MATHEMATICS PAPER - 1A

TIME: 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A, B and C.

SECTION A

Very short answer type questions. **10 X 2 =20**

1. If f : [1, ∞) → [1, ∞) defined by f(x) = $2^{x(x-1)}$ then find $f^{-1}(x)$.
2. If $f(y)=\frac{y}{\sqrt{1-y^2}}$ and $g(y)=\frac{y}{\sqrt{1+y^2}}$ then show that (fog)(y) = y.
3. If $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}, a\neq 0$ then show that $\bar{b}=\bar{c}$.
4. Prove that $\sqrt{3}\csc 20°-\sec 20°=4$
5. Find the range of 7 cos x – 24 sin x + 5.
6. Find a vector in the direction of vector $a=\bar{i}-2\bar{j}$ that has magnitude 7 units.
7. If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the vertices A, B and C respectively of ΔABC

   then find the vector equation of median through the vertex A.
8. Prove that $\cos h^{-1}x=\log_e\left(x-\sqrt{x^2-1}\right)$
9. If A = $\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ then find A^3.

10. If A = $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix, find x.

SECTION - B

Short answer type questions.

Answer any five of the following. **5 X 4 = 20**

11. If $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along the positive directions of the coordinate axes, then show that the four points $4\bar{i}+5\bar{j}+\bar{k}, -\bar{j}-\bar{k}$, $3\bar{i}+9\bar{j}+4\bar{k}$ and $-4\bar{i}+4\bar{j}+4\bar{k}$ are coplanar.

12. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{b}=2\bar{i}+3\bar{j}+\bar{k}$ find

i) The projection vector of $\bar{b}$ on $\bar{a}$ and its magnitude.

ii) The vector components of $\bar{b}$ in the direction of $\bar{a}$ and perpendicular to $\bar{a}$.

13. Prove that $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$.

14. If $\sin 3x+\sin x+2\cos x=\sin 2x+2\cos^2 x$ find the general solution.

15. It $\cos^{-1}p+\cos^{-1}q+\cos^{-1}r=\pi$ prove that $p^2+q^2+r^2+2pqr=1$

16. If a : b : c = 7 : 8 : 9 then find $\cos A=\cos B=\cos C$

17. If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then show that A^n = $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, n is a positive integer.

SECTION - C

Long answer type questions.

Answer any five of the following. **5 X 7 = 35**

18. If $f: A \rightarrow B$, $g: B \rightarrow C$ are two bijections then $(gof)^{-1} = f^{-1} og^{-1}$.

19. Show that $3.5^{2n+1} + 2^{3n+1} + 2^{3n+1}$ is divisible by 17.

20. Solve the following equations by Gauss-Jordan method.

    3x + 4y + 5z = 18

    2x – y + 8z = 13

    5x – 2y + 7z = 20

21. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

22. If $\bar{a}, \bar{b}$ and $\bar{c}$ are non-zero vectors and a is perpendicular to both b and c. If $|\bar{a}| = 2, |\bar{b}| = 3, |\bar{c}| = 4$ and $(\bar{b}, \bar{c}) = \frac{2\pi}{3}$, then find $\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$.

23. Prove that $a^3\cos(B – C) + b^3\cos(C – A) + c^3\cos(A – B) = 3abc$.

24. If A, B, C are angles of a triangle then prove that

    $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Mathematics IA Paper - 2

1. If f: [1, ∞) → [1, ∞) defined by $f(x) = 2^{x(x-1)}$ then find $f^{-1}(x)$.

Sol. f(x): [1 ……∞) → [1 ……∞)

$$f(x) = 2^{x(x-1)}$$
$$f(x) = 2^{x(x-1)} = y$$
$$x(x-1) = \log_2 y$$
$$x^2 - x - \log_2 y = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$
$$f^{-1}(x) = \frac{1 \pm \sqrt{1 + 4\log_2 x}}{2}$$

2. If $f(y) = \frac{y}{\sqrt{1-y^2}}$ and $g(y) = \frac{y}{\sqrt{1+y^2}}$ then show that (fog)(y) = y.

Sol. Given that

$$f(y) = \frac{y}{\sqrt{1-y^2}} \text{ and } g(y) = \frac{y}{\sqrt{1+y^2}}$$

$$\therefore fog(y) = f[g(y)] = f\left[\frac{y}{\sqrt{1-y^2}}\right]$$

$$= \frac{y}{\sqrt{1+y^2}} \Big/ \sqrt{1 - \left(\frac{y}{\sqrt{1+y^2}}\right)^2}$$

$$= \frac{y}{\sqrt{1+y^2}} \times \frac{\sqrt{1+y^2}}{1+y^2-y^2} = y$$

$$\therefore fog(y) = y$$

3. If $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$ and $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}, a \neq 0$ then show that $\bar{b} = \bar{c}$.

