Properties of Triangles

త్రిభుజ ధర్మాలు

1. Sine rule : In ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, R పరివృత్త వ్యాసార్ధం .

   $\Rightarrow a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C$

   $a : b : c = \sin A : \sin B : \sin C$.

2. Cosine rule : In ΔABC,

   $a^2 = b^2 + c^2 - 2bc \cos A$

   $b^2 = c^2 + a^2 - 2ac \cos B$

   $c^2 = a^2 + b^2 - 2ab \cos C$

   or

   $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$; $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ $\Rightarrow \cos A : \cos B : \cos C$.

   $= a(b^2 + c^2 - a^2) : b(c^2 + a^2 - b^2) : c(a^2 + b^2 - c^2)$

3. ΔABC లో

   $a = b \cos C + c \cos B$, $b = c \cos A + a \cos C$, $c = a \cos B + b\cos A$

4. Mollwiede's rule : In ΔABC

   $\frac{a-b}{c} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$, $\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}$.

5. Tangent rule (or) Napier's analogy : In ΔABC

   $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$;

   $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$;

   $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

6. i. $\sin\frac{A}{2} = \frac{\sqrt{(s-b)(s-c)}}{bc}$; $\sin\frac{B}{2} = \frac{\sqrt{(s-c)(s-a)}}{ca}$; $\sin\frac{C}{2} = \frac{\sqrt{(s-a)(s-b)}}{ab}$.

   ii. $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

iii. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-b)(s-c)}{\Delta}}$; $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{(s-a)(s-c)}{\Delta}}$;

$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{(s-a)(s-b)}{\Delta}}$ iv. $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \sqrt{\frac{s(s-a)}{\Delta}}$;

$\cot\frac{B}{2} = \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} = \sqrt{\frac{s(s-b)}{\Delta}}$; $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{s(s-c)}{\Delta}}$

7. $\cot A = \frac{b^2+c^2-a^2}{4\Delta}$; $\cot B = \frac{c^2+a^2-b^2}{4\Delta}$; $\cot C = \frac{a^2+b^2-c^2}{4\Delta}$.

8. ΔABC వైశాల్యం

i. $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

ii. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

iii. $\Delta = \frac{abc}{4R}$

iv. $\Delta = 2R^2\sin A\sin B\sin C$

v. $\Delta = rs$

vi. $\Delta = \sqrt{r\,r_1\,r_2\,r_3}$

9. త్రిభుజం ABC లో

i. $r = \frac{\Delta}{s}$, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

ii. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$r_1 = 4R\sin\frac{A}{2}\cos\cos\frac{C}{2}$

$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

iii. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = \tan(s-c)\frac{C}{2}$

$r_1 = s\tan\frac{A}{2} = (s-b)\cot\frac{C}{2} = (s-c)\cot\frac{B}{2}$

$r_2 = s \tan\frac{B}{2} = (s-c)\cot\frac{A}{2} = (s-a)\cot\frac{C}{2}$

$r_3 = s \tan\frac{C}{2} = (s-a)\cot\frac{B}{2} = (s-b)\cot\frac{A}{2}$

10. i) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

ii) $rr_1r_2r_3 = \Delta^2$

11. i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

ii) $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$.

i) $(r_1 - r)(r_2 + r_3) = a^2$

$(r_2 - r)(r_3 + r_1) = b^2$

$(r_3 - r)(r_1 + r_2) = c^2$

ii) $a = (r_2 + r_3)\sqrt{\frac{rr_1}{r_2r_3}}$, $b = (r_3 + r_1)\sqrt{\frac{rr_2}{r_3r_1}}$, $c =)r_1 + r_2)\sqrt{\frac{rr_3}{r_1r_2}}$

12. i) $r_1 - r = 4R\sin^2\frac{A}{2}$, $r_2 - r = 4R\sin^2\frac{B}{2}$, $r_3 - r = 4R\sin^2\frac{C}{2}$

ii) $r_1 + r_2 = 4R\cos^2\frac{C}{2}$, $r_2 + r_3 = 4R\cos^2\frac{A}{2}$, $r_3 + r_1 = 4R\cos^2\frac{B}{2}$

13. $r_1 + r_2 + r_3 = 4R$, $r + r_2 + r_3 - r_1 = 4R\cos A$, $r + r_3 + r_2 - r_2 = 4R\cos B$,

$r + r_1 + r_2 - r_3 = 4R\cos C$

14. ‘a’ భుజం గా గల సమబాహు త్రిభుజం లో

i) area $= \frac{\sqrt{3}a^2}{4}$ ii) $R = a / \sqrt{3}$

iii) $r = R / 2$ iv) $r_1 = r_2 + r_3 = 3R / 2$

v) $r : R : r_1 = 1 : 2 : 3$

అతిస్వల్ప సమాధాన ప్రశ్నలు

1. ఒక త్రిభుజం యొక్క భుజాలు 3,4,5 ఐతే త్రిభుజ పరివృత్త వ్యాసార్ధాన్ని కనుగొనండి.

Sol. త్రిభుజం యొక్క భుజాలు 3,4,5

$3^2+4^2=5^2$ కావున త్రిభుజం లంబ కోణ త్రిభుజం మరియు దాని విక్ణం =5

పరివృత్త వ్యాసార్ధం = $\frac{1}{2}.(hypotenuse)=\frac{5}{2}$

2. $\sum a(\sin B-\sin C)=0$ అని చూపుము

Solutoin: -

$\sum 2R\sin A(\sin B-\sin C) \because a=2R\sin A$

$2R\{\sin A(\sin B-\sin C)+\sin B(\sin C-\sin A)+\sin C(\sin A-\sin B)\}=0$

3. $a=\sqrt{3}+1\,cms$ $\angle B=30^0$ $\angle C=45^0$ అయితే ని C ను కనుగొనుము

$\angle B=30^0$ $\angle C=45^0$ but $A+B+C=180^0 \Rightarrow A+75^0=180^0$

$A=105^0$

$$\frac{a}{\sin A}=\frac{c}{\sin C}\Rightarrow\frac{\sqrt{3}+1}{\sin 105^0}=\frac{C}{\sin 45^0}\Rightarrow\frac{\sqrt{3}+1}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}=\frac{C}{\left(\frac{1}{\sqrt{2}}\right)}$$

4. a = 2 , b = 3, c = 4 అయితే cos A ను కనుగొనుము

Solution : -

కొసైన్ సూత్రం నుండి $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$$\cos A=\frac{9+16-4}{2\times3\times4}\Rightarrow\cos A=\frac{21}{24}=\frac{7}{8}$$

5. a = 26, b = 30 $\cos C = \frac{63}{65}$ అయితే ని C ను కనుగొనుము

Solution : -

From cosine rule $c^2 = a^2 + b^2 - 2ab\cos c$

$$c^2 = (26)^2 + (30)^2 - 2(\cancel{26})(\cancel{30})\frac{63}{\cancel{65}} \Rightarrow c^2 = 676 + 900 - 1512$$

5

$$c^2 = 64 \Rightarrow c = 8$$

6. ఒక త్రిభుజం లో కోణాలు 1 : 5 : 6 నిష్పత్తి లో ఉంటే భుజాల నిష్పత్తి కనుగొనుము

Solution : -

A = x B = 5x C = 6x

$$A + B + C = 180^0 \Rightarrow 12x = 180^0 \Rightarrow x = 15^0$$

$$\therefore A = 15^0 \quad B = 5 \times 15^0 \quad C = 6 \times 15^0$$

$$A = 15^0 \quad B = 75^0 \quad C = 90^0$$

భుజాల నిష్పత్తి $= a : b : c = 2R\sin A : 2R\sin B : 2R\sin C$

$$= \sin A : \sin B : \sin C$$

$$= \sin 15^0 : \sin 75^0 : \sin 90^0 = \frac{\sqrt{3}-1}{2\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}} : 1$$

$$= (\sqrt{3}-1) : (\sqrt{3}+1) : 2\sqrt{2}$$

7. $2\{bc\cos A + ca\cos B + ab\cos C\} = a^2 + b^2 + c^2$ అని నిరూపించుము.

L.H.S= $2bc\cos A + 2ca\cos B + 2ab\cos c$

From cosine rule $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$$\cos c = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \cancel{2bc}\frac{\{b^2 + c^2 - a^2\}}{\cancel{2bc}} + \cancel{2ac}\left\{\frac{a^2 + c^2 - b^2}{\cancel{2ac}}\right\} + \cancel{2ab}\frac{\{a^2 + b^2 - c^2\}}{\cancel{2ab}}$$

$$b^2 + \cancel{c^2} - \cancel{a^2} + \cancel{a^2} + c^2 - \cancel{b^2} + \cancel{a^2} + \cancel{b^2} - \cancel{c^2} = a^2 + b^2 + c^2$$

8. $\frac{a^2+b^2-c^2}{c^2+a^2-b^2}=\frac{\tan B}{\tan C}$ అని నిరూపించుము.

$$\tan B=\frac{\sin B}{\cos B}=\frac{\left(\frac{b}{2R}\right)}{\frac{a^2+c^2-b^2}{2ac}}=\frac{b}{2\text{R}}\times\frac{2ac}{a^2+c^2-b^2}=\frac{2\left(4R^2\ 4\right)}{2R\left(a^2+c^2-b^2\right)}\{\because abc=4R\Delta\}$$

$$\tan B=\frac{4\Delta}{a^2+c^2-b^2} \qquad |||^{ly}\ \tan C=\frac{4\Delta}{a^2+b^2-c^2}$$

$$\text{R.H.S}\ \frac{\tan B}{\tan C}=\frac{\frac{4\Delta}{a^2+c^2-b^2}}{\frac{4\Delta}{a^2+b^2-c^2}}=\frac{a^2+b^2-c^2}{a^2+c^2-b^2}$$

9. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$ అని నిరూపించుము.

Solution : -

$(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$

$b\cos A+\text{c}\cos A+\text{c}\cos B+a\cos B+a\cos c+b\cos c$

$(b\cos A+a\cos B)+(b\cos c+c\cos B)+(c\cos A+a\cos c)$

c +a +b {from projection rule}

$=a+b+c$

9. $(b-a\cos c)\sin A=a\cos A\sin c$ అని నిరూపించుము.

Solution : -

విక్షేపణ సూత్రం నుండి $b=a\cos c+c\cos A$

L.H.S $=(a\cos C+c\cos A-a\cos C)\sin A=c\cos A\sin A$

$=2R\sin C\cos A\sin A$

$=(2R\sin A)\cos A\sin C$

$=a\cos A\sin C$

10. If 4, 5 are two sides of a triangle and the include angle is 60^0 find the area

Solution: -

Given b = 4 C = 5 A = 60^0

Area of triangle $\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}\times4\times5\sin60^0=5\sqrt{3}$

11. $b\cos^2\frac{C}{2}+\cos^2\frac{B}{2}=S$ అని చూపుము

Solution : -

$$\cos\frac{C}{2}=\sqrt{\frac{S(S-C)}{ab}}\ \frac{\cos B}{2}=\sqrt{\frac{S(S-b)}{ac}}$$

$$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=b\frac{S(S-C)}{ab}+\ c\frac{S(S-b)}{ac}$$

$$=\frac{S}{a}\{s-c+s\}=\frac{s}{a}[2s-b-c]=\frac{s}{a}\{a+\cancel{b}+\cancel{c}-\cancel{b}-\cancel{c}\}$$

$$=S$$

12. $\frac{a}{\cos A}=\frac{b}{\cos B}=\frac{c}{\cos C}$ ఐతే ABC సమబాహు త్రిభుజం అని చూపుము

Solution: -

$$\frac{a}{\cos A}=\frac{b}{\cos B}=\frac{c}{\cos A}\Rightarrow\frac{2R\sin A}{\cos A}=\frac{2R\sin B}{\cos B}=\frac{2R\sin C}{\cos C}$$

$$\tan A=\tan B=\tan C\Rightarrow A=B=C$$

$\therefore$ Triangle ABC is equilateral

13. Σ a cot A = 2(R + r) అని చూపుము

Sol. $\Sigma a\cot A=\Sigma 2R\sin A\frac{\cos A}{\sin A}$

$$=2R\Sigma\cos A$$

$$=2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=2\left(R+4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=2(R+r)$$

14. ΔABC లో $r_1 + r_2 + r_3 - r = 4R$ అని చూపుము.

