PAIR OF LINES

SYNOPSIS AND FORMULAE

1. The equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ represent straight lines, the equation $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ is called joint equation of the two lines.

2. **(i)** If $h^2 > ab$, then $ax^2 + 2hxy + by^2 = 0$ represents two real and different straight lines through the origin.

(ii) The equations of the separate lines are $ax + \left(h + \sqrt{h^2 - ab}\right)y = 0$; $ax + \left(h - \sqrt{h^2 - ab}\right)y = 0$.

(iii) The above lines are coincident if $h^2 = ab$ and imaginary if $h^2 < ab$.

(iv) If the slopes of the two lines represented by $ax^2 + 2hxy + by^2 = 0$ are m_1 and m_2, then $m_1 + m_2 = \frac{-2h}{b}$, $m_1m_2 = \frac{a}{b}$ and $|m_1 - m_2| = \frac{2\sqrt{h^2 - ab}}{|b|}$. The equations of the two lines are $y = m_1x$ and $y = m_2x$.

3. $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines

(a) If the slopes of two lines are in the ratio p : q then $\frac{(p+q)^2}{pq} = \frac{4h^2}{ab}$.

(b) If the slope of one line is square of the slope of the other line then

$ab(a + b) - 6abh + 8h^3 = 0.$

(c) If θ_1, θ_2 are the inclinations of the two lines with x-axis then $\mathrm{Tan}(\theta_1 + \theta_2) = \frac{2h}{a-b}$.

4. If θ is the acute angle between the pair of lines $ax^2 + 2hxy + by^2 = 0$ then

$$\theta = \tan^{-1}\frac{2\sqrt{h^2 - ab}}{|a+b|} = \cos^{-1}\frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}} = \sin^{-1}\frac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}.$$

Note: $a + b = 0$. The lines are perpendicular.

5. The equation of the pair of lines through the origin and perpendicular to the two lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 + 2bxy + ay^2 = 0$

6. The equation of the pair of lines bisecting the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is $h(x^2 - y^2) = (a - b)xy$.

7. If the two pair of lines have the same angular bisector lines, then they are said to be equally inclined to each other.

8. The pair of lines $ax^2 + 2hxy + by^2 = 0$ and $(a + d)x^2 + 2hxy + (b + d)y^2 = 0$ are equally inclined for all values of d.

9. Area of the triangle formed by the line $lx + my + n = 0$ and the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{n^2\sqrt{h^2 - ab}}{\left|bl^2 - 2blm + am^2\right|}$.

10. The line $lx + my + n = 0$ forms an equilateral triangle with the pair of lines $(lx + my)^2 - 3(mx - ly)^2 = 0$ its area $= \dfrac{p^2}{\sqrt{3}} = \dfrac{n^2}{\sqrt{3}(l^2 + m^2)}$ where p is the perpendicular distance from the origin to the lines $lx + my + n = 0$.

11. Let $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$; $S = 0$ represents a pair of lines if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $g^2 \ge ac$, $h^2 \ge ab$, $f^2 \ge bc$.

12. If S = o represents two intersecting lines, their point of intersection is $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$

13. Lengths of the intercepts made by the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gh + 2fy + c = 0$ on the coordinate axes are $\dfrac{2\sqrt{g^2 - ac}}{|a|}$ and $\dfrac{2\sqrt{f^2 - bc}}{|b|}$.

14. $S = 0$ represents two parallel lines if $\Delta = 0$, $f^2 \ge bc$, $g^2 \ge ac$ and $h^2 = ab$, $\dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f}$, $af^2 = bg^2$.

The distance between the parallel lines $= 2\sqrt{\dfrac{g^2 - ac}{a(a + b)}} = 2\sqrt{\dfrac{f^2 - bc}{b(a + b)}}$.

15. Let $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent two straight lines

(i) Pair of lines through origin and parallel to $S = 0$ is $ax^2 + 2hxy + by^2 = 0$

(ii) Pair of lines through (x_1, y_1) and parallel to
$S = 0$ is $a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$.

(iii) Pair of lines through origin and perpendicular to $S = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

(iv) Pair of liens through (x_1, y_1) and perpendicular to
$S = 0$ is $b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$.

16. Equation of the pair of lines through the origin and which are at a distance of d units from (x_1, y_1) is $d^2(x^2 + y^2) = (y_1x - x_1y)^2$.

17. Let (x_1, y_1) be a point of intersection of the pair of lines $S = 0$. Then the equation of the pair of angular bisectors of $S = 0$ is $\frac{(x-x_1)^2-(y-y_1)^2}{a-b} = \frac{(x-x_1)(y-y_1)}{h}$

18. If one of the angular bisectors of $ax^2 + 2hxy + by^2 = 0$ passes through the point of intersection of lines $a_1x^2 + 2h_1xy + b_1y^2 + 2gx + 2fy + c = 0$ then $(a - b)fg + h(f^2 - g^2) = 0$.