Sol. Given that,

$\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c} \Rightarrow \bar{a}(\bar{b}-\bar{c}) = 0 \quad ...(1)$

$\bar{a}\times\bar{b} = \bar{a}\times\bar{c} \Rightarrow \bar{a}\times(\bar{b}-\bar{c}) = 0 \; ...(2)$

From (1) and (2) it is evident that, the vector $(\bar{b}-\bar{c})$ cannot be both perpendicular to $\bar{a}$ and parallel to $\bar{a}$.

Unless it is zero

$\therefore \bar{b}-\bar{c} = 0 \; (\bar{a} \neq 0)$

$\therefore \bar{b} = \bar{c}$

4. Prove that $\sqrt{3}\csc 20° - \sec 20° = 4$

Sol. L.H.S. = $\sqrt{3}\csc 20° - \sec 20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°}$$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cos 20°}$$

$$= \frac{2\cdot\frac{\sqrt{3}}{2}\sin 20° - \frac{1}{2}\sin 20°}{\frac{1}{2}(2\sin 20° \cos 20°)}$$

$$= \frac{4(\sin 60° \cos 20° - \cos 60° \sin 20°)}{\sin 40°}$$

$$= 4\frac{\sin(60° - 20°)}{\sin 40°} = 4\cdot\frac{\sin 40°}{\sin 40°}$$

$$= 4 = \text{R.H.S.}$$

5. Find the range of 7 cos x – 24 sin x + 5.

Sol. Minimum value of $f = C - \sqrt{a^2 + b^2}$

$$= 5 - \sqrt{(-24)^2 + 7^2}$$

$$= 5 - \sqrt{576 + 49}$$

$$= 5 - \sqrt{625} = 5 - 25 = -20$$

Maximum value of $f = C + \sqrt{a^2 + b^2}$

$$= 5 + \sqrt{625} = 5 + 25 = 30$$

6. Find a vector in the direction of vector $a = \bar{i} - 2\bar{j}$ that has magnitude 7 units.

Sol. The unit vector in the direction of the given vector 'a' is

$$\hat{a} = \frac{1}{|a|}a = \frac{1}{\sqrt{5}}(\bar{i} - 2\bar{j}) = \frac{1}{\sqrt{5}}\bar{i} - \frac{2}{\sqrt{5}}\bar{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of a is

$$7a = 7\left(\frac{1}{\sqrt{5}}\bar{i} - \frac{2}{\sqrt{5}}\bar{j}\right) = \frac{7}{\sqrt{5}}\bar{i} - \frac{14}{\sqrt{5}}\bar{j}$$

7. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B and C respectively of ΔABC then find the vector equation of median through the vertex A

Sol: $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ be the given vertices

Let D be the mid point of $BC = \frac{\vec{b} + \vec{c}}{2}$

The vector equation of the line passing through the

points $\vec{a}$, $\vec{b}$ is $\vec{r} = (1-t)\vec{a} + t\vec{b}$

$\therefore$vector equation is $\vec{r} = (1 = t)\vec{a} + t\left(\frac{\vec{b} + \vec{c}}{2}\right)$ where $t \in R$

8. Prove that $\cos h^{-1} x = \log_e\left(x - \sqrt{x^2 - 1}\right)$

Sol: Let $\cos h^{-1} x = y \Rightarrow x = \cos hy$

$$x = \frac{e^y + e^{-y}}{2} \Rightarrow 2x = e^y + \frac{1}{e^y}$$

$$2xe^y = \left(e^y\right)^2 + 1 \Rightarrow \left(e^y\right)^2 - 2xe^y + 1$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \Rightarrow e^y = x \pm \sqrt{x^2 - 1}$$

$$e^y = x + \sqrt{x^2 - 1} \Rightarrow y = \log_e\left(x + \sqrt{x^2 - 1}\right)$$

$$\boxed{\cos h^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right)}$$

9. If A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ **then find** $\mathbf{A^3}$**.**

Sol. $A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10. If A = $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ **is a skew symmetric matrix, find x.**

