)$

$$= \frac{e^x}{2}(\sin 3x - \sin x)$$

x దృష్ట్యా అవకలనము చేయగ

$$y_1 = \frac{1}{2}\left[e^x(3\cos 3x - \cos x) + e^x(\sin 3x - \sin x)\right]$$

x దృష్ట్యా అవకలనము చేయగ

$$+e^x[3\cos 3x - \cos x] + e^x(\sin 3x - \sin x) = \frac{e^x}{2}[-9\sin 3x + \sin x + 3\cos 3x - \cos x$$
$$+3\cos 3x - \cos x + \sin x3x - \sin x]$$

$$= \frac{e^x}{2}[6\cos 3x - 8\sin 3x - 2\cos x]$$

$$= e^x[3\cos 3x - 4\sin 3x - \cos x]$$

ii) **y=** $Tan^{-1}\left(\frac{1+x}{1-x}\right)$

sol : $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

Put $x = \tan\theta$ **then θ = tan⁻¹x**

$$y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}.\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right) = \frac{\pi}{4} + \theta$$

$$\therefore\ f(x) = \frac{\pi}{4} + \tan^{-1}(x)$$

x దృష్ట్యా అవకలనము చేయగ

$$f'(x) = 0 + \frac{1}{1 + x^2}$$

x దృష్ట్యా అవకలనము చేయగ

$$f''(x) = (-1)\left(1 + x^2\right)^{-2}(2x)$$

$$\therefore\ f''(x) = \frac{-2x}{\left(1 + x^2\right)^2}$$

vii) $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

sol : $f(x) = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$$Put\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x$$

$$f(x) = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$\therefore\ f(x) = 3\tan^{-1}(x)$$

x దృష్ట్యా అవకలనము చేయగ

$$f'(x) = 3\left(\frac{1}{1 + x^2}\right) = \frac{3}{1 + x^2}$$

x దృష్ట్యా అవకలనము చేయగ

$$f''(x) = (3)(-1)\left(1 + x^2\right)^{-2}(2x) \Rightarrow\ f''(x) - \frac{-6x}{\left(1 + x^2\right)^2}$$

19. $y = \sin 2x \sin 3x \sin 4x,$ అయితే y'' కనుగొనుము

sol : $y = \sin 2x \sin 3x \sin 4x$

$$= \frac{1}{2}\sin 2x[2\sin 4x. \sin 3x]$$

$$= \frac{1}{2}\sin 2x[\cos x - \cos 7x]$$

$$= \frac{1}{2}[\sin 2x.\cos x - \cos 7x. \sin 2x]$$

$$= \frac{1}{2}\times\frac{1}{2}[2\sin 2x.\cos x - 2\cos 7x.\sin 2x]$$

$$= \frac{1}{4}[(\sin 3x + \sin x) - (\sin 9x - \sin 5x)]$$

$$= \frac{1}{4}[-\sin 9x + \sin 5x + \sin 3x + \sin x]$$

x దృష్ట్యా అవకలనము చేయగ

$$y_1 = \frac{1}{4}[-9\cos 9x + 5\cos 5x + 3\cos 3x + \cos x]$$

x దృష్ట్యా అవకలనము చేయగ

$$y_2 = \frac{1}{4}[81\sin 9x - 25\sin 5x - 9\sin 3x - \sin x]$$

20) $ax^2 + 2hxy + by^2 = 1,$ అయితే $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}$ అని చూపుము

sol : $ax^2 + 2hxy + by^2 = 1$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d}{dx}\left(ax^2 + 2hxy + by^2\right) = 0$$

$$\Rightarrow a.2x + 2h\left(x.\frac{dy}{dx} + y\right) + b.2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2ax + 2hx.\frac{dy}{dx} + 2hy + 2by.\frac{dy}{dx} = 0$$

$$\Rightarrow 2(hx+by).\frac{by}{dx} = -2(ax+hy)$$

$$\frac{dy}{dx}=\frac{-2(ax+hy)}{2(hx+by)}=-\frac{(ax+hy)}{(hx+by)} \quad .....(1)$$

x దృష్ట్యా అవకలనము చేయగ

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}\frac{(ax+hy)}{(hx+by)}$$

$$\frac{d^2y}{dx^2}$$

$$=\frac{\left[(hx+by)\left(a+h\frac{dy}{dx}\right)-(ax+hy)\left(h+b.\frac{by}{dx}\right)\right]}{(hx+by)^2}$$

$$=\frac{(ax+hy)\left[h-b.\frac{(ax+hy)}{hx+by}\right]-(hx+by)\left[\frac{(ax+hy)}{hx+by}\right]}{(hx+by)^2}(ax+hy)\left(h^2x+bhy-abx-bhy\right)$$

$$=\frac{-(hx+by)\left(ahx+aby-ahx-h^2y\right)}{(hx+by)^3}$$

$$=\frac{\left(h^2-ab\right)[x(ax+hy)]+\left(h^2-ab(y(hx+by))\right)}{(hx+by)^3}=\frac{\left(h^2-ab\right)[x(ax+hy)+y(hx+by)]}{(hx+by)^3}$$

$$=\frac{\left(h^2-ab\right)\left[ax^2-2hxy+by^2\right]}{(hx+by)^3}$$

$$=\frac{h^2-ab}{(hx+by)^3}\left[\because\ ax^2+2hxy+by^2=1\right]$$

21. $y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$,0 < |x| < 1 అయితే $\frac{dy}{dx}$ కనుగొనుము .

Sol. Put $x^2 = \cos 2\theta$

$$y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(-1)}{\sqrt{1-x^4}}\times 2x = \frac{-x}{\sqrt{1-x^4}}$$

22. x = a(t – sint), y = a(1 + cos t), అయితే $\frac{d^2y}{dx^2}$ కనుగొనుము.

Sol. $\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = -a\sin t$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-a\sin t}{a(1-\cos t)}$$

$$= \frac{-2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = -\cot\left(\frac{t}{2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{1}{2}\csc^2\frac{t}{2} \times \frac{1}{a(1-\cos t)} = \frac{1}{4a\sin^4\frac{t}{2}}$$