MATHEMATICS PAPER - 1B

COORDINATE GEOMETRY (2D &3D) AND CALCULUS.

TIME: 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A, B and C.

SECTION A

Very short answer type questions. **10x2 =20**

1. Find the condition for the points (a, 0), (h, k) and (0,b)where $ab \neq 0$ to be collinear.

2. Find the angle between the following straight lines.

3. Find the distance between the midpoint of the line segment $\overline{AB}$ and the point (3,–1, 2) where A = (6, 3, –4) and B = (–2, –1, 2).

4. Find the equation of the plane passing through the point (1, 1, 1) and parallel to the plane x + 2y + 3z – 7 = 0.

5. $\underset{x\to 0}{Lt} \dfrac{\sin(a+bx)-\sin(a-bx)}{x}$

6. $\underset{x\to 0}{Lt} \left[\dfrac{3^x-1}{\sqrt{1+x}-1}\right]$

7. $y = \log_7(\log x)(x>0)$

8. If $y = \cot^{-1}(\operatorname{cosec} 3x)$ find $\dfrac{dy}{dx}$.

9. The time t of a complete oscillation of a simple pendulum of length l is given by the equation t = where g is gravitational constant. Find the approximate percentage error in the calculated g, corresponding to an error of 0.01 percent is the value of t.

10. Verify Rolle's theorem for the following functions. $x^2 - 1$ on $[-1, 1]$

SECTION-B

Short answer type questions

Answer any five of the following. **5 X 4 = 20**

11. A(5, 3) and B(3, –2) are two fixed points. Find the equation of locus of P, so that the area of triangle PAB is 9.

12. Show that the axes are to be rotated through an angle of $\frac{1}{2}\text{Tan}^{-1}\left(\frac{2h}{a-b}\right)$ so as to remove the xy term from the equation $ax^2 + 2hxy + by^2 = 0$, if $a \neq b$ and through the angle $\frac{\pi}{4}$, if a = b

13. A straight line parallel to the line $y = \sqrt{3}x$ passes through $Q(2,3)$ and cuts the line $2x + 4y - 27 = 0$ at P. Find the length of PQ.

14. Evaluate $\underset{x \to a}{Lt}\left[\frac{\left(\sqrt{a+2x} - \sqrt{3x}\right)}{\left(\sqrt{3a+x} - 2\sqrt{x}\right)}\right]$

15. A balloon which always remains spherical on inflation is being inflated by pumping on 900 cubic centimeters of gas per second. Find the rate at which the radius of balloon increases when the radius in 15 cm.

16. Find the angle between the curve $\mathbf{2y = e^{\frac{-x}{2}}}$ and y-axis.

17. If f and g are two differentiable functions at x then the product function f.g is differentiable at x and

(fg)' (x) = f '(x). g(x) + f(x) . g'(x)

SECTION-C

Long answer type questions
Answer any five of the following. **5 x 7 = 35**

18. Find the equation of the straight lines passing through the point (-3, 2) and making an angle of $45°$ with the straight line 3x – y + 4 =0.

19. Write down the equation of the pair of straight lines joining the origin to the points of intersection of the lines 6x – y + 8 = 0 with the pair of straight lines $3x^2 + 4xy - 4y^2 - 11x + 2y + 6 = 0$. Show that the lines so obtained make equal angles with the coordinate axes.

20. Show that the straight lines represented by $3x^2 + 48xy + 23y^2 = 0$ and 3x – 2y + 13 =0 form an equilateral triangle of area $\frac{13}{\sqrt{3}}$ sq. units.

21. If a ray makes angle α, β, γ and δ with the four diagonals of a cube find $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

22. If $y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ for 0 < |x| < 1, find $\frac{dy}{dx}$.

23. If the tangent at any point on the curve $\mathbf{x^{2/3} + y^{2/3} = a^{2/3}}$ intersects the coordinate axes in A, B show that the length AB is constant,

24. Show that when curved surface area of a cylinder inscribed in a sphere of radius R is a maximum, then the height of the cylinder is $\sqrt{2}\,R$.

SECTION-A

1. Find the condition for the points (a, 0), (h, k) and (0, b) where $ab \neq 0$ to be collinear.

Sol. A(a, 0), B(h, k), C(0, b) are collinear.