Hint : A is a skew symmetric matrix

$\Rightarrow A^T = -A$

Sol. A is a skew symmetric matrix

$\Rightarrow A^T = A$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 2 \\ 1 & -x & 0 \end{bmatrix}$$

Equating second row third column elements we get x = 2.

11. If $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along the positive directions of the coordinate axes, then show that the four points $4\bar{i}+5\bar{j}+\bar{k}, -\bar{j}-\bar{k}$, $3\bar{i}+9\bar{j}+4\bar{k}$ and $-4\bar{i}+4\bar{j}+4\bar{k}$ are coplanar.

Sol. Let O be a origin, then

$\overline{OA} = 4\bar{i}+5\bar{j}+\bar{k}, \ \overline{OB} - \bar{j} - \bar{k}$

$\overline{OC} = 3\bar{i}+9\bar{j}+4\bar{k}$, and $\overline{OD} = -4\bar{i}+4\bar{j}+4\bar{k}$

$\overline{AB} = \overline{OB} - \overline{OA} = -4\bar{i} - 6\bar{j} - 2\bar{k}$

$\overline{AC} = \overline{OC} - \overline{OA} = -\bar{i} + 4\bar{j} + 3\bar{k}$

$\overline{AD} = \overline{OD} - \overline{OA} = -8\bar{i} - \bar{j} + 3\bar{k}$

$$\left[\overline{AB} \quad \overline{AC} \quad \overline{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$= -4[12+3] + 6[-3+24] - 2[1+32]$

$= -4\times 15 + 6\times 21 - 2\times 33$

$= -60 + 126 - 66$

$= -126 + 126 = 0$

Hence given vectors are coplanar.

12. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{b}=2\bar{i}+3\bar{j}+\bar{k}$ find

i) The projection vector of $\bar{b}$ on $\bar{a}$ and its magnitude.

ii) The vector components of $\bar{b}$ in the direction of $\bar{a}$ and perpendicular to $\bar{a}$.

Sol. Given that $\bar{a}=\bar{i}+\bar{j}+\bar{k}$, $\bar{b}=2\bar{i}+3\bar{j}+\bar{k}$

i) Then projection of $\bar{b}$ on $\bar{a}=\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|^2}\cdot\bar{a}$

$$=\frac{(\bar{i}+\bar{j}+\bar{k})\cdot(2\bar{i}+3\bar{j}+3\bar{k})}{|\bar{i}+\bar{j}+\bar{k}|^2}\cdot|\bar{i}+\bar{j}+\bar{k}|$$

$$=\frac{2+3+1}{(\sqrt{3})^2}\cdot\bar{i}+\bar{j}+\bar{k}$$

$$=\frac{6(\bar{i}+\bar{j}+\bar{k})}{3}=2(\bar{i}+\bar{j}+\bar{k})$$

Magnitude = $\frac{|\bar{a}\cdot\bar{b}|}{|\bar{a}|}=\frac{|(\bar{i}+\bar{j}+\bar{k})\cdot(2\bar{i}+3\bar{j}+\bar{k})|}{|\bar{i}+\bar{j}+\bar{k}|}$

$$=\frac{|2+3+1|}{|\sqrt{3}|}=\frac{6}{\sqrt{3}}=2\sqrt{3}\text{ unit}$$

ii) The component vector of $\bar{b}$ in the direction of $\bar{a}=\frac{(\bar{a}\cdot\bar{b})}{|\bar{a}|^2}\cdot\bar{a}$

$=2(\bar{i}+\bar{j}+\bar{k})$ ($\because$ from 10(i))

The vector component of $\bar{b}$ perpen-dicularto $\bar{a}$.

