LIMITS

SYNOPSIS

LIMIT OF A REAL FUNCTION:

f(x) be a function defined in a deleted nbd of 'a' and $l \in$ R, for each $\in > 0$, there exists $\delta > 0$ such that $0 < |xa| < a < x < a + \delta \Rightarrow |f(x) - l \} k < \in$ then l is called the limit of f(x) at 'a' then we write it as $\underset{x \to a}{Lt} f(x) = l$

RIGHT LIMIT OF 'F' AT 'A':

For each . $\in > 0$, there exist $\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - l| < \in$, then l is called right limit of f(x) at 'a'. Then we write it $\underset{x \to a+}{Lt} f(x) = l$.

LEFT LIMIT OF 'F' AT 'A':

for each $\in > 0$, there exists a $\delta > 0$ such that $a - \delta < x < a \Rightarrow |f(x) - l| < \in$ then l is called left limit of 'f' at 'a'. Then we write it is $\underset{x \to a-}{Lt} f(x) = l$.

Note: if $\underset{x \to a-}{Lt} f(x) = \underset{x \to a+}{Lt} f(x) = l$, then we say that limit of f(x) exists at x = a in this case limit is denoted by $\underset{x \to a}{Lt} f(x)$: $\underset{x \to a}{Lt} f(x) = l$.

INFINITE LIMIT:

Let 'f' be a function defined in a deleted nbd of 'a'. If for every k > 0 (how ever larger) $\exists\ \delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > k$ then we write $\underset{x \to a}{Lt} f(x) = \infty$.

1. $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = n.a^{n-1}, n \in Q$
2. $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{x} = 1,\ \underset{x \to 0}{Lt} \dfrac{e^{mx} - 1}{x} = m,\ \underset{x \to 0}{Lt} \dfrac{e^{mx} - 1}{e^{nx} - 1} = \dfrac{m}{n}$.
3. $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x} = \log_e a \left(\text{or } \log a\right)$.
4. $\underset{x \to 0}{Lt} \dfrac{a^x - b^x}{x} = \log\left(\dfrac{a}{b}\right)$
5. $\underset{x \to 0}{Lt} (1+x)^{\frac{1}{x}} = e,\ \underset{x \to 0}{Lt} (1+ax)^{\frac{1}{x}} = e^a$
6. $\underset{n \to \infty}{Lt} \left(1 + \dfrac{1}{n}\right)^n = e,\ \underset{n \to \infty}{Lt} \left(1 + \dfrac{a}{n}\right)^n = e^a$,

$$\underset{n\to\infty}{Lt}\left(1+\frac{1}{an+b}\right)^{cn+d}=e^{c/a},$$

$$\underset{x\to\infty}{Lt}\left(\frac{x+p}{x+q}\right)^{x+n}=e^{p-q}.$$

7. $$\underset{x\to\infty}{Lt}\left(\frac{x^2+px+r}{x^2+qx+d}\right)^{cx+d}=e^{c(p-q)}$$

8. $$\underset{x\to-\infty}{Lt}\left(1-\frac{1}{x}\right)^{x}=e^{-1}$$

9. $$\underset{x\to 0}{Lt}\frac{\log(1+px)}{x}=p$$

10. $$\underset{x\to 0}{Lt}\left(\frac{a_1^x+a_2^x+\ldots\ldots a_n^x}{n}\right)^{\frac{1}{x}}=\sqrt[n]{a_1a_2a_3\ldots a_n}$$

11. $$\underset{x\to\infty}{Lt}\left(\sqrt{x^2+ax+b}-x\right)=\frac{a}{2}$$

12. If $a>b$, then $\underset{x\to\infty}{Lt}\left(a^x+b^x\right)^{1/x}=a$

(ii) $$\underset{x\to\infty}{Lt}\frac{[ax+b]}{x}=a$$

13. (i) $$\underset{n\to\infty}{Lt}\frac{\sum n^2}{n^3}=\frac{1}{3}$$

(ii) $$\underset{n\to\infty}{Lt}\frac{\sum_{r=1}^{n} r^{\lambda}}{n^{(\lambda+1)}}=\frac{1}{\lambda+1}$$

(iii) $$\underset{n\to\infty}{Lt}\frac{\sum_{r=1}^{n}[r^{\lambda}x]}{n^{\lambda+1}}=\frac{x}{\lambda+1}$$

14. $\underset{\theta\to 0}{Lt}\frac{\sin\theta}{\theta}=1,\ \underset{\theta\to 0}{Lt}\frac{\tan\theta}{\theta}=1$ (θ is measured in radians)

15. $$\underset{\theta\to 0}{Lt}\frac{\sin a\theta}{\theta}=a,\ \underset{\theta\to 0}{Lt}\frac{\tan a\theta}{\theta}=a.$$

16. $$\underset{\theta\to 0}{Lt}\frac{\sin a\theta}{\sin b\theta}=\frac{a}{b},\ \underset{\theta\to 0}{Lt}\frac{\tan a\theta}{\tan b\theta}=\frac{a}{b}$$

17. $$\underset{\theta\to 0}{Lt}\frac{\sin\theta^0}{\theta}=\frac{\pi}{180},\ \underset{\theta\to 0}{Lt}\frac{\sin a\,\theta^0}{\theta}=\frac{a\,\pi}{180}.$$

18. If $\theta\to\infty,\ \frac{\sin\theta}{\theta}\to 0.$

If $\theta \to \infty, \dfrac{\cos\theta}{\theta} \to 0$.