Sol. $r_1 + r_2 + r_3 - r = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$-4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= 4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right] + 4R\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right]$

$= 4R\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right) + 4R\cos\frac{A}{2}\sin\left(\frac{B+C}{2}\right)$

$= 4R\sin\frac{A}{2}\sin\frac{A}{2} + 4R\cos\frac{A}{2}\cos\frac{A}{2}$

$= 4R\left[\sin^2\frac{A}{2} + \cos^2\frac{A}{2}\right]$

$= 4R\ \left(\because \sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1\right)$

15. $\frac{c - b\cos A}{b - \cos A} = \frac{\cos B}{\cos C}$ అని చూపుము

Hint: విక్షేపణ సూత్రాలనుండి $C = b\ \mathrm{coA} + a\ \mathrm{cosB}$

$b = a\cos C + C\cos A$

16. $a\{b\cos C - c\cos B\} = b^2 - c^2$ అని చూపుము

Solution : -

$a\{b\cos c - c\cos B\} = ab\cos c - ac\cos B$

R.H.S లో $\cos C = \frac{a^2 + b^2 - c^2}{ab}$, $\cos B = \frac{a^2 + c^2 - b^2}{ac}$ ప్రతి క్షేపించండి.

17. ΔABC లో $\Sigma(b + c)\cos A = 2s$ అని చూపుము

Sol. L.H.S.

$= (b + c)\cos A + (c + a)\cos B + (a + b)\cos C$

$= (b\cos A + a\cos B) + (c\cos B + b\cos C) + (a\cos C + c\cos A)$

$= c + a + b = 2s =$ R.H.S.

18. ΔABC లో (a + b + c), (b + c – a) = 3bc అయితే A కనుగొనుము .

Sol. $2s(2s - 2a) = 3bc$

$$\Rightarrow \frac{s(s-a)}{bc} = \frac{3}{4} \Rightarrow \cos^2\frac{A}{2} = \frac{3}{4}$$

$$\Rightarrow \cos\frac{A}{2} = \frac{\sqrt{3}}{2} = \cos 30°$$

$$\therefore \frac{A}{2} = 30° \Rightarrow A = 60°$$

18. ΔABC లో $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$ అని చూపుము

Sol. L.H.S. = $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$

$$= \frac{3s-(a+b+c)}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r} = \text{R.H.S.}$$

19. ΔABC లో $rr_1r_2r_3 = \Delta^2$ అని చూపుము

Sol. L.H.S. = $rr_1r_2r_3 = \frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}$

$$= \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{R.H.S.}$$

20. ఒక సమభాహు త్రిభుజం లో $\frac{r}{R}$ విలువ కనుగొనుము

Sol. $\frac{r}{R} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = 4\sin^2 30°$

$$= 4\left(\frac{1}{2}\right)^3 = \frac{1}{2} \quad (\because A = B = C = 60°)$$

21. If $rr_2 = r_1r_3$ఐతే, Bకనుగొనుము.

Sol. $rr_2 = r_1r_3 \Rightarrow \frac{\Delta}{s}\cdot\frac{\Delta}{s-b} = \frac{\Delta}{s-a}\cdot\frac{\Delta}{s-c}$

$$\Rightarrow (s-a)(s-c) = s(s-b)$$

$$\Rightarrow \frac{(s-c)(s-a)}{s(s-b)} = 1 \Rightarrow \tan^2\frac{B}{2} = 1$$

$$\Rightarrow \tan\frac{B}{2} = 1 \Rightarrow \frac{B}{2} = 45° \Rightarrow B = 90°$$

22. A = 90° ఐతే 2(r + R) = b + c అని చూపుము

Sol. L.H.S. = 2r + 2R = 2(s – a) $\tan\frac{A}{2} + 2R\cdot 1$

= 2(s – a) tan 45° + 2R sin A

= (2s – 2a)1 + a (∵ A = 90°)

= b + c

= R.H.S.

23. త్రిభుజం లో $\sum r_1 \cot\frac{A}{2}$ ను S లో వ్రాయండి S

Solution : -

$$\sum r_1 \cot A/2 = r_1 \cot A/2 + r_2 \cot B/2 + r_3 \cot\frac{C}{2}$$

$$r_1 = s\tan A/2 \quad r_2 = s\tan B/2 \quad r_3 = s\tan C/2$$

$$= s\tan\frac{A}{2}\cot A/2 + s\tan B/2\cot B/2 + s\tan\frac{C}{2}\cot\frac{C}{2}$$

$$= s + s + s = 3s$$

24. $\sum \frac{r_1}{(s-b)(s-c)} = \frac{3}{r}$ అని చూపుము

Solution : - $\sum \frac{r_1}{(s-b)(s-c)} = \frac{r_1}{(s-b)(s-c)} + \frac{r_2}{(s-c)(s-a)} + \frac{r_3}{(s-a)(s-b)}$

But $r_1 = \frac{\Delta}{s-a}$ $r_2 = \frac{\Delta}{s-b}$ $r_3 = \frac{\Delta}{s-c}$

$$= \frac{\Delta}{(s-a)(s-b)(s-c)} + \frac{\Delta}{(s-a)(s-b)(s-c)} + \frac{\Delta}{(s-a)(s-b)(s-c)}$$

$$= \frac{3\Delta s}{s(s-a)(s-b)(s-c)} = \frac{3\not\Delta s}{\Delta^2} = \frac{3}{\left(\frac{\Delta}{s}\right)} = \frac{3}{r}$$

స్వల్ప సమాధాన ప్రశ్నలు

1. త్రిభుజం ABC లో $a\cos A + b\cos B + C\cos c = 4R\sin A\sin B\sin C$ అని చూపుము

Solution : -

L.H.S $= a\cos A + b\cos B + C\cos C = 2R\sin A\cos A + 2\sin B\cos B + 2R\sin C\cos C$

$= R\left\{\sin 2A + \sin^2 B + \sin^2 C\right\}$

Given that $A + B + C = 180^0$

$\therefore L.H.S = R\left\{\sin(A+B)\cos(A-B) + \sin 2C\right\}$

$= R\left\{2\sin C\cos(A-B) + 2\sin C\cos C\right\} = 2R\sin C\left\{\cos(A-B) + \cos C\right\}$

$= 2R\sin C\left\{\cos(A-B) + \cos\left(180^0 - \overline{A+B}\right)\right\}$

$= 2R\sin C\left\{\cos(A-B) - \cos(A+B)\right\} = 4R\sin A\sin B\sin C$

2. $\sum a^3\sin(B-C) = 0$ అని చూపుము

Solution : -

$\sum a^3\sin(B-C) = \sum (2R\sin A)^3\sin(B-C) = 8R^3\sum \sin^3 A\sin(B-C)$

$8R^3\sum \sin^2 A\sin A\sin(B-C) = 8R^3\sum \sin^2 A\sin\left(180^0 - B + \overline{C}\right)\sin(B-C)$

$\left\{\because A = 180^0 - \overline{B+C}\right\}$

$= 8R^3\sum \sin^2 A\sin(B+C).\sin(B-C) = 8R^3\sum \sin^2 A\left(\sin^2 B - \sin^2 C\right)$

$= 8R^3\left\{\sin^2 A\left(\sin^2 B - \sin^2 C\right) + \sin^2 B\left(\sin^2 C - \sin^2 A\right) + \sin^2 C\left(\sin^2 A - \sin^2 B\right)\right\}$

= 0

3. త్రిభుజంABC లో $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{C^2-a^2}=\frac{C\sin(A-B)}{a^2-b^2}$ అని చూపుము

Solution : -

$$\frac{a\sin(B-C)}{b^2-c^2}=\frac{2R\sin A\sin(B-C)}{4R^2\{\sin^2 B-\sin^2 C\}}\left\{\begin{array}{l}\because a=2R\sin A : b=2R\sin B\\ C=2R\sin C \text{ from } \sin e \text{ Rule}\end{array}\right\}$$

$$\frac{1}{2R}\frac{\sin(B+C).\sin(B-C)}{\sin^2 B-\sin^2 C}\{\because \sin A=\sin(B+C) \text{ In a triangle } ABC\}$$

$$\frac{1}{2R}\frac{\sin^2 B-\sin^2 C}{\sin^2 B-\sin^2 C}=\frac{1}{2R}$$

ఇదేవిధంగా $\frac{b\sin(C-A)}{c^2-a^2}=\frac{1}{2R}$ and $\frac{C\sin(A-B)}{a^2-b^2}=\frac{1}{2R}$ అని చూపగలము

4. $\frac{a}{bc}+\frac{\cos A}{a}=\frac{b}{ca}+\frac{\cos B}{b}=\frac{c}{ab}+\frac{\cos C}{c}$ అని చూపుము

Solution : -

$$\frac{a}{bc}+\frac{\cos A}{a}=\frac{a^2+bc\cos A}{abc}$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\frac{a^2+bc\cos A}{abc}=\frac{a^2+\frac{bc\left[b^2+c^2-a^2\right]}{2bc}}{abc}=\frac{2a^2+b^2+c^2-a^2}{2abc}$$

$$\frac{a^2+b^2+c^2}{2abc}$$

$$|||^{ly}\ \frac{b}{ca}+\frac{\cos B}{b}=\frac{b^2+ac\cos B}{abc}=\frac{b^2+\frac{ac\left(a^2+c^2-b^2\right)}{2ac}}{abc}=\frac{a^2+b^2+c^2}{2abc}$$

$$|||^{ly}\ \frac{c}{ab}+\frac{\cos c}{c}=\frac{a^2+b^2+c^2}{2abc}$$

5. $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{a^2+b^2}{a^2+c^2}$ అని చూపుము

Solution : -

త్రిభుజంABC లో $A+B+C=180^0 \Rightarrow C=180^0-\overline{A+B}$

$B=\left(180^0-\overline{A+C}\right)$

$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{1+\cos(A-B)\cos\left(180^0-\overline{A+D}\right)}{1+\cos(A-C).\cos\left(180^0-A+C\right)}$

$=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}=\frac{1-\left\{\cos^2 A-\sin^2 B\right\}}{1-\left\{\cos^2 A-\sin^2 C\right\}}$

$=\frac{1-\cos^2 A+\sin^2 B}{1-\cos^2 A+\sin^2 C}=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{\frac{a^2}{4R^2}+\frac{b^2}{4R^2}}{\frac{a^2}{4R^2}+\frac{c^2}{4R^2}}=\frac{a^2+b^2}{a^2+c^2}$

6. $C=60^0$ అయితే (i) $\frac{a}{b+c}+\frac{b}{c+a}=1$ (ii) $\frac{b}{c^2-a^2}+\frac{a}{c^2-b^2}=0$ అని చూపుము