$\Rightarrow$ Slope of AB = Slope of AC

$$\frac{k-0}{h-a} = \frac{-b}{a} \Rightarrow ak = -bh + ab$$

$$bh + ak = ab \Rightarrow \frac{h}{a} + \frac{k}{b} = 1$$

2. Find the angle between the following straight lines

$\mathbf{y} = -\sqrt{3}x + 5, y = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$

Sol. slope of 1st line is $m_1 = -\sqrt{3}$

Slope of 2nd line is $m_2 = \frac{1}{\sqrt{3}}$. $m_1 m_2 = \left(-\sqrt{3}\right)\frac{1}{\sqrt{3}} = -1.$

The lines are perpendicular $\theta = \frac{\pi}{2}$

3. Find the distance between the midpoint of the line segment $\overrightarrow{AB}$ and the point (3,–1, 2) where A = (6, 3, –4) and B = (–2, –1, 2).

Sol. Given points are

A= (6, 3, –4), B = (–2, –1, 2)

Coordinates of Q are :

$$\left(\frac{6-2}{2}, \frac{3-1}{2}, \frac{-4+2}{2}\right) = (2,1,-1)$$

Coordinates of P are: (3, –1, 2)

$$PQ = \sqrt{(3-2)^2 + (-1-1)^2 + (2+1)^2}$$

$$= \sqrt{1+4+9} = \sqrt{14} \text{ units.}$$

4. Find the equation of the plane passing through the point (1, 1, 1) and parallel to the plane x + 2y + 3z – 7 = 0.

Sol. Equation of the given plane is

$$x + 2y + 3z - 7 = 0$$

Equation of the parallel plane is

$$x + 2y + 3z = k$$

This plane passing through the point P(1, 1, 1)

$\Rightarrow 1 + 2 + 3 = k \Rightarrow k = -6$

Equation of the required plane is

$$x + 2y + 3z = 6$$

5. $\underset{x\to 0}{Lt} \dfrac{\sin(a+bx)-\sin(a-bx)}{x}$

Sol: $\underset{x\to 0}{Lt} \dfrac{\sin(a+bx)-\sin(a-bx)}{x} = \underset{x\to 0}{Lt} \dfrac{2\cos a.\sin bx}{x} = \underset{x\to 1}{Lt}\, 2\cos a.\ \underset{x\to 0}{Lt} \dfrac{\sin bx}{bx}.b$

$= 2\cos a.\ \ b = 2b.\cos a.$

6. $\underset{x\to 0}{Lt}\left[\dfrac{3^x-1}{\sqrt{1+x}-1}\right]$

Sol: $\underset{x\to 0}{Lt} \dfrac{3^x-1}{\sqrt{1+x}-1}$

$= \underset{x\to 0}{Lt} \dfrac{3^x-1}{\sqrt{1+x}-1} \times \dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$(rationalise$ Dr.)

$$= \underset{x\to 0}{Lt}\frac{(3^x-1)\left(\sqrt{1+x}+1\right)}{1+x-1}$$

$$= \underset{x\to 0}{Lt}\frac{3^x-1}{x}.\underset{x\to 0}{Lt}\left(\sqrt{1+x}+1\right)$$

$$= (\log 3)\left(\sqrt{1+0}+1\right) = 2.\log 3$$

7. $y = \log_7(\log x)(x > 0)$

Sol: $y = \log_7(\log x)(x > 0)$

$$\frac{dy}{dx} = \frac{1}{\log_7}.\frac{1}{\log x}.\frac{1}{x}$$

$$= \frac{1}{x(\log x)\left(\log_e^7\right)} = \frac{\log_7^e}{x\log_e^x}$$

8. **If** $y = \cot^{-1}(\operatorname{cosec} 3x)$ **find** $\frac{dy}{dx}$**.**

Sol: $\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}(\operatorname{cosec} 3x)$

$$= -\frac{1}{1+\operatorname{cosec}^2}.\frac{d}{dx}(\operatorname{cosec} 3x)$$

$$= -\frac{1}{1+\operatorname{cosec}^2 3x}(-\operatorname{cosec} 3x.\cot 3x)\frac{d}{dx}(3x) = \frac{3.\operatorname{cosec} 3x.\cot 3x}{1+\operatorname{cosec}^2 3x}$$

9. **The time t of a complete oscillation of a simple pendulum of length l is given by the equation t = where g is gravitational constant. Find the approximate percentage error in the calculated g, corresponding to an error of 0.01 percent is the value of t.**