19. i) $\underset{x\to 0}{Lt}\dfrac{\cos ax - \cos bx}{x^2} = \dfrac{b^2 - a^2}{2}$.

ii) $\underset{x\to 0}{Lt}\dfrac{\tan^n x - \sin^n x}{x^{n+2}} = \dfrac{n}{2}$

20. $\underset{x\to 0}{Lt}\dfrac{\cos ax - \cos bx}{\cos cx - \cos dx} = \dfrac{b^2 - a^2}{d^2 - c^2} = \dfrac{a^2 - b^2}{c^2 - d^2}$.

21. $\underset{x\to 0}{Lt}\dfrac{\sin^{-1} x}{x} = 1,\ \underset{x\to 0}{Lt}\dfrac{\tan^{-1} x}{x} = 1$.

22. $\underset{x\to 0}{Lt}\log_{\sin qx}\sin px = 1$

23. $\underset{x\to 0}{Lt}\dfrac{1-\cos mx}{x^2} = \dfrac{m^2}{2},\ \underset{x\to 0}{Lt}\dfrac{1-\cos mx}{1-\cos nx} = \dfrac{m^2}{n^2}$.

24. If $|r| < 1$ then $r^n \to 0$ as $n \to \infty$.

25.(i) If $r > 1$, $r^n \to \infty$ as $n \to \infty$.

(ii) $\underset{x\to\infty}{Lt}\dfrac{a_0 x^m + a_1 x^{m-1} + ... + a_n}{b_0 x^n + b_1 x^{n-1} + ... + b_n} = \dfrac{a_o}{b_o}$ $(m = n)$

$= 0$ $(m < n)$

$= \infty$ $(m>n, a_0 > 0)$

$= -\infty$ $(m>n, a_0 < 0)$

(iii) If m and n are positive integers then $\underset{x\to 0}{Lt}\dfrac{\sin x^n}{(\sin x)^m} = 1 (m = n)$

$\underset{x\to 0}{Lt}\dfrac{\sin x^n}{(\sin x)^m} = 0$ $(n > m)$

$\underset{x\to o}{Lt}\dfrac{\sin x^n}{(\sin x)^m} = \infty$ $(n < m)$

26. As $x \to \infty$, $e^x \to \infty$ and $e^{-x} \to 0$.

27. $\underset{x\to a}{Lt} f(x)$ exists if $\underset{x\to a-}{Lt} f(x) = \underset{x\to a+}{Lt} f(x)$.

28. $\underset{x\to 0-}{Lt}\dfrac{1}{x} = -\infty,\ \underset{x\to 0+}{Lt}\dfrac{1}{x} = \infty.\ \ \underset{x\to 0}{Lt}\dfrac{1}{x}$ does not exist.

29. $\underset{x\to 0-}{Lt} e^{\frac{1}{x}} = 0,\ \underset{x\to 0+}{Lt} e^{\frac{1}{x}} = \infty$.

$\underset{x\to 0}{Lt} e^{\frac{1}{x}}$ does not exist.

30. $\underset{x\to 0}{Lt}\dfrac{|x|}{x}$ does not exist. $\underset{x\to 0^+}{Lt}\dfrac{|x|}{x} = 1,\ \underset{x\to 0^-}{Lt}\dfrac{|x|}{x} = -1$

31. $\underset{x\to 0}{Lt}\, x\sin\dfrac{1}{x} = 0$ while $\underset{x\to 0}{Lt}\sin\dfrac{1}{x}$ does not exist.

32. $\underset{x \to n}{Lt}[x]$ does not exist if n is an integer and $\underset{x \to n}{Lt}[x]$ exists if n is not an integer.

33. Indeterminate forms

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, 0^\infty$

34. L. Hospital rule

If $\frac{f(a)}{g(a)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$; then

$\underset{x \to a}{Lt}\frac{f(x)}{g(x)} = \frac{f^1(a)}{g^1(a)}, g^1(a) \neq 0$

Suppose $\frac{f^1(a)}{g^1(a)}$ is also of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$\underset{x \to a}{Lt}\frac{f(x)}{g(x)} = \frac{f^{11}(a)}{g^{11}(a)}$ and so on till we get the limit

35. The indeterminate form $0.\infty$: -

Let $\underset{x \to a}{Lt} f(x) = 0, \underset{x \to a}{Lt} g(x) = \infty$, then

$\underset{x \to a}{Lt}\left[f(x).g(x)\right] = \underset{x \to a}{Lt}\frac{f(x)}{\left[g(x)\right]^{-1}}$; which is in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Now the limit easily can be found.

36. The indeterminate forms:- $0^0, 0^\infty, \infty^0, 1^\infty$

The limit of above indeterminate forms can be found as given below

$\underset{x \to a}{Lt}[f(x)]^{g(x)} = e^k$, where k is given by k = $\underset{x \to a}{Lt} g(x).\log f(x)$

37. If $\underset{x \to a}{lt} f(x) = 1$ and $\underset{x \to a}{lt} g(x) = \infty$, then $\underset{x \to a}{lt}[f(x)]^{g(x)} = e^{\underset{x \to a}{lt} g(x)[f(x)-1]}$