Solution : -

Given $C=60^0$

$C^2=a^2+b^2-2ab\cos C$

$C^2=a^2+b^2-\cancel{2}ab\left(\frac{1}{\cancel{2}}\right)$

$C^2+ab=a^2+b^2$

$\frac{a}{b+c}+\frac{b}{c+a}=$

$\frac{ac+a^2+bc+b^2}{(b+c)(c+a)}=\frac{a^2+b^2+ac+bc}{bc+ab+c^2+ac}$

But $a^2+b^2=c^2+ab$

$\frac{c^2+ab+ac+bc}{bc+ab+c^2+ac}=1$

(ii) $\frac{b}{c^2-a^2}+\frac{a}{c^2-b^2}=0 \Rightarrow \frac{b\left\{c^2-b^2\right\}+a\left\{c^2-a^2\right\}}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$

$$=\frac{bc^2-b^3+ac^2-a^3}{\left(c^2-a^2\right)\left(c^2-b^2\right)}=\frac{c^2\left(a+b\right)-\left(a+b\right)\left\{a^2+b^2-ab\right\}}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$$

$$=\frac{\left(a+b\right)\left[c^2-\left\{a^2+b^2-ab\right\}\right]}{\left(c^2-a^2\right)\left(c^2-b^2\right)}=\frac{\left(a+b\right)\left[c^2-\left(c^2+\cancel{ab}-\cancel{ab}\right)\right]}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$$

= 0

7. త్రిభుజంABC లో $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ ఐతే $c=60^0$ అని చూపుము

Solution : -

Given $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$

$$\frac{a+b+c}{a+c}+\frac{a+b+c}{b+c}=3$$

$$\frac{\left(a+c\right)+b}{a+c}+\frac{a+\left(b+c\right)}{b+c}=3 \Rightarrow \frac{\cancel{a}+c}{\cancel{a}-c}+\frac{b}{a+c}+\frac{a}{b+c}+\frac{\cancel{b+c}}{\cancel{b+c}}=3$$

$$\frac{b}{a+c}+\frac{a}{b+c}=1 \Rightarrow \frac{b^2+bc+a^2+ac}{ab+ac+bc+c^2}=1$$

$$a^2+b^2-c^2=ab \Rightarrow 2ab\cos C=ab\left\{\because a^2+b^2-c^2=2abc\cos C\right\}$$

$$\cos C=\frac{ab}{2ab}=\frac{1}{2}\Rightarrow C=60^0 \cos C=\frac{\cancel{ab}}{\cancel{2ab}}=\frac{1}{2}\Rightarrow C=60^0$$

8. a : b : c = 7 : 8 : 9 ఐతే $\cos A:\cos B:\cos C$ కనుగొనుము

Soltion : -

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{64k^2+81k^2-49k^2}{2\left(8k\right)\left(9k\right)}=\frac{\cancel{96k^2}^{12}}{2\times\cancel{8k}\times 9\cancel{k}}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{49k^2+81k^2-64k^2}{2\times 7k\times 9k}=\frac{66^{33}}{\cancel{2}\times 63}=\frac{11}{21}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{49k^2+64k^2-81k^2}{2\times 7k\times 8k}=\frac{32k^2}{20k\times 8k}=\frac{2}{7}$$

$$\therefore \cos A=\cos B=\operatorname{co}C=\frac{2}{3}=\frac{11}{21}=\frac{2}{7}=\frac{2}{3}\times 21=\frac{11}{21}\times 21=\frac{2}{7}\times 21$$

$$=14:11:6$$

9. $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$ అని చూపుము

LHS $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{bc\cos A+ca\cos B+ab\cos C}{abc}$

$=\frac{2bc\cos A+2ca\cos B+2ab\cos C}{2bac}$

$=\frac{b^2+c^2-\cancel{a^2}+\cancel{a^2}+\cancel{c^2}-\cancel{b^2}+a^2+\cancel{b^2}-\cancel{c^2}}{2abc}$

$\left\{\begin{array}{l}\because 2bc\cos A=b^2+c^2-a^2 : 2ac\cos B=a^2+c^2-b^2\\ 2ab\cos c=a^2+b^2+c^2\end{array}\right\}$

$=\frac{a^2+b^2+c^2}{2abc}$

10. $(b-a)\cos c+c(\cos B-\cos A)=c\sin\left(\frac{A-B}{2}\right)\cos ec\left(\frac{A+B}{2}\right)$ అని చూపుము

Solution : -

$(b-a)\cos c+c\{\cos B-\cos A\}$

$b\cos c-a\cos c+\cos B-\cos A=(b\cos c+\cos B)-ca\cos C+\cos A$

$=a-b$ (విక్షేపణ సూత్రాలనుండి)

$=2R\{\sin A-\sin B\}$

$=2R\left\{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right\}$

$\frac{2R\left\{2\sin\frac{C}{2}\sin\left(\frac{A-R}{2}\right)\right\}\cos\frac{C}{2}}{\cos\frac{C}{2}}=\frac{2R\left\{2\sin\frac{C}{2}\cos\frac{C}{2}\right\}\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$

$=\frac{2R\sin C\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}=c\sin C\sin\left(\frac{A-B}{2}\right)\cos ec\left(\frac{A+B}{2}\right)$

$\left\{\begin{array}{l}\because \textit{In a triangle ABC}\\ \frac{C}{2}=\left(90^0-\overline{\frac{A+B}{2}}\right);\ \frac{A+B}{2}=90^0-\frac{C}{2}\end{array}\right\}$

11. $a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}=a\frac{(s-a)(s-b)}{ab}+c\frac{(s-b)(s-c)}{bc}$

$$\frac{(s-b)\{s-a+s-c\}}{b}$$

$$\frac{(s-b)\{2s-ac\}}{b}=\frac{(s-b)\{a+b+c-a-c\}}{b}=(s-b)$$

12. $b+c=3a$ ఐతే $\cot\frac{B}{2}\cot\frac{c}{2}$ కనుగొనుము

Solution : -

$$\cot\frac{B}{2}\cot\frac{c}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}\cdot\frac{s(s-c)}{(s-a)(s-b)}}=\frac{s}{s-a}=\frac{2s}{2(s-a)}$$

$$=\frac{b+c+a}{b+c-a}=\frac{3a+a}{3a-a}=\frac{4a}{2a}=2$$

13. $(b-c)^2\cos^2\frac{A}{2}+(b+c)^2\sin^2\frac{A}{2}=a^2$ అని చూపుము

Solution : -

$$(b-c)^2\cos^2\frac{A}{2}+(b+c)^2\sin^2\frac{A}{2}$$

$$\{b^2+c^2-2bc\}\cos^2\frac{A}{2}+\{b^2+c^2+2bc\}\sin^2\frac{A}{2}$$

$$b^2\left\{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right\}+c^2\left\{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right\}-2bc\left\{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right\}$$

$b^2+c^2-2bc\cos A=a^2$ (from cosine rule)

$|||^{ly}$ prove that (i) $(c-a)^2\cos^2\frac{C}{2}+(c+a)^2\sin^2\frac{B}{2}=b^2$

(ii) $(a-b)^2\cos^2\frac{c}{2}+(a+b)^2\sin^2\frac{C}{2}=c^2$]

14. త్రిభుజంABC లో $r + r_1 + r_2 - r_3 = 4R\cos C$ అని చూపుము

Sol.

$$r + r_1 + r_2 - r_3 = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} - 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right] + 4R\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2} - \cos\frac{B}{2}\cos\frac{C}{2}\right]$$

$$= 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right) + 4R\cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right)$$

$$= 4R\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) + 4R\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= 4R\left[\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) + \sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)\right]$$

$$= 4R\cos\left(\frac{B+C}{2} - \frac{B-C}{2}\right)$$

$$= 4R\cos\left(\frac{B+C-B+C}{2}\right)$$

$$= 4R\cos\frac{2C}{2} = 4R\cos C$$

15. $r_1 + r_2 + r_3 = r$ ఐతే $\angle C = 90^\circ$ అని చూపుము

Sol. $r_1 + r_2 = r - r_3 \Rightarrow \frac{r_1 + r_2}{r - r_3} = 1 \quad \ldots(1)$

$$r_1 + r_2 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$= 4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4R\cos\frac{C}{2}\left[\sin\frac{A+B}{2}\right]$$

$$= 4R\cos\frac{C}{2}\cdot\cos\frac{C}{2}$$

$$= 4R\cos^2\frac{C}{2}$$

$$r - r_3 = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2} - \cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$= 4R\sin\frac{C}{2}\left[-\cos\left(\frac{A+B}{2}\right)\right]$$

$$= 4R\sin\frac{C}{2}\left[-\sin\frac{C}{2}\right]$$

$$= -4R\sin^2\frac{C}{2}$$

$$\frac{r - r_3}{r_1 + r_2} = \frac{4R\sin^2\frac{C}{2}}{4R\cos^2\frac{C}{2}} = \tan^2\frac{C}{2}$$

$$\therefore \tan^2\frac{C}{2} = \tan 45° \qquad \text{From(1)}$$

$$\frac{C}{2} = 45° \qquad \therefore \angle C = 90°$$

16. $4(r_1r_2 + r_2r_3 + r_3r_1) = (a + b + c)^2$ అని నిరూపించండి .

Sol. $4(r_1r_2 + r_2r_3 + r_3r_1) = 4\left[\frac{\Delta}{S-a}\cdot\frac{\Delta}{S-b} + \frac{\Delta}{S-b}\cdot\frac{\Delta}{S-c} + \frac{\Delta}{S-c}\cdot\frac{\Delta}{S-a}\right]$

$$= 4\Delta^2\left[\frac{S-c+S-a+S-b}{(S-a)(S-b)(S-c)}\right]$$

$$= 4\Delta^2\left[\frac{3S-(a+b+c)}{(S-a)(S-b)(S-c)}\right]$$

$$= 4\Delta^2\left[\frac{S^2}{S(S-a)(S-b)(S-c)}\right]$$

$$= 4\Delta^2\frac{S^2}{\Delta^2} = 4S^2$$

$$= (2S)^2 = (a+b+c)^2$$

17. $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2S^2}$ అని నిరూపించండి .

Sol. $\frac{1}{r} - \frac{1}{r_1} = \frac{S}{\Delta} - \frac{S-a}{\Delta} = \frac{S-S+a}{\Delta} = \frac{a}{\Delta}$

ఇదేవిధంగా

$$\frac{1}{r}-\frac{1}{r_2}=\frac{b}{\Delta} \text{ and } \frac{1}{r}-\frac{1}{r_3}=\frac{c}{\Delta}$$

$$\text{L.H.S.}=\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)$$

$$=\frac{a}{\Delta}\frac{b}{\Delta}\frac{c}{\Delta}=\frac{abc}{\Delta^3}$$

$$=\frac{4R\cdot\Delta}{\Delta^3}=\frac{4R}{\Delta^2}=\frac{4R}{(rS)^2}=\text{R.H.S.}$$

18. $\dfrac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_3}}=a$ అని నిరూపించండి

Solution : -

LHS

$$\frac{\dfrac{\Delta}{s-a}\left\{\dfrac{\Delta}{s+b}+\dfrac{\Delta}{s-c}\right\}}{\sqrt{\dfrac{\Delta^2}{(s-a)(s-b)}+\dfrac{\Delta^2}{(s-b)(s-c)}+\dfrac{\Delta^2}{(s-c)(s-a)}}}=\frac{\dfrac{\Delta^2}{(s-a)(s-b)(s-c)}\{s-b+s-c\}}{\sqrt{\dfrac{\Delta^2}{(s-a)(s-b)(s-c)}(s-a)(s-b)+(s-c)}}$$

$$\frac{\Delta^{2}}{(s-a)(s-b)(s-c)}\times\frac{\sqrt{(s-a)(s-b)(s-c)}}{\Delta}\times\frac{a}{\sqrt{3s-2s}}$$

$$\frac{a\Delta}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{a\Delta}{\Delta}=a$$

19. $r(r_1+r_2+r_3)=ab+bc+ca-S^2$ అని నిరూపించండి

Sol. L.H.S. $= r(r_1+r_2+r_3)$

$$=\frac{\Delta}{S}\left(\frac{\Delta}{S-a}+\frac{\Delta}{S-b}+\frac{\Delta}{S-c}\right)$$

$$=\frac{\Delta^2}{S}\left[\frac{(S-b)(S-c)+(S-a)(S-c)+(S-a)(S-b)}{(S-a)(S-b)(S-c)}\right]$$

$$=\frac{\Delta^2}{\Delta^2}\left[\begin{array}{l}S^2-Sc-Sb+bc+S^2-Sc-Sa\\+ac+S^2-Sb-Sa+ab\end{array}\right]$$

$= 3S^2 - 2S(a+b+c) + bc + ca + ab$

$= 3S^2 - 4S^2 + bc + ca + ab$

$= ab + bc + ca - S^2$

$=$ R.H.S.