19. In the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$.

(i) Orthocenter = P = (kl, km), where $k = \frac{-n(a+b)}{am^2 - 2hlm + bl^2}$.

(ii) $op = |k|\sqrt{l^2+m^2}$.

(iii) If $ax^2 + 2hxy + by^2 = 0$ represents two sides of ΔABC and the orthocenter is (c, d), then the 3^{rd} side is $(a + b)(cx + dy) = ad^2 - 2hcd + bc^2$.

20. **(i)** To find the centroid of the triangle formed by $ax^2 + 2hx + by^2 = 0$ and $lx + my + n = 0$, let $f = bx^2 - 2hxy + ay^2$.

$$G = \frac{-n}{3f}\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \text{ at } x = l, y = m.$$

(ii) To find the 3rd side when G of ΔOAB is (x_1, y_1) and OA, OB combined equation is $ax^2 + 2hxy + by^2 = 0$. Midpoint of AB = $\left(\frac{3x_1}{2}, \frac{3y_1}{2}\right)$ and the slope of AB is $\left(\frac{ax_1 + by_1}{hx_1 + by_1}\right)$.

Equation of AB is $y = \frac{3y_1}{2} = -\left(\frac{ax_1 + by_1}{hx_1 + by_1}\right)\left(x - \frac{3x_1}{2}\right)$.

21. The lines $ax^2 + 2hxy + by^2 = 0$ and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ form a

(i) Rhombus $\Leftrightarrow \frac{g^2 - f^2}{a - b} = \frac{fg}{h}$ and $a + b \neq 0$.

(ii) Square $\Leftrightarrow \frac{g^2 - f^2}{a - b} = \frac{fg}{h}$ and $a + b = 0$.

(iii) Rectangle $\Leftrightarrow \frac{g^2 - f^2}{a - b} \neq \frac{fg}{h}$ and $a + b = 0$.

(iv) Parallelogram $\Leftrightarrow \frac{g^2 - f^2}{a - b} \neq \frac{fg}{h}$ and $a + b \neq = 0$.

22. The lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ represent an isosceles triangle of $\frac{l^2 - m^2}{lm} = \frac{a - b}{h}$.

23. **(a)** The two pairs of lines $a_1x^2 + 2h_1xy + b_1y_2 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 = 0$ have one line in common then $\begin{vmatrix} a_1 & 2h_1 \\ b_2 & 2h_2 \end{vmatrix} \begin{vmatrix} 2h_1 & b_1 \\ -2h_2 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2$

(b) One line in the first pair is perpendicular to one line in he second pair then

$\begin{vmatrix} a_1 & 2h_1 \\ b_2 & 2h_2 \end{vmatrix} \begin{vmatrix} 2h_1 & b_1 \\ -2h_2 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ b_2 & a_2 \end{vmatrix}^2$.

24. **(a)** The product of the perpendicular distances from (x_1, y_1) to the pair of lines represented by $ax^2 + 2hxy + by = 0$ is $\frac{\left|ax_1^2 + 2hx_1y_1 + by_1^2\right|}{\sqrt{(a-b)^2 + 4h^2}}$

(b) The product of the perpendicular distances from $(x_1\ y_1)$ to the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\dfrac{|ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c|}{\sqrt{(a-b)^2 + 4h^2}}$.

(c) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are two lines which are at equidistance from the origin then $f^4 - g^4 = (bf^2 - ag^2)$.

25. Equation of the pair of lines through the origin and making an angle 'α' with the line $lx + my + n = 0$ is $(lx + my)^2 - \tan^2 \alpha(mx - ly)^2 = 0$ and the area of the triangle is $\dfrac{n^2}{\tan\alpha(l^2 + m^2)}$.

26. **(a)** One of the lines of $S \equiv 0$ is parallel to $lx + my + n = 0$ then $bl^2 - 2hlm + am^2 = 0$.

(b) One of the lines of $S \equiv 0$ is perpendicular to $lx + my + n = 0$ then $al^2 + 2hlm + bm^2 = 0$.

27. **(a)** The area of the parallelogram formed by $S \equiv ax^2 + 2hxy + by^2 = 0$ and $S^1 = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\dfrac{|c|}{2\sqrt{h^2 - ab}}$.

(b) Equation of the diagonal not passing through the origin is $2gx + 2fy + c = 0$.

(c) Equation of the diagonal passing through the origin is $(hf - bg)y = (gh - af)x$.

28. Area of parallelogram is $\dfrac{p_1p_2}{\sin\alpha}$ where p_1, p_2 is product of perpendicular distances between parallel lines and α is angle between non parallel lines.

29. Point of intersection of diagonals of a rectangle formed by the pairs $a_1x^2 + b_1x + c_1 = 0$ and $a_2y^2 + b_2y + c_2 = 0$ is $\left(\dfrac{-b_1}{2a_1}, \dfrac{-b_2}{2a_2}\right)$.

30. The condition that one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the coordinate axes is $(a + b)^2 = 4h^2$.