$$=\bar{b}-\frac{(\bar{a}\cdot\bar{b})\bar{a}}{|\bar{a}|^2}=(2\bar{i}+3\bar{j}+\bar{k})-2(\bar{i}+\bar{j}+\bar{k})$$

$$=2\bar{i}+3\bar{j}+\bar{k}-2\bar{i}-2\bar{j}-2\bar{k}=\bar{j}-\bar{k}$$

13. Prove that $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$

Sol: $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$

$$\left(\sin\frac{\pi}{8}\right)^4+\left\{\sin^2\frac{\pi}{2}-\frac{\pi}{8}\right\}^4+\left\{\sin^2\left(\frac{\pi}{2}+\frac{\pi}{8}\right)\right\}^2+\left\{\sin\left(\pi-\frac{\pi}{8}\right)\right\}^4$$

$$\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}$$

$$2\left\{\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right\}=2\left\{\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)^2-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right\}$$

$$=2-4\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$$

$$=2-\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2=2-\sin^2\frac{\pi}{4}=2-\frac{1}{2}=\frac{3}{2}$$

14. **If** $\sin 3x+\sin x+2\cos x=\sin 2x+2\cos^2 x$ **find the general solution.**

Sol: $\sin 3x+\sin x+2\cos=\sin 2x+2\cos^2 x$

$(2\sin 2x+2\cos x)=2\sin\cos x+2\cos^2 x$

$2\sin 2x\cos x-2\sin x\cos x+2\cos x-2\cos^2 x=0$

$2\cos x\{\sin 2x-\sin x+1-\cos x\}=0$

$\cos x=0 \quad 2\cos\frac{3x}{2}.\sin\frac{x}{2}+2\sin^2\frac{x}{2}=0$

$\cos x=0 \quad 2\sin\frac{x}{2}\left\{\cos\frac{3x}{2}+\sin\frac{x}{2}\right\}=0$

$\sin\frac{x}{2}=0 \qquad \cos\frac{3x}{2}=-\sin\frac{x}{2}$

$n=(2n+1)\pi/2 \qquad \frac{x}{2}=n\pi \qquad \cos\frac{3x}{2}=\cos(\pi/2+x/2)$

$x=2n\pi \qquad \frac{3x}{2}=2n\pi\pm(\pi/2+x/2)$

$\frac{3x}{2}-\frac{x}{2}=2n\pi+\pi/2 \ : \frac{3x}{2}+\frac{x}{2}=2n\pi-\frac{\pi}{2}$

$x=(2n+1)\pi/2; \quad x=2n\pi \ ; x=2n\pi+\ \pi/2 \quad ; \quad x=n\pi-\pi/4\ n\in z$

15. **It** $\cos^{-1}p+\cos^{-1}q+\cos^{-1}r=\pi$ **prove that** $p^2+q^2+r^2+2pqr=1$

Sol: Let $\cos^{-1}p=\alpha \qquad \cos^{-1}q=\beta \qquad \cos^{-1}r=\delta$

$p=\cos\alpha \qquad q=\cos\beta \qquad r=\cos\delta$

Given $\alpha+\beta+\gamma=\pi$ $\quad\quad \cos(\alpha+\beta)=\cos(\pi-\gamma)$

$\cos\alpha\cos\beta-\sin\alpha\sin\gamma=\cos\gamma$

$pq=-r+\sqrt{1-p^2}\ \sqrt{1-q^2}\Rightarrow pq+r=\sqrt{1-p^2}\sqrt{1-q^2}$

Squaring on both such

$p^2q^2+r^2+2pqr=1-p^2-q^2+p^2q^2\Rightarrow p^2+q^2+r^2+2pqr=1$

16. If a : b : c = 7 : 8 : 9 then find $\cos A=\cos B=\cos C$

Sol:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{64k^2+81k^2-49k^2}{2(8k)(9k)}=\frac{96k^2}{2\times 8k\times 9k}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{49k^2+81k^2-64k^2}{2\times 7k\times 9k}=\frac{66^{33}}{2\times 63}=\frac{11}{21}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{49k^2+64k^2-81k^2}{2\times 7k\times 8k}=\frac{32k^2}{20k\times 8k}=\frac{2}{7}$$

$$\therefore \cos A=\cos B=\text{co}\,C=\frac{2}{3}=\frac{11}{21}=\frac{2}{7}=\frac{2}{3}\times 21=\frac{11}{21}\times 21=\frac{2}{7}\times 21$$

$=14:11:6$

17. If A = $\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$ **then show that** $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$**, n is a positive integer.**

Sol. We shall prove the result by Mathematical Induction.

$$A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$$

$$n=1\Rightarrow A'=\begin{bmatrix}1+2 & -4\\ 1 & 1-2\end{bmatrix}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$$

The result is true for n = 1

Suppose the result is true for n = k

$$A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$$

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

$\therefore$ The given result is true for n = k + 1

By Mathematical Induction, given result is true for all positive integral values of n.