20. $(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = c$ అని నిరూపించండి

Sol. $(r_1 + r_2)\tan\frac{C}{2}$

$$= 4R\cos^2\frac{C}{2}\tan\frac{C}{2}$$

$$= 4R\cos^2\frac{C}{2}\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$$

$$= 4R\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2R\sin C = c \qquad ...(1)$$

$$(r_3 - r)\cot\frac{C}{2} = 4R\sin\frac{C}{2}\cot\frac{C}{2}$$

$$= 4R\sin^2\frac{C}{2}\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$= 4R\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2R\sin C = c \qquad ...(2)$$

(1) , (2) ల నుండి

$$(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = c$$

21. ΔABC లో a, b, c లు A.P. లో ఉండటానికి ఆవశ్యక పర్యాప్తం r_1, r_2, r_3 లు H.P.లో ఉంటాయని చూపుము

Sol. r_1, r_2, r_3 లు H.P. ఉన్నాయి $\Leftrightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ లు A.P. ఉంటాయి.

$\Leftrightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$. లు A.P. ఉంటాయి.

$\Leftrightarrow s-a, s-b, s-c$. లు A.P. ఉంటాయి.

$\Leftrightarrow -a, -a, -c$ లు A.P. ఉంటాయి.

$\Leftrightarrow$ a, b, c లు A.P. ఉంటాయి.

22. A, A_1, A_2, A_3 లు త్రిభుజంయొక్క అంతర మరియు బాహ్య వృత్తాల వైశాల్యాలు అయితే $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$ అని చూపుము

Solution : -

Here $A=\pi r^2 \quad A_1=\pi r_1^2 \; ; A_2=\pi r_2^2 \; ; A_3=\pi r_3^2$

LHS $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{\pi r_1^2}}+\frac{1}{\sqrt{\pi r_2^2}}+\frac{1}{\sqrt{\pi r_3^2}}$

$$\frac{1}{\sqrt{\pi}}\left\{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right\}=\frac{1}{\sqrt{\pi}}\left\{\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right\}$$

$$\frac{1}{\sqrt{\Delta}}\left\{\frac{3s-(a+b+c)}{\Delta}\right\}=\frac{1}{\sqrt{\pi}}\frac{s}{\Delta}=\frac{1}{\sqrt{\pi}}\times\frac{1}{\left(\frac{\Delta}{s}\right)}$$

$$\frac{1}{\sqrt{\pi}}.\frac{1}{r}=\frac{1}{\sqrt{\pi r^2}}=\frac{1}{\sqrt{A}}$$

ధీర్ఘ సమాధాన ప్రశ్నలు

1. ΔABC లో , $\cot A + \cot B + \cot C = \dfrac{a^2 + b^2 + c^2}{4\Delta}$ అని చూపుము

Solution : -

$$\cot A = \frac{\cos A}{\sin A}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cot = \frac{\dfrac{b^2 + c^2 - a^2}{2bc}}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc \sin A} = \frac{b^2 + c^2 - a^2}{4\left\{\dfrac{1}{2} bc \sin A\right\}}$$

$$\cot A = \frac{b^2 + c^2 - a^2}{4\Delta} \left\{\because \Delta = \frac{1}{2} bc \sin A\right\}$$

$$|||^{ly} \quad \cot B = \frac{a^2 + c^2 - b^2}{4\Delta} \quad \cot c = \frac{a^2 + b^2 - c^2}{4\Delta}$$

$$\cot A + \cot B + \cot C = \frac{b^2 + c^2 - a^2}{4\Delta} + \frac{a^2 + c^2 - b^2}{4\Delta} + \frac{a^2 + b^2 - c^2}{4\Delta}$$

$$= \frac{b^2 + c^2 - \cancel{a^2} + \cancel{a^2} + \cancel{c^2} - \cancel{b^2} + a^2 + \cancel{b^2} - \cancel{c^2}}{4\Delta}$$

$$= \frac{a^2 + b^2 + c^2}{4\Delta}$$

2. $\cot \dfrac{A}{2} : \cot \dfrac{B}{2} \ \cot \dfrac{C}{2} = 3:5:7$ అయితే a : b : c = 6 : 5 : 4 అని చూపుము

Solution : -

$$\cot \frac{A}{2} : \cot \frac{B}{2} : \cot \frac{C}{2} = \frac{s(s-a)}{\Delta} : \frac{s(s-c)}{\Delta} = 6:5:4$$

$$\therefore s - a : s - b : s - c = 6:5=4$$

Let $s - a = 3k \quad s - b = 5k \quad s - c = 7k$

$$(s-a) + (s-b) + (s-c) = 3k + 5k + 7k$$

$$3s - 2s = 15k \Rightarrow s = 15k$$

$$s - a = 3k \Rightarrow 15k - a = 3k \Rightarrow a = 12k$$

$$s - b = 5k \Rightarrow 15k - 5k = b \Rightarrow b = 10k$$

$$s - c = 7k \Rightarrow 15k - 7k = c \Rightarrow c = 8k$$

$a:b:c=12k:10k:8k \Rightarrow a:b:c=6:5:4$

3. త్రిభుజంABC లో $(a+b+c)\left\{\tan\frac{A}{2}+\tan\frac{B}{2}\right\}=2c\cot\frac{C}{2}$ అని చూపుము

Solution : -

$$(a+b+c)\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]=2s\left\{\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}\right\}$$

$$=\frac{2s(s-c)}{\Delta}\{s-b+s-a\}$$

$$=2\cot\frac{C}{2}\{2s-b-a\}\Rightarrow 2\cot\frac{C}{2}\{\not a+\not b+c-\not b-\not a\}$$

$$=2c\cot\frac{C}{2}$$

4. cos A + cos B + cos C = $1+\frac{r}{R}$ అని చూపుము

Sol. L.H.S. = cos A + cos B + cos C

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=1+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}$$

$$=1+\frac{r}{R}=\text{R.H.S.}$$

5. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R}$ అని చూపుము

Sol. L.H.S. = $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$$=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(from transformationsPROVE THIS IN EXAM)

$$=2+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2R}$$

$$=2+\frac{r}{2R}=\text{R.H.S.}$$

6. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$ అని చూపుము

Solution : -

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$=\frac{S}{\Delta}\{s-a+s-b+s-c\}=\frac{s}{\Delta}\{3s-(a+b+c)\}=\frac{s}{\Delta}(3s-2s)$$

$$=\frac{s}{\Delta}(s)=\frac{s^2}{\Delta}$$

7. $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{bc+ca+ab-s^2}{\Delta}$ అని చూపుము

Solution : -

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}$$

$$\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}+\frac{(s-a)(s-b)}{\Delta}$$

$$\frac{s^2-cs-bs+bc+s^2-as-cs+ac+s^2-bs-as+ab}{\Delta}$$

$$\frac{3s^2-2as-2bs-2cs+bc+ca+ab}{\Delta}$$

$$\frac{bc+ca+ab+3s^2-2s(a+b+c)}{\Delta}=\frac{bc+ca+ab+3s^2-2s(2s)}{\Delta}$$

$$\frac{bc+ca+ab+3s^2-4s^2}{\Delta}=\frac{bc+ca+ab-s^2}{\Delta}$$

8. $\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$ అని చూపుము

Solution : - $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$

$$\frac{s}{\Delta}\{s-a+s-b+s-c\}=\frac{s}{\Delta}\{3s-a-b-c\}=\frac{s}{\Delta}\times(3s-2s)=\frac{s^2}{\Delta}$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\frac{b^2+c^2-a^2}{2bc}}{\sin A} = \frac{b^2+c^2-a^2}{bc\sin A} = \frac{b^2+c^2-a^2}{\Delta\left\{\frac{1}{2}bc\sin A\right\}}$$

$$= \frac{b^2+c^2-a^2}{4\Delta} \quad \left\{\because \frac{1}{2}bc\sin A = \Delta\right\}$$

$$\cot A + \cot B + \cot C = \frac{b^2+c^2-a^2}{4\Delta} + \frac{c^2+a^2-b^2}{4\Delta} + \frac{a^2+b^2-c^2}{4\Delta}$$

$$= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{4\Delta} = \frac{a^2+b^2+c^2}{4\Delta}$$

$$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C} = \frac{\frac{s^2}{\Delta}}{\frac{a^2+b^2+c^2}{4\Delta}} = \frac{s^2}{\cancel{\Delta}} \times \frac{4\cancel{\Delta}}{a^2+b^2+c^2}$$

$$= \frac{4s^2}{a^2+b^2+c^2} = \frac{(2s)^2}{a^2+b^2+c^2} = \frac{(a+b+c)^2}{a^2+b^2+c^2}\,]$$

9. $\sum (a+b)\tan\left(\frac{A-B}{2}\right) = 0$ అని చూపుము

Solution : -

$\sum (a+b)\tan\left(\frac{A-B}{2}\right)$ from Napiries Rules

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\therefore \sum (a+b)\tan\left(\frac{A-B}{2}\right) = \sum \left(\cancel{a+b}\right)\left(\frac{a-b}{\cancel{a+b}}\right)\cot\frac{c}{2}$$

$$\sum \frac{(a-b)s(s-c)}{\Delta} = \frac{s}{\Delta}\sum (a-b)(s-c)$$

$$\frac{s}{\Delta}\sum s(a-b)-c(a-b)=0$$

$$\frac{s}{\Delta}\left[s(a-b)+s(b-c)+x(c-a)\right]-\left\{c(a-b)+b(c-a)+a(b-c)\right\}=0$$

10. $\frac{b-c}{b+c}\cot\frac{A}{2}+\frac{b+c}{b-c}\tan\frac{A}{2}=2\operatorname{cosec}(B-C)$ అని చూపుము

Solution : -

$$\frac{b-c}{b+c}\cot A/2+\frac{b+c}{b-c}\tan A/2$$

From Napiers Rule $\frac{b-c}{b+c}\cot\frac{A}{2}=\tan\left(\frac{B-C}{2}\right)$

$$\frac{b+c}{b-c}\tan\frac{A}{2}=\cot\left(\frac{B-C}{2}\right)$$

$$\therefore\ LHS=\tan\left(\frac{B-C}{2}\right)+\cot\left(\frac{B-C}{2}\right)$$

$$=\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}+\frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{B-C}{2}\right)}=\frac{\sin^2\left(\frac{B-C}{2}\right)+\cos^2\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{B-C}{2}\right)}$$

$$\frac{2}{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B-C}{2}\right)}=\frac{2}{\sin(B-C)}=2\operatorname{cosec}(B-C)$$

11. $\sin\theta=\frac{a}{b+c}$ అయితే $\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$ అని చూపుము