Sol: percentage error in t is =0.01

Given t=

Taking logs on both sides

log t = log ( ) + {(log (l) – log g}

Taking differentials on both sides,

$= 0 +$

Multiplying with 100,

$\rightarrow$

Percentage error in g = -0.02

10. Verify Rolle's theorem for the following functions. $x^2 - 1$ on $[-1, 1]$

Sol. Let $f(x) = x^2 - 1$

f is continuous on $[-1, 1]$

since $f(-1) = f(1) = 0$ and

f is differentiable on $[-1, 1]$

$\therefore$ By Rolle's theorem $\exists c \in (-1,1)$

Such that $f'(c) = 0$

$f'(x) = 2x = 0$

$\therefore = f'(c) = 0$

$2c = 0 \Rightarrow c = 0$

The point $c = 0 \in (-1, 1)$

Then Rolle's theorem is verified.

SECTION B

11. A(5, 3) and B(3, –2) are two fixed points. Find the equation of locus of P, so that the area of triangle PAB is 9.

Sol. Given points are A(5, 3), B(3, –2)

Let P(x, y) be a point in the locus.

Given, area of ΔAPB = 9.

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x-5 & y-3 \\ 3-5 & -2-3 \end{vmatrix} = 9$$

$$\Rightarrow \begin{vmatrix} x-5 & y-3 \\ -2 & -5 \end{vmatrix} = 18$$

$$\Rightarrow |-5x+25+2y-6| = 18$$

$$\Rightarrow |-5x+2y+19| = 18$$

$$\Rightarrow -5x+2y+19 = \pm 18$$

$$\Rightarrow -5x+2y+19 = 18 \text{ or } -5x+2y+19 = 18$$

$$\Rightarrow 5x-2y-1 = 0 \text{ or } 5x-2y-37 = 0$$

∴ Locus of P is (5x – 2y – 1)(5x – 2y – 37) = 0.

12. Show that the axes are to be rotated through an angle of $\frac{1}{2}\mathrm{Tan}^{-1}\left(\frac{2h}{a-b}\right)$ so as to removethexy term from the equation $ax^2+2hxy+by^2=0$, if $a \neq b$ and through the angle $\frac{\pi}{4}$, if a = b

Sol: Given equation is $ax^2+2hxy+by^2=0$

Since the axes are rotated through an angle θ, then $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$

Now the transformed equation is

$$a\left(X\cos\theta - Y\sin\theta\right)^2 + 2h\left(X\cos\theta - Y\sin\theta\right)\left(X\sin\theta + Y\cos\theta\right) + b\left(X\sin\theta + Y\cos\theta\right)^2 = 0$$

$$\Rightarrow a\left(X^2\cos^2\theta + Y^2\sin^2\theta - 2XY\cos\theta\sin\theta\right) + 2h\left[X^2\cos\theta\sin\theta + XY\left(\cos^2\theta - \sin^2\theta\right) - Y^2\sin\theta\,\mathrm{co}\right.$$

$$+ b\left(X^2\sin^2\theta + Y^2\cos^2\theta + 2XY\cos\theta\sin\theta\right) = 0$$

$\Rightarrow$ It is in the form

$$AX^2 + 2XY\left[-a\cos\theta\sin\theta + h\left(\cos^2\theta - \sin^2\theta\right) + b\cos\theta\sin\theta\right] + BY^2 = 0$$

Since XY term is to be eliminated, $(b-a)\cos\theta\sin\theta + h\left(\cos^2\theta - \sin^2\theta\right) = 0$

$\Rightarrow 2h\cos 2\theta = 2(a-b)\sin\theta\cos\theta = (a-b)\sin 2\theta$ -------------(1)

$$\Rightarrow \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2h}{a-b}$$

$\Rightarrow$ Angle of rotation $\theta = \frac{1}{2}Tan^{-1}\left(\frac{2h}{a-b}\right)$

If a=b, then from (1) , $\Rightarrow 2h\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

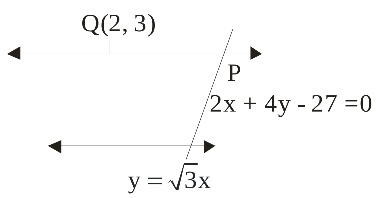
13. A straight line parallel to the line $y = \sqrt{3}x$ passes through $Q(2,3)$ and cuts the line $2x + 4y - 27 = 0$ at P. Find the length of PQ.