18. **If** $f : A \to B$, $g : B \to C$ are two bijections then $(gof)^{-1} = f^{-1} og^{-1}$.

Proof: $f : A \to B, g : B \to C$ are bijections $\Rightarrow gof : A \to C$ is bijection $\Rightarrow (gof)^{-1} : C \to A$ is a bijection.

$f : A \to B$ is a bijection $\Rightarrow f^{-1} : B \to A$ is a bijection

$g : B \to C$ is a bijection $\Rightarrow g^{-1} : C \to B$ is a bijection

$g^{-1} : C \to B$, $g^{-1} : B \to A$ are bijections $\Rightarrow f^{-1} og^{-1} : C \to A$ is a bijection

Let $z \in C$

$z \in C, g : B \to C$ is onto $\Rightarrow \exists y \in B \ni g(y) = z \Rightarrow g^{-1}(z) = y$

$y \in B, f : A \to B$ is onto $\Rightarrow \exists x \in A \ni f(x) = y \Rightarrow f^{-1}(y) = x$

$(gof)(x) = g[f(x)] = g(y) = z \Rightarrow (gof)^{-1}(z) = x$

$\therefore (gof)^{-1}(z) = x = f^{-1}(y) = f^{-1}[g^{-1}(z)] = (f^{-1}og^{-1})(z)$ $\qquad \therefore (gof)^{-1} = f^{-1}og^{-1}$

19. **Show that** $3.5^{2n+1} + 2^{3n+1} + 2^{3n+1}$ **is divisible by 17.**

Sol: Let $S_{(n)} = 3.5^{2n+1} + 2^{3n+1}$ be the given statement.

$S_{(1)} = 3.5^3 + 2^4 = 375 + 16 = 391 = 17 \times 23$

This is divisible by 17

Assume S_k is true

$S_k = 3.5^{2k+1} + 2^{3k+1}$ is divisible by 17

Let $3.5^{2k+1} + 2^{3k+1} = 17m$

$3.5^{2k+1} = 17m - 2^{3k+1}$

$S_{k+1} = 3.5^{2k+3} + 2^{3k+4}$

$= 3.5^{2k+1}.^{2} + 2^{3k+1}.2^3$

$= 25\left\{17m - 2^{3k+1}\right\} + 2^{3k+1}.8$

$= 25 \times 17m - 17(2^{3k+1}) = 17\left\{25m - 2^{3k+1}\right\}$ is divisible by 17

Hence S_{k+1} is true

$\therefore S_n$ is true for all $n \in N$

20. Solve the following equations by Gauss-Jordan method.

3x + 4y + 5z = 18

2x – y + 8z = 13

5x – 2y + 7z = 20

Sol: The augmented matrix is $\begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_1 \to R_1 - R_2$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_2 \to R_2 - 2R_3, R_3 \to R_3 - 5R_1$

$$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

$R_2 \to -5R_2 + 2R_3$

$$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & 1 & -26 & 25 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2, R_3 \to R_3 + 27R_2$

$$= \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & -680 & -680 \end{bmatrix}$$

$R_3 \to R_3 + (-680)$

$$= \begin{bmatrix} 1 & 0 & 127 & 135 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 127R_3, R_2 \to R_2 + 26R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence the solution is x = 3, y = 1, z = 1.