Solution : -

$$\sin\theta=\frac{a}{b+c}\Rightarrow\cos^2\theta=1-\sin^2\theta$$

$$\therefore\cos^2\theta=1-\frac{a^2}{(b+c)^2}\Rightarrow\cos^2\theta=\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$\cos^2\theta=\frac{b^2+c^2+2bc-a^2}{(b+c)^2}=\frac{(b^2+c^2-a^2)+2bc}{(b+c)^2}$$

$$=\frac{2bc\cos A+2bc}{(b+c)^2}=\frac{2bc\{1+\cos A\}}{(b+c)^2}$$

$$\cos^2\theta = \frac{2bc \times 2\cos^2\frac{A}{2}}{(b+c)^2}$$

$$\cos\theta = \frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$$

12. $a = (b+c)\cos\theta$ అయితే $\sin\theta = \frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$ అని చూపుము

Solution : -

$$a = (b+c)\cos\theta \Rightarrow \frac{a}{b+c} = \cos\theta$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\frac{a^2}{(b+c)^2} \Rightarrow \sin^2\theta = \frac{(b+c)^2 - a^2}{(b+c)^2}$$

$$\sin^2\theta = \frac{b^2+c^2+2bc-a^2}{(b+c)^2} \Rightarrow \sin^2\theta = \frac{(b^2+c^2-a^2)+2bc}{(b+c)^2}$$

$$\sin^2\theta = \frac{2bc\cos A + 2bc}{(b+c)^2} \Rightarrow \sin^2\theta = \frac{2bc\{1+\cos A\}}{(b+c)^2}$$

$$\sin^2\theta = \frac{2bc\left(2\cos^2\frac{A}{2}\right)}{(b+c)^2} \Rightarrow \sin\theta = \frac{2\sqrt{bc}}{b-c}\cos\frac{A}{2}$$

13. If $a = (b-c)\sec\theta$ అయితే $\tan\theta = \frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$ అని చూపుము

Solution : -

$$\sec\theta = \frac{a}{b-c} \Rightarrow \tan^2\theta = \sec^2\theta - 1 = \frac{a^2}{(b-c)^2} = 1$$

$$\tan^2\theta = \frac{a^2-(b-c)^2}{(b-c)^2} \Rightarrow \tan^2\theta = \frac{a^2-b^2-c^2+2bc}{(b-c)^2}$$

$$\tan^2\theta = \frac{2bc-\{b^2+c^2-a^2\}}{(b-c)^2}$$

$$\tan^2\theta = \frac{2bc - 2bc\cos A}{(b-c)^2} = \frac{2bc[1-\cos A]}{(b-c)^2}$$

$$\tan^2\theta = \frac{2bc\left(2\sin^2\frac{A}{2}\right)}{(b-c)^2} \Rightarrow \tan\theta = \frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$$

14. $a\cos\theta = b\cos(C+\theta) + c\cos(B-\theta)$ అయితే

Soluton : -

$$RHS = b\cos(c+\theta) + c\cos(B-\theta) = b\{\cos c\cos\theta - \sin c\sin\theta\}$$

$$+ c\{\cos B\cos\theta + \sin B\sin\theta\}$$

$$= b\cos c\cos\theta \; - b\sin\sin\theta + c\cos B\cos\theta + c\sin B\sin\theta$$

$$= \cos\theta\{b\cos c + c\cos B\} - \frac{bc}{2R}/\sin\theta + \frac{cb}{2R}\sin\theta = a\cos\theta$$

15. త్రిభుజంABC లోని కోణాలు A.P లో ఉ ండి మరియు

$b:c = \sqrt{3}:\sqrt{2}$ అయితే $A = 75^0$ అని చూపుము

Solution : -

త్రిభుజంABC లోని కోణాలు AP లో ఉన్నాయి.

$\therefore 2B = A + C \Rightarrow 3B = A + B + C$ but $A + B + C = 180^0$

$\therefore 3B = 180^0 \Rightarrow B = 60^0$

$b:c = \sqrt{3}:\sqrt{2}$

$2R\sin B = 2R\sin C = \sqrt{3}:\sqrt{2} \Rightarrow \sin B = \sin C = \sqrt{3} \; = \sqrt{2}$

$\sin 60^0 = \sin C = \sqrt{3}:\sqrt{2}$

$\frac{\sqrt{3}}{2}:\sin C = \sqrt{3}:\sqrt{2} = \frac{\cancel{\sqrt{3}}}{2}\times\sqrt{2} = \cancel{\sqrt{3}}\sin C$

$\Rightarrow \sin c = \frac{1}{\sqrt{2}} \Rightarrow C = 45^0$

$A + B + C = 180^0 \Rightarrow A + 60^0 + 45^0 = 180^0 \Rightarrow A = 75^0$c (proved)

16. $\frac{a^2+b^2}{a^2-b^2}=\frac{\sin C}{\sin(A-B)}$ ఐతే త్రిభుజంABC సమద్విబాహు లేదా లంబ కోణ త్రిభుజం అని చూపుము

Solution : -

Given $\frac{a^2+b^2}{(a^2-b^2)}=\frac{\sin C}{\sin(A-B)}$

$\Rightarrow (a^2+b^2)\sin(A-B)=(a^2-b^2)\sin C$

Using sine rule we have

$4R^2\{\sin^2 A+\sin^2 B\}\sin(A-B)=4R^2\{\sin^2 A-\sin^2 B\}\sin C$

$\{\sin^2 A+\sin^2 B\}\sin(A-B)-\sin(A-B)\sin(A+B)\sin C=0$

But in triangle ABC $\sin(A+B)=\sin C$

$\therefore (\sin^2 A+\sin^2 B)\sin(A-B)-\sin(A-B)(\sin C)(\sin C)=0$

$\sin(A-B)\{\sin^2 A+\sin^2 B-\sin^2 C\}=0$

$\sin(A-B)=0 \text{ or } \sin^2 A+\sin^2 B=\sin^2 C$

$A=B \text{ or } a^2+b^2=c^2$

$\therefore$ triangle either isosceles or right angled

17. $\cos A+\cos B+\cos C=\frac{3}{2}$ ఐతే త్రిభుజంABC సమబాహు త్రిభుజం అని చూపుము

Solution: -

$\cos A+\cos B+\cos C=\frac{3}{2} \Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$

$2\cos\left(90^0-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$

$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=3/2$

$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-2\sin^2\frac{C}{2}=\frac{1}{2}$

$$4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-4\sin^2\frac{C}{2}=1\Rightarrow 1+4\sin^2\frac{C}{2}-4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)=0$$

$$\left(2\sin\frac{C}{2}\right)^2-2\left(2\sin\frac{C}{2}\cos\frac{A-B}{2}\right)+\cos^2\left(\frac{A-B}{2}\right)-\cos^2\left(\frac{A-B}{2}\right)+1=0$$

$$\left\{2\sin\frac{C}{2}-\cos\left(\frac{A-B}{2}\right)\right\}^2+\sin^2\left(\frac{A-B}{2}\right)=0$$

$$\therefore 2\sin\frac{C}{2}-\cos\left(\frac{A-B}{2}\right)=0 \text{ and } \sin\frac{A-B}{2}=0$$

$$\therefore 2\sin\frac{C}{2}=\cos\left(\frac{A-B}{2}\right) \text{ and } A-B=0$$

$$\therefore 2\sin\frac{C}{2}=1\Rightarrow\frac{C}{2}=30^0\Rightarrow c=60^0$$

$A=B\therefore A=B=60^0$

Hence trangile is equilateral

18. $\cos^2 A+\cos^2 B+\cos^2 C=1$ ఐతే త్రిభుజంABC లంబ కోణ త్రిభుజం అని చూపుము

$$\cos^2 A+\cos^2 B+\cos^2 C=1\Rightarrow\cos^2 A+\cos^2 B-1+\cos^2 C=0$$

$$\cos^2 A-\sin^2 B+\cos^2 C=0\Rightarrow\cos(A-B)\cos(A-B)+\cos^2 C=0$$

$$\cos(180^0-c).\cos(A-B)+\cos^2 C=0$$

$$-\cos C\cos(A-B)+\cos C=0$$

$$-\cos C\{\cos(A-B)-\cos C\}=0$$

$$-\cos C\{\cos(A-B)-\cos C\}=0$$

$$-\cos c\{\cos(A-B)-\cos(180^0-\overline{A+B})\}=0$$

$$-\cos c\{\cos(A-B)+\cos(A+B)\}=0$$

$$-\cos c\{\cos(A-B)+\cos(A+B)\}=0$$

$$2\cos A\cos B\cos C=0$$

$$\Rightarrow\cos A=0 \text{ or } \cos B=0\ (or)\ \cos C=0$$

$$A=90^0\ (or)\ B=90^0 \text{ or } C=90^0$$

$\therefore$ Triangle is right angled triangle

19. $a^2+b^2+c^2=8R^2$ ఐతే త్రిభుజంABC లంబ కోణ త్రిభుజం అని చూపుము

Solutin : -

Given $a^2+b^2+c^2=8R^2 \Rightarrow 4R^2\{\sin^2 A+\sin^2 B+\sin^2 C\}=8R^2$

$\sin^2 A+\sin^2 B+\sin^2 C=2 \Rightarrow 1-\cos^2 A+\sin^2 B+\sin^2 C=2$

$1-\{\cos^2 A-\sin^2 B\}+1-\cos^2 C=2$

$-\cos(A-B)\cos(A-B)-\cos^2 C=0$

$\cos C\cos(A-B)-\cos^2=0 \Rightarrow \cos C\{\cos(A-B)-\cos C\}=0$

$\cos C\{\cos(A-B)+\cos(A+B)\}=0 \Rightarrow 2\cos A\cos B\cos C=0$

$\cos A=0\ or\ \cos B=0\ (or)\cos C=0 \Rightarrow A=90^0\ (or)\ B=90^0\ (or)\ C=90^0$

20. $\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}$ లు AP ఉంటే a, b,c లు AP లో ఉంటాయని చూపుము

Solution ; -

$\cot\frac{A}{2},\cot\frac{B}{2}\cot\frac{C}{2}$ are in AP

$2\cot\frac{B}{2}=\cot\frac{A}{2}+\cot\frac{C}{2} \Rightarrow \frac{2s(s-b)}{\Delta}=\frac{s(s-a)}{\Delta}+\frac{s(s-c)}{\Delta}$

$\Rightarrow 2(s-b)=(s-a)+(s-c) \Rightarrow a-b+c=2s-a-c$

$a+c-b=\cancel{a}+\cancel{b}+\cancel{c}-\cancel{a}-\cancel{c} \Rightarrow a+c=2b$

21. $\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}$ లు HP ఉంటే a, b,c లు HP లో ఉంటాయని చూపుము

Solution : -

$\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}$ are in HP

$\frac{(s-b)(s-c)}{bc},\frac{(s-c)(s-a)}{ac},\frac{(s-a)(s-b)}{ab}$ are in HP

$\frac{bc}{(s-b)(s-c)},\frac{ac}{(s-a)(s-c)},\frac{ab}{(s-a)(s-b)}$ are in HP

Multiplying with $\frac{(s-a)(s-b)(s-c)}{abc}$ we have

$$\frac{bc(s-a)(s-b)(s-c)}{abc(s-b)(s-c)}, \frac{ac(s-a)(s-b)(s-c)}{abc(s-a)(s-c)}, \frac{ab(s-a)(s-b)(s-c)}{abc(s-a)(s-b)}$$ and P

$\frac{s-a}{a}, \frac{s-b}{b}, \frac{s-c}{c}$ are in AP

Adding '1' to every term we here

$\frac{s-a}{a}+1, \frac{s-b}{b}+1, \frac{s-c}{c}+1$ are in AP

$\frac{s}{a}, \frac{s}{b}, \frac{s}{c}$ are in AP $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

a, b, c are in HP

22. $a^2 \cot A + b^2 \cot B + c^2 \cot C = \frac{abc}{R}$ అని చూపుము

Solution: -

$a^2 \cot A + b^2 \cot B + c^2 \cot C$

$4R^2 \sin^2 A \times \frac{\cos A}{\sin A} + 4R^2 \sin^2 B \frac{\cos B}{\sin B} + 4R^2 \sin^2 C \frac{\cos C}{\sin C}$