Sol: PQ is parallel to the straight line $y = \sqrt{3}x$

$\tan\alpha = \sqrt{3} = \tan 60^\circ$

$\alpha = 60^\circ$

$Q(2,3)$ **is a given point**



Co-ordinates of any point P are

$\left(x_1 + r\cos\alpha,\ y_1 + r\sin\alpha\right)$

$\left(2+r\cos 60^{o},\ 3+r\sin 60^{o}\right)$

$= P\left(2+\frac{r}{2},\ 3+\frac{\sqrt{3}}{2}r\right)$

P is a point on the line $2x+4y-27=0$

$2\left(2+\frac{r}{2}\right)+4\left(3+\frac{\sqrt{3}}{2}r\right)-27=0$

$4+r+12+2\sqrt{3}r-27=0$

$r\left(2\sqrt{3}+1\right)=27-16=11$

$=\frac{11}{2\sqrt{3}+1}\cdot\frac{2\sqrt{3}-1}{2\sqrt{3}-1}=\frac{11\left(2\sqrt{3}-1\right)}{11}$

14. $\underset{x\to a}{Lt}\left[\frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)}\right]$

Sol. Rationalize both nr.and dr.

$$\underset{x\to a}{Lt}\frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$
$$\times\frac{\left(\sqrt{3a+x}+\sqrt{4x}\right)}{\left(\sqrt{3a+x}-\sqrt{4x}\right)\left(\sqrt{3a+x}+\sqrt{4x}\right)}$$

$$=\underset{x\to a}{Lt}\frac{a+2x-3x}{\sqrt{a+2x}+\sqrt{3x}}\times\frac{\sqrt{3a+x}+\sqrt{4x}}{3a+x-4x}=\underset{x\to a}{Lt}\frac{(a-x)\left(\sqrt{3a+x}+\sqrt{4x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)3(a-x)}$$

$$=\frac{2(2a)}{2\left(\sqrt{3a}\right)3}=\frac{2}{3\sqrt{3}}$$

15. A balloon which always remains spherical on inflation is being inflated by pumping on 900 cubic centimeters of gas per second. Find the rate at which the radius of balloon increases when the radius in 15 cm.

Sol. $\frac{dv}{dt}=900\,c.c./\sec$

r = 15 cm

Volume of the sphere $v = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} \Rightarrow 900 = 4\pi(15)^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{900}{4 \times 225\pi} = \frac{dr}{dt} \Rightarrow \frac{900}{900\pi} = \frac{dr}{dt}$$

$$\frac{1}{\pi} = \frac{dr}{dt} \qquad \therefore \frac{dr}{dt} = \frac{1}{\pi} cm/s$$

16. Find the angle between the curve $2y = e^{\frac{-x}{2}}$ **and y-axis.**

Sol: Equation of y-axis is x = 0

The point of intersection of the curve $2y = e^{\frac{-x}{2}}$ and $x = 0$ is $P\left(0, \frac{1}{2}\right)$

The angle ψ **made by the tangent to the curve** $2y = e^{\frac{-x}{2}}$ **at P with x – axis is**

given by $\tan\psi = \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = \left.\frac{-1}{4} e^{\frac{-x}{2}}\right|_{\left(0,\frac{1}{2}\right)} = \frac{-1}{4}$

Further, if ϕ **is the angle between the** $y-$axis and $2y = e^{\frac{-x}{2}}$ **, then we have**

$$\tan\phi = \left|\tan\left(\frac{\pi}{2} - \psi\right)\right| - |\cot\psi| = 4$$

$\therefore$ **The angle between the curve and the y-axis is** $\tan^{-1} 4$ **.**

17. If f and g are two differentiable functions at x then the product function f.g is differentiable at x and (fg)' (x) = f '(x). g(x) + f(x) . g'(x)

Proof:

Since f and g are differentiable at x,therefore f '(x)and g'(x) exist and

f '(x) = $\underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h}$

and $\underset{h\to 0}{Lt}\dfrac{g(x+h)-g(x)}{h}$ = g '(x)