Show that

21. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol: L.H.S = $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2 - a^2 & b^2 - a^3 \\ 0 & c^2 - a^2 & c^3 - a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2 + ba + a^2 \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$= -(a-b)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b-c & b^2 - c^2 + a(b-c) \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$$

$$= -(a-b)(c-a)(b-c)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & 1 & b+c+a \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$$

$= -(a-b)(b-c)(c-a)$

$[(c^2 + ca + a^2) - (b+c+a)(c+a)]$

$= -(a-b)(b-c)(c-a)$

$[c^2 + ca + a^2 - b(c+a) - (c+a)^2 =$

$= -(a-b)(b-c)(c-a)$

$[c^2 + ca + a^2 - bc - ab - c^2 - 2ca - a^2]$

$= -(a-b)(b-c)(c-a)[-ab - bc - ca]$

$= (a-b)(b-c)(c-a)(ab + bc + ca)$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

22. If $\bar{a}, \bar{b}$ and $\bar{c}$ are non-zero vectors and a is perpendicular to both b and c. If $|\bar{a}| = 2, |\bar{b}| = 3, |\bar{c}| = 4$ and $(\bar{b}, \bar{c}) = \frac{2\pi}{3}$, then find $\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$.

Sol: If $\bar{a}$ is perpendicular to $\bar{b}$ and $\bar{c}$.

$\Rightarrow \bar{a}$ is parallel to $\bar{b}\times\bar{c}$

$\Rightarrow \left[\bar{a}, \bar{b}\times\bar{c}\right] = 0$

$\Rightarrow \bar{b}\times\bar{c} = |\bar{b}||\bar{c}|\sin(\bar{b},\bar{c})\hat{a}$

$\Rightarrow |\bar{b}\times\bar{c}| = 3\times 4\sin\frac{2\pi}{3}\hat{a}$

$\Rightarrow |\bar{b}\times\bar{c}| = 12\sin 120.1 = 12\times\frac{\sqrt{3}}{2} = 6\sqrt{3}$

$\therefore \left[|\bar{a}\ \bar{b}\ \bar{c}|\right] = |\bar{a}\cdot(\bar{b}\times\bar{c})| = |\bar{a}||\bar{b}\times\bar{c}|\cos(\bar{a}\ \bar{b}\ \bar{c})$

$= (2\cdot 6\sqrt{3})\cos 0 = 12\sqrt{3}$

$\therefore |\bar{a}\cdot\bar{b}\times\bar{c}| = (2\cdot 6\sqrt{3}) = 12\sqrt{3}$

23. Prove that $a^3\cos(B - C) + b^3\cos(C - A) + c^3\cos(A - B) = 3abc$.

Sol. L.H.S. $= \Sigma a^3\cos(B - C)$

$= \Sigma a^2(2R\sin A)\cos(B - C)$

$= R\Sigma a^2\cdot[2\sin(B + C)\cos(B - C)]$

$= R\Sigma a^2(\sin 2B + \sin 2C)$

$= R\Sigma a^2(2\sin B\cos B + 2\sin C\cos C)$

$= \Sigma[a^2(2R\sin B)\cos B + a^2(2R\sin C)\cos C]$

$= \Sigma(a^2 b\cos B + a^2 c\cos C)$

$= (a^2 b\cos B + a^2 c\cos C)$

$\quad +(b^2 c\cos C + b^2 a\cos A)$

$\quad\quad +(c^2 a\cos A + c^2 b\cos B)$

$= ab(a\cos B + b\cos A) + bc(b\cos C + c\cos B)$

$\quad +ca(c\cos A + a\cos C)$

$= ab(c) + bc(a) + ca(b)$

$= 3abc = R.H.S.$

24. If A,B,C are angles of a triangle then prove that

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Sol: A+B+C = 180^0

$$LHS = \sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2}$$

$$= \sin^2\frac{A}{2} + \sin\left(\frac{B}{2}+\frac{C}{2}\right).\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$= \sin^2\frac{A}{2} + \sin\left(90-\frac{A}{2}\right).\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$=1\text{-}\cos^2\frac{A}{2} + \cos\frac{A}{2}.\sin\left(\frac{B}{2}-\frac{C}{2}\right)$$

$$=1-\cos\frac{A}{2}\left(\cos\frac{A}{2}-\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right)$$

$$=1-\cos\frac{A}{2}\left(\cos\left(90-\left(\frac{B}{2}+\frac{C}{2}\right)\right)-\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right)$$

$$=1-\cos\frac{A}{2}\left(\sin\left(\frac{B}{2}+\frac{C}{2}\right)-\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right)$$

$$=1-\cos\frac{A}{2}\left(2\cos\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = RHS$$