$2R^2 \{\sin 2A + \sin 2B + \sin 2C\}$

$2R^2 \{2\sin(A+B)\cos(A-B) + \sin 2C\}$

$2R^2 \{2\sin C \cos(A-B) + 2\sin C \cos C\}$

$2R^2 \left[2\sin C \{\cos(A-B) + \cos C\}\right] = 4R \sin C \{\cos(A-B) - \cos(A+B)\}$

$4R^2 \sin C \sin A \sin B = \frac{2\{2R^2 \sin A \sin B \sin C\}}{R} =$

$\frac{(2R\sin A)(2R\sin B)(2R\sin C)}{R} = \frac{abc}{R}$

23. $a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}=s+\frac{\Delta}{R}$ అని చూపుము

Solution: - $a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}$

$$\frac{a(1+\cos A)}{2}+\frac{b(1+\cos B)}{2}+\frac{c(1+\cos C)}{2}$$

$$\frac{a+a\cos A+b+b\cos B+C+c\cos C}{2}=\frac{(a+b+c)+}{2}$$

$$\frac{(a+b+c)+\{a\cos A+b\cos B+\text{c}\cos C\}}{2}$$

$$\frac{2s+2R\sin A\cos A+2R\sin B\cos B+2R\sin C\cos C}{2}$$

$$\frac{2S+R(\sin 2A+\sin 2B+\sin 2C)}{2}$$

$$\frac{2S+R\{2\sin(A+B)\cos(A-B)+\sin 2C\}}{2}$$

$$\frac{2S+R\{2\sin C\cos(A-B)+2\sin C\cos C\}}{2}$$

$$\frac{2S+2R\sin C\{\cos(A-B)+\cos C\}}{2}$$

$$\frac{2S+2R\sin C\{\cos(A-B)-\cos(A+B)\}}{2}$$

$$\frac{2S+4R\sin A\sin B\sin C}{2}=S+2\text{R}\sin A\sin B\sin C$$

$$\frac{S+2R^2\sin A\sin B\sin C}{R}=S+\frac{\Delta}{R}$$

24. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}$ అని చూపుము

.

Sol. L.H.S. = $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$

$$=\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\frac{1-\cos C}{2}$$

$$=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)$$

$$=\frac{3}{2}-\frac{1}{2}\left[1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

($\because$ from transformations)

$$=\frac{3}{2}-\frac{1}{2}\left[1+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right]$$

$$=\frac{3}{2}-\frac{1}{2}\left[1+\frac{r}{R}\right]$$

$$=\frac{3}{2}-\frac{1}{2}-\frac{r}{2R}=1-\frac{r}{2R}=\text{R.H.S.}$$

25. i. $\Delta=r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$. ii. $a=(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}$ అని చూపుము

Sol. i)R.H.S. $r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$

$$=r_1r_2\sqrt{\frac{4R-(r_1+r_2)}{r_1+r_2}}$$

$$\left(\because r_1+r_2=4R\cos^2\frac{C}{2}\right)$$

$$=r_1r_2\sqrt{\frac{4R\left(1-\cos^2\frac{C}{2}\right)}{4R\cos^2\frac{C}{2}}}$$

$$= r_1 r_2 \sqrt{\frac{\sin^2 \frac{C}{2}}{\cos^2 \frac{C}{2}}} = r_1 r_2 \tan\frac{C}{2}$$

$$= \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \tan\frac{C}{2}$$

$$= \frac{\Delta^2}{(S-a)(S-b)} \sqrt{\frac{(S-b)(S-a)}{S(S-c)}}$$

$$= \frac{\Delta^2}{\sqrt{S(S-a)(S-b)(S-c)}} = \frac{\Delta^2}{\Delta} = \Delta = \text{R.H.S.}$$

ii) RHS $= (r_2 + r_3)\sqrt{\frac{rr_1}{r_2 r_3}} = \left(\frac{\Delta}{s-b} + \frac{\Delta}{s-c}\right)\sqrt{\frac{\frac{\Delta^2}{s(s-a)}}{\frac{\Delta^2}{(s-b)(s-c)}}}$

$$= \frac{\Delta\{s-c+s-b\}}{(s-b)(s-c)} \sqrt{\frac{\Delta^2}{s(s-a)} \cdot \frac{(s-b)(s-c)}{\Delta^2}}$$

$$= \frac{\Delta.(2s-b-c)}{\left\{\sqrt{(s-b)(s-c)}\right\}^2} \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}} = \frac{\Delta\{a+b+c-b-c\}}{\sqrt{s(s-a)(s-b)(s-c)}} = a$$

26. $r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 - (a^2 + b^2 + c^2)$ అని చూపుము

Sol.

$(r_1 + r_2 + r_3 - r)^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1 r_2 + r_2 r_3 + r_3 r_1)$ ...(1)

But $[r_1 + r_2 + r_3 - r] = 4R$ and $r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2$

$16R^2 = [r_1 + r_2 + r_3 - r]^2$

$16R^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1 r_2 + r_2 r_3 + r_3 r_1)$

$= r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2S^2$

$16R^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2(rr_1 + rr_2 + rr_3) + 2S^2$ Consider $2(rr_1 + rr_2 + rr_3) =$

$$= 2\left[\frac{\Delta^2}{S(S-a)} + \frac{\Delta^2}{S(S-b)} + \frac{\Delta^2}{S(S-c)}\right]$$

$$= 2\Delta^2 \frac{[(S-b)(S-c)+(S-a)(S-c)+(S-a)(S-b)]}{S(S-a)(S-b)(S-c)}$$

$$= \frac{2\Delta^2}{\Delta^2}\left[S^2 - Sc - Sb + bc + S^2 - Sc - Sa + ac + S^2 - Sb - Sa + ab\right]$$

$$= 2\left[3S^2 - 2S(a+b+c) + ab + bc + ca\right]$$

$$= 2\left[3S^2 - 4S^2 + ab + bc + ca\right]$$

$$= 2\left[ab + bc + ca - S^2\right]$$

$$= 2\left[ab + bc + ca\right] - 2S^2$$

From (2)

$$\Rightarrow r_1^2 + r_2^2 + r_3^2 + r^2$$

$$= 16R^2 + 2r(r_1 + r_2 + r_3) - 2S^2$$

$$= 16R^2 - 2S^2 + 2(ab + bc + ca) - 2S^2$$

$$= 16R^2 - 4S^2 + 2(ab + bc + ca)$$

$$= 16R^2 - 4\left(\frac{a+b+c}{2}\right)^2 + 2(ab + bc + ca)$$

$$= 16R^2 - \left[(a+b+c)^2 - 2(ab + bc + ca)\right]$$

$$= 16R^2 - (a^2 + b^2 + c^2)$$

27. P_1, P_2, P_3 లు శీర్షాల నుండి భుజాలకు గీసిన ఉన్నతులు అయితే

(i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r}$ (ii) $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{1}{r_3}$

(iii) $P_1P_2P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta^3}{abc}$ అని చూపుము

Sol.

$$\Delta = \frac{1}{2}aP_1, \Delta = \frac{1}{2}bP_2, \Delta = \frac{1}{2}cP_3$$

$$\Rightarrow P_1 = \frac{2\Delta}{a}, P_2 = \frac{2\Delta}{b} \text{ and } P_3 = \frac{2\Delta}{c}$$

i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta}$

$$= \frac{a+b+c}{2\Delta} = \frac{2S}{2\Delta} = \frac{1}{r}$$

ii) $\frac{1}{P_1}+\frac{1}{P_2}-\frac{1}{P_3}=\frac{a}{2\Delta}+\frac{b}{2\Delta}-\frac{c}{2\Delta}$

$$=\frac{a+b-c}{2\Delta}=\frac{2S-2c}{2\Delta}=\frac{S-c}{\Delta}=\frac{1}{r_3}$$

iii) $P_1P_2P_3=\frac{2\Delta}{a}\cdot\frac{2\Delta}{b}\cdot\frac{2\Delta}{c}=\frac{8\Delta^3}{abc}$

28. $a=13$ $b=14$ $c=15$ ఐతే $R=\frac{65}{8}$ $r=4$ $r_1=\frac{21}{2}$ $r_2=12$ *and* $r_3=14$ అని చూపుము

Sol. a = 13, b = 14, c = 15

$$S=\frac{a+b+c}{2}=\frac{13+14+15}{2}=\frac{42}{2}=21$$

$$S-a=21-13=8, S-b=21-14=7$$

$$S-c=21-15=6$$

$$\Delta=\sqrt{S(S-a)(S-b)(S-c)}$$

$$=\sqrt{21(8)(7)(6)}=\sqrt{21\times16\times21}$$

$$=\sqrt{21\times21\times4\times4}=21\times4=84$$

$$R=\frac{abc}{4\Delta}=\frac{13\times14\times15}{4\times84}=\frac{65}{8}$$

$$r=\frac{\Delta}{S}=\frac{84}{21}=4$$

$$r_1=\frac{\Delta}{S-a}=\frac{84}{8}=\frac{21}{2}$$

$$r_2=\frac{\Delta}{S-b}=\frac{84}{7}=12$$

$$r_3=\frac{\Delta}{S-c}=\frac{84}{6}=14$$

29. $r_1 = 2\ r_2 = 3\ r_3 = 6$ ఐతే $r = 1\ \ a = 3\ \ b = 4\ c = 5$ అని చూపుము

Sol. $\Delta^2 = rr_1r_2r_3 = 1\cdot 2\cdot 3\cdot 6 = 36$

$$\Delta^2 = 36 \Rightarrow \Delta = 6$$

$$r = \frac{\Delta}{S} = \frac{6}{S} \Rightarrow S = 6 \qquad (\because r = 1)$$

$$r_1 = \frac{\Delta}{S-a} \Rightarrow S - a = \frac{\Delta}{r_1}$$

$$\therefore a = S - \frac{\Delta}{r_1} = 6 - \frac{6}{2} = 6 - 3 = 3$$

$$r_2 = \frac{\Delta}{S-b} \Rightarrow S - b = \frac{\Delta}{r_2}$$

$$\therefore b = S - \frac{\Delta}{r_2} = 6 - 3 = 3$$

$$r_3 = \frac{\Delta}{S-c} \Rightarrow S - c = \frac{\Delta}{r_3}$$

$$\therefore c = S - \frac{\Delta}{r_3} = 6 - 1 = 5$$

30. $a^2\cot A + b^2\cot B + c^2\cos C = \frac{abc}{R}$ అని చూపుము

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Sol. L.H.S. $a^2\cot A + b^2\cot B + c^2\cos C$

$$= 4R^2\sin^2 A\frac{\cos A}{\sin A} + 4R^2\sin^2 B\frac{\cos B}{\sin B} \qquad +4R^2\sin^2 C\frac{\cos C}{\sin C}\text{(by sine rule)}$$

$$= 2R^2(2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C)$$

$$= 2R^2(\sin 2A + \sin 2B + \sin 2C)$$

$$= 2R^2(4\sin A\sin B\sin C)$$

$$= \frac{1}{R}(2R\sin A)(2R\sin B)(2R\sin C)$$

$$= \frac{abc}{R} = \text{R.H.S.}$$

31. ΔABC లో $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ ఐతే C = 60° అని చూపుము

Sol. $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$

$\Rightarrow \frac{b+c+a+c}{(a+c)(b+c)}=\frac{3}{a+b+c}$

$\Rightarrow 3(a+c)(b+c)=(a+b+2c)(a+b+c)$

$\Rightarrow 3(ab+ac+bc+c^2)$

$=(a^2+b^2+2ab)+3c(a+b)+2c^2$

$\Rightarrow ab=a^2+b^2-c^2=2ab\cos C$

(from cosine rule)

$\Rightarrow \cos C=\frac{1}{2}\Rightarrow C=60°$

32. $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$ అని చూపుము

Sol. L.H.S. = $\Sigma \cot A=\Sigma\frac{\cos A}{\sin A}$

$=\Sigma\left(\frac{b^2+c^2-a^2}{2bc\sin A}\right)$(by cosine rule)

$=\Sigma\frac{b^2+c^2-a^2}{4\Delta}\left[\because \Delta=\frac{1}{2}bc\sin A\right]$

$=\frac{1}{4\Delta}[b^2+c^2-a^2+c^2+a^2$

$-b^2+a^2+b^2-c^2]$

$=\frac{a^2+b^2+c^2}{4\Delta}=$ R.H.S.