$\underset{h\to 0}{Lt}\dfrac{(fg)(x+h)-(fg)(x)}{h}$ =

$= \underset{h\to 0}{Lt}\dfrac{f(x+h)g(x+h)-f(x)g(x)}{h}$

$= \underset{h\to 0}{Lt}\dfrac{1}{h}[f(x+h).g(x+h)-f(x)g(x+h)]+\underset{h\to 0}{Lt}\dfrac{1}{h}[f(x)g(x+h)-f(x)g(x)]$

$= \underset{h\to 0}{Lt}\left[\dfrac{f(x+h)-f(x)}{h}\right]\underset{h\to 0}{Lt}\,g(x+h)$ + f(x). $\underset{h\to 0}{Lt}\left[\dfrac{g(x+h)-g(h)}{h}\right]$

$\therefore$ (fg)'(x)= f '(x) g(x) + f(x) g'(x)

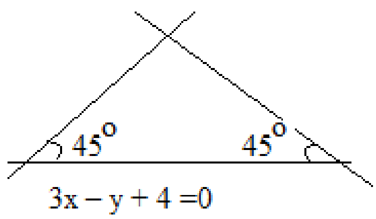
SECTION-C

18. Find the equation of the straight lines passing through the point (-3,2) and making an angle of 45° with the straight line 3x – y + 4 =0.

Sol. Given point P (-3,2)

Given line 3x – y + 4 = 0 ------(1)

Slope $m_1 = -\dfrac{a}{b} = 3$



Let m be the slope of the required line.

Then $\tan 45^\circ = \dfrac{m-3}{1+3m}$

$\Rightarrow \left|\frac{m-3}{1+3m}\right| = 1 \Rightarrow \frac{m-3}{1+3m} = 1$

$\Rightarrow$m-3 = 1+ 3m $\Rightarrow$2m = -4 or m = -2

$\Rightarrow \frac{m-3}{1+3m}$ **= - $\Rightarrow$m – 3 = - 1 – 3m**

$\Rightarrow$4m = 2$\Rightarrow$ m= ½

case (1) m = - 2 and point (-3,2)

Equation of the line is

y -2 = -2(x + 3) = -2x – 6 $\Rightarrow$2x + y + 4 = 0

case (2) m = $\frac{1}{2}$,point (-3,2)

Equation of the line is

y – 2 = $\frac{1}{2}(x+3)$ $\Rightarrow$2y – 4 = x + 3 $\Rightarrow$x -2y + 7 = 0

19. Write down the equation of the pair of straight lines joining the origin to the points of intersection of the lines 6x – y + 8 = 0 with the pair of straight lines $3x^2+4xy-4y^2-11x+2y+6=0$. Show that the lines so obtained make equal angles with the coordinate axes.

Sol. Given pair of line is $3x^2+4xy-4y^2-11x+2y+6=0\ldots(1)$

Given line is $6x-y+8=0 \Rightarrow \frac{6x-y}{-8}=1 \quad \Rightarrow \frac{y-6x}{8}=1$-----(2)

Homogenising (1) w.r.t (2)

$$3x^2+4xy-4y^2-(11x-2y)\left(\frac{y-6x}{8}\right)+6\left(\frac{y-6x}{8}\right)^2=0$$

$$64\left[3x^2+4xy-4y^2\right]-8\left[11xy-66x^2-2y^2+12xy\right] \quad +6\left[y^2+36x^2-12xy\right]=0$$

$$\Rightarrow 936x^2+256xy-256xy-234y^2=0$$

$\Rightarrow 468x^2-117y^2=0 \qquad \Rightarrow 4x^2-y^2=0$----(3)

is eq. of pair of lines joining the origin to the point of intersection of (1) and (2).

The eq. pair of angle bisectors of (3) is $h\left(x^2-y^2\right)-(a-b)xy=0$

$\Rightarrow 0\left(x^2-y^2\right)-(4-1)xy=0 \quad \Rightarrow xy=0$

x = 0 or y = 0 which are the eqs. is of co-ordinates axes

$\therefore$ The pair of lines are equally inclined to the co-ordinate axes

20. Show that the straight lines represented by $3x^2 + 48xy + 23y^2 = 0$ and $3x – 2y + 13 = 0$ form an equilateral triangle of area $\frac{13}{\sqrt{3}}$ sq. units.

Sol. Equation of pair of lines is $3x^2+48xy+23y^2=0$ …………(1)

Equation of given line is 3x – 2y + 13 = 0 ……(2)

$\Rightarrow$ slope = 3/2

$\therefore$ the line (2) is making an angle of $\tan^{-1}\frac{3}{2}$ with the positive direction of x-axis.

Therefore no straight line which makes an angle of 60^0 with (2) is vertical.

Let m be the slope of the line passing through origin and making an angle of 60^0 with line (2).

$$\therefore \tan 60^0=\left|\frac{m_1-m_2}{1+m_1m_2}\right| \Rightarrow \sqrt{3}=\left|\frac{\frac{3}{2}-m}{1+\frac{3}{2}m}\right| \Rightarrow \sqrt{3}=\left|\frac{3-2m}{2+3m}\right|$$

Squaring on both sides , $3=\frac{(3-2m)^2}{(2+3m)^2} \Rightarrow 23m^2+48m+3=0$,which is a quadratic equation in m.

Let the roots of this quadratic equation be m_1,m_2, which are the slopes of the lines.

Now, $m_1+m_2=\frac{-48}{23}$ and $m_1.m_2=\frac{3}{23}$.

The equation of the lines passing through origin and having slopes m_1,m_2 are $m_1x – y = 0$ and $m_2x – y = 0$.

Their combined equation is $(m_1x – y)(m_2x – y)=0$

$$\Rightarrow m_1m_2x^2-(m_1+m_2)xyy^2=0$$

$$\Rightarrow \frac{3}{23}x^2-\left(-\frac{48}{23}\right)xy+y^2=0$$

$$\Rightarrow 3x^2+48xy+23y^2=0$$

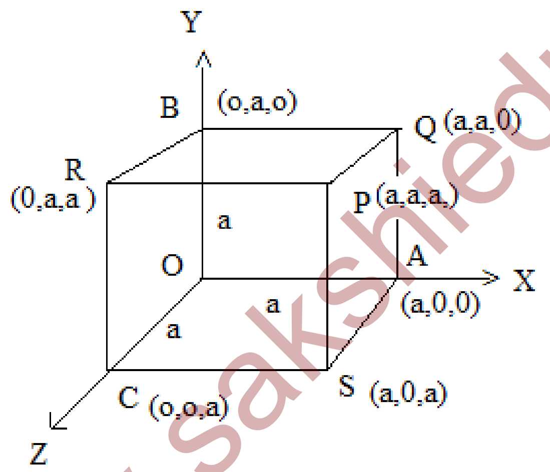
Which is the given pair of lines.

Therefore, given lines form an equilateral triangle.

$$\therefore \text{ Area of } \Delta=\left|\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|=\frac{169\sqrt{576-69}}{\left|3(-2)^2-48.3(-2)+23(3)^2\right|}$$

$$=\frac{169\sqrt{507}}{|12+288+207|}=\frac{169.13\sqrt{3}}{507}=\frac{13\sqrt{3}}{3}=\frac{13}{\sqrt{3}}\text{sq.units.}$$

21. If a ray makes angle α,β,γ and δ with the four diagonals of a cube find $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$



Sol:

Let OABC;PQRS be the cube.

Let a be the side of the cube. Let one of the vertices of the cube be the origin O and the co-ordinate axes be along the three edges $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ passing through the origin.

The co-ordinate of the vertices of the cube with respect to the frame of reference OABC are as shown in figure are A (a,o,o), B(o,a,o), C(0,o,a) P(a,a,a) Q(a,a,o) R(o,a,a) and S(a,o,a)

The diagonals of the cube are $\overline{OP},\overline{CQ},\overline{AR} and \overline{BS}$. and their d.rs are respectively (a, a, a), (a, a, -a), (-a, a, a) and (a, -a, a).

Let the direction cosines of the given ray be (l, m, n).

Then $l^2 + m^2 + n^2 = 1$

If this ray is making the angles α, β, γ *and* δ with the four diagonals of the cube, then

$$\cos\alpha = \frac{|a\times l + a\times m + a\times n|}{\sqrt{a^2+a^2+a^2}.1} = \frac{|l+m+n|}{\sqrt{3}}$$

Similarly, $\cos\beta = \dfrac{|l+m-n|}{\sqrt{3}}$

$\cos\gamma = \dfrac{|-l+m+n|}{\sqrt{3}}$ *and* $\cos\delta = \dfrac{|-l+m+n|}{\sqrt{3}}$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$

$\frac{1}{3}\{|l+m+n|^2 + |l+m-n|^2 + |-l+m+n|^2 + |l-m+n|^2\}$

$\frac{1}{3}[(l+m+n)^2 + (l+m-n)^2 + (-l+m+n)^2 + (l-m+n)^2]$

$\frac{1}{3}[4(l^2+m^2+n^2)] = \frac{4}{3}$ (since $l^2 + m^2 + n^2 = 1$)

22.If $y = \mathrm{Tan}^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ for

$0 < |x| < 1$, find $\dfrac{dy}{dx}$.