33. ΔABC లో a cos A = b cos B ఐతే ఆ త్రిభుజం సమద్విబాహు త్రిభుజం లేదా లంబకోణ త్రిభుజం అని చూపుము

Sol. a cos A = b cos B

$\Rightarrow$ 2R sin A cos A = 2R sin B cos B

$\Rightarrow \sin 2A = \sin 2B = \sin(180° - 2B)$

Hence $2A = 2B$ or $2A = 180° - 2B$

$\Rightarrow A = B$ or $A = (90° - B)$

$\Rightarrow a = b$ or $(A + B) = 90°$

$\Rightarrow a = b$ or $C = 90°$

$\therefore$ The triangle is isosceles or right angled.

34. $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3:5:7$ ఐతే $a:b:c = 6:5:4$ అని చూపుము

Sol. $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3:5:7$

$$\Rightarrow \frac{s(s-a)}{\Delta} \cdot \frac{s(s-b)}{\Delta} \cdot \frac{s(s-c)}{\Delta} = 3:5:7$$

$$\Rightarrow (s-a):(s-b):(s-c) = 3:5:7$$

$$\Rightarrow \frac{s-a}{3} = \frac{s-b}{5} = \frac{s-c}{7} = k \text{ (say)}$$

Then $s - a = 3k$, $s - b = 5k$, $s - c = 4k$

Adding these equations,

$3s - (a + b + c) = 3k + 5k + 7k = 15k$

$\Rightarrow 3s - 2s = 15k \Rightarrow s = 15k$

Hence $a = 12k$, $b = 10k$, $c = 8k$

$\therefore a : b : c = 12k : 10k : 8k = 6 : 5 : 4$

35. $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$ అని చూపుము

Sol. L.H.S. $= \Sigma a^3\cos(B-C)$

$= \Sigma a^2(2R\sin A)\cos(B-C)$

$= R\Sigma a^2 \cdot [2\sin(B+C)\cos(B-C)]$

$= R\Sigma a^2(\sin 2B + \sin 2C)$

$= R\Sigma a^2(2\sin B\cos B + 2\sin C\cos C)$

$= \Sigma[a^2(2R\sin B)\cos B + a^2(2R\sin C)\cos C]$

$= \Sigma(a^2b\cos B + a^2c\cos C)$

$= (a^2b\cos B + a^2c\cos C) + (b^2c\cos C + b^2a\cos A) + (c^2a\cos A + c^2b\cos B)$

$= ab(a\cos B + b\cos A) + bc(b\cos C + c\cos B) + ca(c\cos A + a\cos C)$

$= ab(c) + bc(a) + ca(b)$

$= 3abc = \text{R.H.S.}$

36. P_1, P_2, P_3 లు శీర్షాల నుండి భుజాలకు గీసిన ఉన్నతులు అయితే

$$\frac{1}{P_1^2} + \frac{1}{P_2^2} + \frac{1}{P_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta^2}$$ అని చూపుము

Sol. P_1, P_2, P_3 లు శీర్షాల నుండి భుజాలకు గీసిన ఉన్నతులు కావున

$\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$\Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$

Now $\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{a^2+b^2+c^2}{4\Delta^2}$

$= \frac{1}{\Delta}(\cot A + \cot B + \cot C) = \text{R.H.S.}$

$[\because \cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}]$

37. త్రిభుజంABC లో $(r_2-r_1)(r_3-r_1) = 2r_2r_3$ ఐతShow that A = 90° అని చూపుము

.

Sol. $(r_2 - r_1)(r_3 - r_1) = 2r_2r_3$

$$\Rightarrow \left[\frac{\Delta}{(s-b)} - \frac{\Delta}{(s-a)}\right]\left[\frac{\Delta}{(s-c)} - \frac{\Delta}{(s-a)}\right]$$

$$= 2\frac{\Delta}{(s-b)}\frac{\Delta}{(s-c)}$$

$$\Rightarrow \Delta\left[\frac{s-a-s+b}{(s-b)(s-a)}\right]\cdot\Delta\left[\frac{s-a-s+c}{(s-c)(s-a)}\right]$$

$$= \frac{2\Delta^2}{(s-b)(s-c)}$$

$$\Rightarrow (b-a)(c-a) = 2(s-a)^2$$

$$\Rightarrow (b-a)(c-a) = 2\left(\frac{b+c-a}{2}\right)^2$$

$$\Rightarrow 2(bc - ca - ab + a^2)$$

$$= b^2 + c^2 + a^2 + 2bc - 2ca - 2ab$$

$$\Rightarrow 2a^2 = b^2 + c^2 + a^2$$

$$\Rightarrow b^2 + c^2 = a^2$$

Hence ΔABC is right angled and A = 90°.

38. త్రిభుజంABC లో $\sum (r + r_1)\tan\left(\frac{B-C}{2}\right) = 0$ అని చూపుము

Solution: -

$$r_1 r_2 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\sum (r_1 + r)\tan\left(\frac{B-C}{2}\right) = \sum 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$$

$$\sum 3R\,2\sin\left(90^0 - \frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)$$

$$\sum 2R\left\{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)\right\}$$

$$\sum 2R\{\sin B - \sin C\} = \sum 2R\sin B - 2R\sin C$$

$$\sum b-c = b-c+c-a+a-b=0$$

39. $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$ అని చూపుము

Sol. L.H.S. $= \frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$

$$=\frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}$$

$$=\frac{1}{\Delta^2}\left[s^2+(s-a)^2+(s-b)^2+(s-c)^2\right]$$

$$=\frac{1}{\Delta^2}\left[s^2+s^2-2as+a^2+s^2-2bs+b^2+s^2-2cs+c^2\right]$$

$$=\frac{1}{\Delta^2}\left[4s^2-2s(a+b+c)+a^2+b^2+c^2\right]$$

$$=\frac{1}{\Delta^2}\left[4s^2-2s(2s)\right]+\frac{a^2+b^2+c^2}{\Delta^2}$$

$$=\frac{a^2+b^2+c^2}{\Delta^2}=\text{R.H.S.}$$

40. $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$ అని చూపుము

Sol. L.H.S. $= \frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{abc}[ar_1+br_2+cr_3]$

$$=\frac{1}{abc}\left[\Sigma a\cdot s\tan\frac{A}{2}\right]=\frac{s}{abc}\Sigma 2R\sin A\tan\frac{A}{2} \qquad \left(\because \Delta=\frac{abc}{4R}\right)$$

$$=\frac{s}{abc}\Sigma\left[2R\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}\cdot\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\right)\right](\because r=\Delta/s)$$

$$=4\frac{Rs}{abc}\Sigma\left(\sin^2\frac{A}{2}\right)=\frac{s}{\Delta}\Sigma\left(\frac{1-\cos A}{2}\right)$$

$$= \frac{1}{2r}(1 - \cos A + 1 - \cos B + 1 - \cos C)$$

$$= \frac{1}{2r}[3 - (\cos A + \cos B + \cos C)]$$

$$= \frac{1}{2r}\left[3 - \left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\right]$$

$$= \frac{1}{2r}\left[2 - \left(\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right)\right]$$

$$= \frac{1}{2r}\left[2 - \frac{r}{R}\right] = \frac{1}{r} - \frac{1}{2R} = \text{R.H.S.}$$

41. $r : R : r_1 = 2 : 5 : 12$ ఐతే $A = 90^0$ అని చూపుము

Sol. If $r : R : r_1 = 2 : 5 : 12$, then $r = 2k$, $R = 5k$ and $r_1 = 12k$

$$r_1 - r = 12k - 2k = 10k = 2(5k) = 2R$$

$$\Rightarrow 4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right] = 2R$$

$$\Rightarrow 2\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right) = 1 \Rightarrow \sin^2\frac{A}{2} = \frac{1}{2}$$

$$\left[\because \cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}\right]$$

$$\Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{2}} = \sin 45°$$

$$\Rightarrow \frac{A}{2} = 45° \Rightarrow A = 90°$$

త్రిభుజం ఒక లంబ కోణ త్రిభుజం.

42. $r + r_3 + r_1 - r_2 = 4R\cos B$ అని చూపుము

Sol. $r + r_3$

$$= 4R\sin\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2} + \cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$= 4R\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)$$

$r_1 - r_2$

$$= 4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4R\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)$$

$\therefore\ r + r_3 + r_1 - r_2$

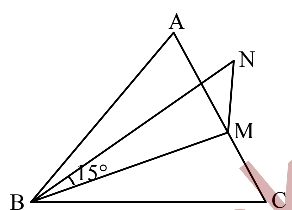
$$= 4R\left[\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + \cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 4R\sin\left(\frac{C}{2} + \frac{A}{2} - \frac{B}{2}\right) = 4R\sin\left(90° - \frac{B}{2} - \frac{B}{2}\right)$$

$$= 4R\cos B$$

42. ఒక త్రిభుజాకార స్థలం ABC లో మధ్యబిందువు M వద్ద ఒక దీపస్తంభం ఉంది. BC = 7 m, CA = 8 m , AB = 9 m ఇంకా B వద్ద దీప స్తంభం చేసే కోణం 15° ఐతే దీపస్తంభం ఎత్తు ఎంత?

Sol.



MN = దీపస్తంభం ఎత్తు

MN = h (?)

$\angle NBM = 15°$

ΔABC లో $\cos C = \dfrac{b^2 + c^2 - a^2}{2abc}$

$$= \frac{64 + 49 - 81}{2\times 8\times 7} = \frac{16\times 2}{16\times 7} = \frac{32}{112} = \frac{2}{7}$$

$$\therefore \cos C = \frac{2}{7}$$

BM = x

ΔBCM లో, $\cos C = \dfrac{7^2 + 4^2 - x^2}{2\times 7\times 4}$

$$\frac{2}{7} = \frac{49+16-x^2}{7\times 8}$$

$$16 = 65 - x^2$$

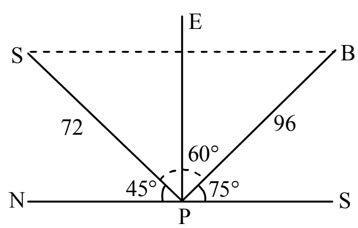
$$x^2 = 65 - 16 \Rightarrow x = 7$$

ΔBMN లో $\tan 15° = \frac{h}{x}$

$$h = x\tan 15° = 7(2-\sqrt{3})$$

43. ఒక రేవు వద్ద రెండు ఓడలుఒకే సమయంలో బయలుదేరాయి. ఒకటి గంటకు 24 km వేగం తో N 45°E దిశ లో మరొకటి 32 km వేగంతో S 75° E దిశ లో ప్రయణిస్తే 3 గంటల తర్వాత ఓడల మధ్య దూరం ఎంత?

Sol.



P is the position of the port.

A is the position of the North-East traveled ship after 3 hours is = 72 km

Position of the South-East traveled ship after 3 hours is 3 × 32 = 96 km

From the data $\angle APB = 60°$

In ΔAPB,

$$\cos P = \frac{AP^2 + BP^2 - AB^2}{2AP\,BP}$$

$$\cos 60° = \frac{(72)^2 + (96)^2 - AB^2}{2\times 72\times 96}$$

$$\frac{1}{2} = \frac{72^2 + 96^2 - AB^2}{2\times 72\times 96}$$

$$1 = \frac{5184 + 9216 - AB^2}{72\times 96}$$

$$1 = \frac{14400 - AB^2}{6912}$$

$$6912 = 14400 - AB^2$$

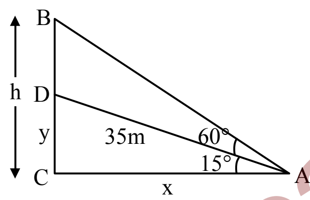
$$AB^2 = 14400 - 6912$$

$$AB^2 = 7488$$

$$AB = \sqrt{7488} = 86.53 = 86.4\,\text{km}$$

31. A tree stands vertically on the slant of the hill. From a point A on the ground 35 meters down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. If the angle of elevation of the foot of the tree from A is 15°, then find the height of the tree.

Sol.



BD is the height of the tree and A is the point of observation.

Let CD = y

AC = x

Given that, ∠CAD = 15°, ∠CAB = 60° and AD = 35 m.

In ΔCAD, $\sin 15° = \dfrac{y}{35}$

$$y = 35\sin 15° = \frac{35(\sqrt{3}-1)}{2\sqrt{2}} \qquad \ldots(1)$$

$$\cos 15° = \frac{x}{35}$$

$$x = \frac{\sqrt{3}+1}{2\sqrt{2}} \times 35 \qquad \ldots(2)$$

In ΔCAB, $\tan 60° = \dfrac{h}{x}$

$$h = x\sqrt{3} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times 35 \times \sqrt{3}$$

Height of the tree = h – y

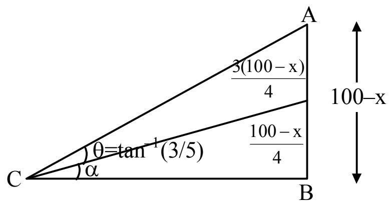
$$\frac{\sqrt{3}+1}{2\sqrt{2}}\times 35\sqrt{3}-\frac{\sqrt{3}-1}{2\sqrt{2}}\times 35=$$

$$=\frac{\sqrt{3}+1}{2\sqrt{2}}\left[3+\sqrt{3}-\sqrt{3}+1\right]$$

$$=\frac{35\times 4}{2\sqrt{2}}=35\sqrt{2}\text{ m}$$

32. The upper 3/4th portion of a vertical pole subtends an angle $Tan^{-1}3/5$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. Given that the vertical pole is at a height less than 100 m from the ground, find its height.

Sol.



AB is the height of the tree.

AD is the 3/4th part of upper part of the tree.

DB is the 1/4th lower part of the tree.

Let AB = 100 – x

C is the point of observation.

In ΔBCD,

Let $\angle DCA:\ \theta=\tan^{-1}\frac{3}{5}\Rightarrow \tan\theta=\frac{3}{5}$

$$\tan\alpha=\frac{100-x}{4}\times\frac{1}{40}=\frac{100-x}{160}$$

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\frac{100-x}{40}=\frac{\frac{3}{5}+\frac{100-x}{160}}{1-\frac{3}{5}\times\frac{100-x}{160}}$$

$$\frac{100-x}{40}=\frac{480+5(100-x)}{800-3(100-x)}$$

$$\frac{100-x}{40}=\frac{480+500-5x}{800-300+3x}$$

$$\frac{100-x}{40}=\frac{980-5x}{500+3x}$$

$$[100-x][500+3x]=40[980-5x]$$

$$50000+300x-500x-3x^2=39200-200x$$

$$\Rightarrow 3x^2+500x-400x=50000-39200$$

$$3x^2=10800$$

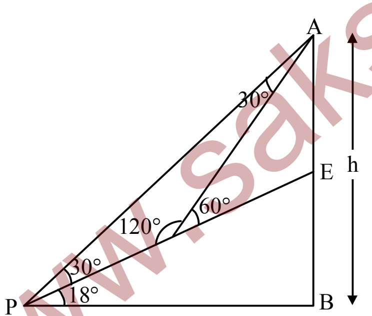
$$x^2=\frac{10800}{3}=3600$$

$$x=\sqrt{3600}=60$$

Height of the tree = 100 – x = 40 m.

33. Let an object be placed at some height h cm and let P and Q be two points of observation which are at a distance 10 cm apart on a line inclined at angle 15° to the horizontal. If the angles of elevation of the object from P and Q are 30° and 60° respectively then find h.

Sol.



A is the position of the object.

Given that AB = h cm

P and Q are points of observation.

Given that, PQ = 10 cm

We have,

$\angle BPE = 15°$, $\angle EPA = 30°$, $\angle EQA = 60°$

In ΔPQA,

P = 30°, Q = 120° and A = 30°

$\therefore$ By sine rule,

$$\frac{AP}{\sin 120^\circ} = \frac{PQ}{\sin 30^\circ}$$

$$\frac{AP}{\sin(180^\circ - 60^\circ)} = \frac{10}{1/2}$$

$$\frac{AP}{\sin 60^\circ} = 20^\circ \Rightarrow \frac{AP}{\sqrt{3}/2} = 20^\circ$$

$$AP = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$

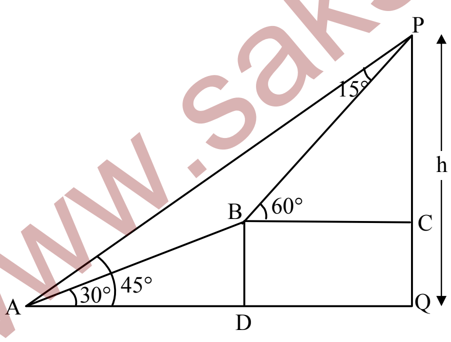
In ΔPBA, $\sin 45^\circ = \frac{AB}{AP}$

$$\frac{1}{\sqrt{2}} = \frac{h}{10\sqrt{3}}$$

$$h = \frac{10\sqrt{3}}{\sqrt{2}} = \frac{5 \cdot 2 \cdot \sqrt{3}}{\sqrt{2}} = 5\sqrt{2}\sqrt{3} = 5\sqrt{6}\,\text{cm}$$

34. The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45° and from a point B is 60°, where B is a point at a distance 30 meters from the point A measured along the line AB which makes an angle 30° with AQ. Find the height of the tower.

Sol.



In the figure

PQ = h, ∠PAQ = 45°

∠BAQ = 30° and ∠PBC = 60°

Also, AB = 30 m

$\therefore$ ∠BAP = ∠APB = 15°

This gives, BP = AB = 30 and

h = PC + CD = BP sin 60° + AB sin 30°

$= 15\sqrt{3} + 15 = 15(\sqrt{3}+1)$ meters.

Theorem : - - త్రిభుజం ABC లో

(i) $\sin\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$ (ii) $\cos\frac{A}{2} = \sqrt{\frac{S(S-a)}{bc}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(S-b)(S-C)}{S(S-a)}} = \frac{\Delta}{S(S-a)} = \frac{(S-b)(S-C)}{\Delta}$

(iv) $\cot A/2 = \sqrt{\frac{S(S-a)}{(S-b)(S-c)}} = \frac{\Delta}{(S-b)(S-c)} = \frac{S(S-a)}{\Delta}$

Proof (i) కొసైన్ సూత్రం నుండి

$$a^2+b^2+c^2 = 2bc\cos A \Rightarrow 2bc\cos = b^2+c^2-a^2$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$2\sin^2\frac{A}{2} = 1-\cos A$$

$$= 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$\therefore 2\sin^2\frac{A}{2} = \frac{a^2-\{b^2+c^2-2bc\}}{2bc} \Rightarrow \sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc}$$

$$\sin^2\frac{A}{2} = \frac{(a+b-c)(a-b+c)}{4bc}$$

$\because a+b+c = 2S$ we have $2S-2C = a+b-c$

$$\therefore \sin^2\frac{A}{2} = \frac{\cancel{2}(S-C)\cancel{2}(S-b)}{\cancel{4}bc} \Rightarrow \sin\frac{A}{2} = \sqrt{\frac{(S-b)(S-C)}{bc}}$$

Proof (ii)

$$2\cos^2\frac{A}{2} = 1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc}$$

$$2\cos^2\frac{A}{2} = \frac{2bc+b^2+c^2-a^2}{2bc} \Rightarrow 2\cos^2\frac{A}{2}\frac{(b+c)^2-a^2}{2bc}$$

$$\cos^2\frac{A}{2} = \frac{(b+c-a)(b+c+a)}{4bc}$$

$a+b+c=2S$; $2S-2a=b+c-a$

$$\therefore \cos^2\frac{A}{2}=\frac{2(S-a)2S}{4bc} \Rightarrow \cos\frac{A}{2}=\sqrt{\frac{S(S-a)}{bc}}$$

Proof (iii) $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{\frac{(S-b)(S-c)}{b-c}}{\frac{S(S-a)}{b-c}}}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

$$\tan\frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}\frac{(S-b)(S-c)}{(S-b)(S-c)}}=\frac{(S-b)(S-c)}{\sqrt{S(S-a)(S-b)(S-c)}}=\frac{(S-b)(S-c)}{\Delta}$$

$$\tan\frac{A}{2}\sqrt{\frac{(S-b)(S-c)}{S(S-a)}\times\frac{S(S-a)}{S(S-a)}}=\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-a)}=\frac{\Delta}{S(S-a)}$$

Proof of (iv)

$\tan A/2$ యొక్క వ్యుత్క్రమ విలువను తీసుకుంటే $\cot A/2$ విలువ వస్తుంది.

$\sin\frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}}$ $\quad \sin B/2=\sqrt{\frac{(S-c)(S-a)}{ac}}$ $\quad \sin\frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$

$\cos\frac{A}{2}=\sqrt{\frac{S(S-a)}{bc}}$ $\quad \cos\frac{B}{2}=\sqrt{\frac{S(S-b)}{ac}}$ $\quad \cos\frac{C}{2}\sqrt{\frac{S(S-c)}{ab}}$

$$\tan\frac{A}{2}=\sqrt{\frac{(S-b)(S-C)}{S(S-a)}}=\frac{\Delta}{S(S-a)}=\frac{(S-b)(S-c)}{\Delta}$$

$$\tan\frac{B}{2}=\sqrt{\frac{(S-c)(S-a)}{S(S-b)}}=\frac{\Delta}{S(S-b)}=\frac{(S-c)(S-a)}{\Delta}$$

$$\tan\frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}=\frac{\Delta}{S(S-c)}=\frac{(S-c)(S-a)}{\Delta}$$

$$\cot\frac{A}{2}=\sqrt{\frac{S(S-a)}{(S-b)(S-c)}}=\frac{\Delta}{(S-b)(S-c)}=\frac{S(S-a)}{\Delta}$$

$$\cot\frac{B}{2}=\sqrt{\frac{S(S-b)}{(S-a)(S-c)}}=\frac{\Delta}{(S-a)(S-c)}=\frac{S(S-b)}{\Delta}$$

$$\cot\frac{C}{2}=\sqrt{\frac{S(S-C)}{(S-a)(S-b)}}=\frac{\Delta}{(S-a)(S-b)}=\frac{S(S-C)}{\Delta}$$