Sol.Put $x^2 = \cos 2\theta$

$$y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

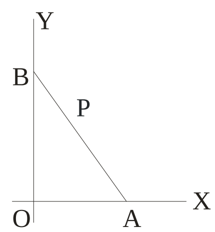
$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(-1)}{\sqrt{1-x^4}}\times 2x = \frac{-x}{\sqrt{1-x^4}}$$

23. If the tangent at any point on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ intersects the coordinate axes in A, B show that the length AB is constant,



Sol: Equation of the curve is $x^{2/3}+y^{2/3}=a^{2/3}$

Let $x = a\cos^3\theta,\ y = a\sin^3\theta$

be the parametric equations of the curve.

Then any point P on the curve is $\left(a\cos^3\theta, a\sin^3\theta\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{a.3\sin^2\theta.(\cos\theta)}{a.3\cos^2\theta.(-\sin\theta)} = \frac{-\sin\theta}{\cos\theta}$$

Equation of the tangent at P is

$$y - a\sin^3\theta = \frac{-\sin\theta}{\cos\theta}\left(x - a\cos^3\theta\right)$$

$$\Rightarrow \frac{y}{\sin\theta} - a\sin^2\theta = -\frac{x}{\cos\theta} + a\cos^2\theta$$

$$\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a\left(\sin^2\theta + \cos^2\theta\right) = a$$

$$\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$$

$\Rightarrow x - \text{int ercept } OA = a\cos\theta$

$y - \text{int ercept } OB = a\sin\theta$

Now $AB^2 = OA^2 + OB^2$

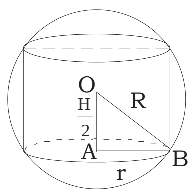
$= (a\cos\theta)^2 + (a\sin\theta)^2$

$= a^2\left(\sin^2\theta + \cos^2\theta\right) \quad = a^2$

$\Rightarrow AB = a, a \text{ cons tan t}$

24. Show that when curved surface area of a cylinder inscribed in a sphere of radius R is a maximum, then the height of the cylinder is $\sqrt{2}\,R$.

Sol: let r be the radius and h be the height of the cylinder.



From ΔOAB, $OA^2 + AB^2 = OB^2$

$$\Rightarrow r^2 + \frac{h^2}{4} = R^2;\ r^2 = R^2 - \frac{h^2}{4}$$

Curved surface area $= 2\pi rh$

$$= 2\pi\sqrt{R^2 - \frac{h^2}{4}}.h$$

$$= \pi h\sqrt{4R^2 - h^2}$$

Let $f(h) = \pi h\sqrt{4R^2 - h^2}$

$$f'(h) = \pi\left[h.\frac{1}{2\sqrt{4R^2 - h^2}}(-2h) + \sqrt{4R^2 - h2}.1\right]$$

$$= \pi.\frac{-h^2 + 4R^2 - h^2}{\sqrt{4R^2 - h^2}} = \frac{2\pi\left(2R^2 - h^2\right)}{\sqrt{4R^2 - h^2}}$$

For max or min $f'(h) = 0$

$$\Rightarrow \frac{2\pi\left(2R^2 - h^2\right)}{\sqrt{4R^2 - h^2}} = 0$$

$$\therefore 2R^2 - h^2 = 0$$

$$\Rightarrow h^2 = 2R^2 \Rightarrow h = \sqrt{2}R$$

$$\Rightarrow \sqrt{4R^2 - h^2}(-2h) + \left(2R^2 - h^2\right)$$

$$f''\left(\text{when } h = \sqrt{2}R\right) = 2\pi\frac{\frac{d}{dh}\sqrt{4R^2 - h^2}}{4R^2 - h^2}$$

$$= -\frac{4\pi h + 0}{\sqrt{4r^2 - h^2}} < 0$$

$f(h)$ is greatest when $h = \sqrt{2}R$

i.e., Height of the cylinder $= \sqrt{2}R$