THEORY OF EQUATIONS

SYNOPSIS

1. **Polynomial Function:** If $a_0, a_1, a_2 \ldots\ldots a_n$ are real and n is a positive integer, then $f)x) = a_0 + a_1x + a_2x^2 + \ldots.. + a_nx^n$ is called polynomial function.

2. **Degree of the Polynomial:** The highest power of x for which the coefficient is non-zero in a polynomial function, is called the degree of the function.

3. **Constant Polynomial:** If the degree of the polynomial function is zero, then that polynomial is called a constant function.

4. **Zero Polynomial:** If the coefficients of a polynomial are all zeros, then that polynomial is called zero polynomial.Zero polynomial has no degree.

   The domain of a zero polynomial is R. The range is a subset of R

5. **Polynomial Equation:** If $a_0, a_1, a_2, \ldots.. a_n$ are real and n is a positive integer then $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n = 0$ is called polynomial equation in x, with real coefficients. If $a_n \neq 0$, then f(x) = 0 is an equation of degree n.

6. **Linear, Quadratic, Cubic, Biquadratic equations:** polynomial equations of degree 1,2,3,4 are respectivelycalled as a linear, quadratic, cubic, biquadratic equations.

7. **Division Algorithm:** Let f(x), g(x) be the polynomials of degree n and m respectively such that m < n. Then there exist two polynomials q(x) and r(x) uniquely such that f(x) = q(x). g(x) + r(x).

   The degree of q(x) is (n – m).

8. **Remainder Theorem:** If a polynomial f(x) is divided by(x – a) then the remainder is f(a).

9. **Fundamental Theorem:** Every polynomial equation of degree n$\geq$ 1 has atleast one root real or imaginary.It has only n roots real or imaginary and no more.

10. Every n^{th} degree equation has exactly n roots real or imaginary.

12. If $\alpha_1, \alpha_2, \alpha_3 .....\alpha_n$ are the roots of $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$, then $S_1 = -\frac{a_1}{a_0}$, $S_2 = \frac{a_2}{a_0}$, $S_3 = -\frac{a_3}{a_0}$, ..., $S_n = (-1)^n \frac{a_n}{a_0}$.

Where S_n stands for the sum of the products of the roots taken 'r' at a time.

13. For the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$

i) $\sum\alpha^2 = p_1^2 - 2p_2$

ii) $\sum\alpha^3 = -p_1^3 + 3p_1p_2 - 3p_3$

iii) $\sum\alpha^4 = p_1^4 - 4p_1^2p_2 + 2p_2^2 + 4p_1p_3 - 4p_4$ iv) $\sum\alpha^2\beta = 3p_3 - p_1p_2$

v) $\sum\alpha^2\beta\gamma = p_1p_3 - 4p_4$

Note: For the equation $x^3 + p_1x^2 + p_2x + p_3 = 0$ $\sum\alpha^2\beta^2 = p_2^2 - 2p_1p_3$

14. For the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_n = 0$,

i) $S_r + S_{r-1}p_1 + S_{r-2}p_2 + ... + S_1p_{r-1} + rp_r = 0$, if $r \leq n$ and

ii) $S_r + S_{r-1}p_1 + S_{r-2}p_2 + ... + S_{r-n}p_n = 0$, if $r > n$.

Where S_r denotes the sum of the r^{th} powers of the roots of the equation.

15. The sum of the r^{th} powers of the roots of the equation $x^n + p_1x^{n-1}+p_2x^{n-2}$ ....$+p_{n-1}x +p_n =0$ is the coefficient of x^{-r} in the expansion of $\frac{xf^1(x)}{f(x)}$ in descending powers of x.

16. To remove the second term from a n^{th} degree equation, the roots must be diminished by $\frac{-a_1}{na_0}$ and the resultant equation will not contain the term with x^{n-1}.

17. If $\alpha_1, \alpha_2, ...\alpha_n$ are the roots of f(x) = 0, the equation

i) Whose roots are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2} .... \frac{1}{\alpha_n}$ is $f\left(\frac{1}{x}\right) = 0$.

ii) Whose roots are $k\alpha_1$, $k\alpha_2$,...,$k\alpha_n$ is $f\left(\frac{x}{k}\right) = 0$.

iii) Whose roots are $\alpha_1 - h, \alpha_2 - h, ...\alpha_n - h$ is f(x+h) = 0.

iv) Whose roots are $\alpha_1 + h, \alpha_2 + h, ...\alpha_n + h$ is f(x-h) = 0.

v) Whose roots are $\alpha_1^2, \alpha_2^2 .... \alpha_n^2$ is $f\left(\sqrt{y}\right) = 0$

18. If α, β are the roots of the quadratic equation f(x) = 0, then the equation whose roots are $a\alpha + b, a\beta+b$ is $f\left(\frac{x-b}{a}\right) = 0$.

19. An equation in which the reciprocal of every root of the equation is also its root is called a reciprocal equation. In such an equation, the coefficients from one end are equal to coefficients from the other end (or) Equal in magnitude and opposite in sign.

20. In any equation with rational coefficients, irrational roots occur in conjugate pairs.

21. In any equation with real coefficients, complex roots occur in conjugate pairs.

22. If α is r $^{-}$ multiple root of f(x) = 0, then α is a (r-1) - multiple root of $f^{1}(x)=0$ and (r-2) - multiple root of $f^{11}(x) = 0$ and non multiple root of $f^{r-1}(x) = 0$.

23. i) If $f(x) = x^n + p_1x^{n-1} + ... + p_{n-1}x + p_n$ and f(a) and f(b) are of opposite sign, then atleast one real root of f(x) =0 lies between a and b.

24. **Descarate's rule of signs**

If f(x) is a polynomial (with real coefficients with the terms arranged in descending powers of x), the number of real positive roots of f(x) = 0 does not exceed the number of changes in signs of the coefficients of f(x), and the number of negative real roots of f(x) = 0 does not exceed the number of changes of signs of f(-x).

25. i) The equation of lowest degree with rational coefficients having a root $\sqrt{a}+i\sqrt{b}$ is $x^4-2(a- b)x^2+(a+b)^2 = 0$.

ii) The equation of lowest degree with rational coefficients having a root $\sqrt{a}+\sqrt{b}$ is $x^4-2(a+ b)x^2+(a-b)^2 = 0$.

26. The condition that the roots of $ax^3 + bx^2 + cx + d = 0$ may be in A.P. is

$2b^3+27a^2d = 9abc$.

27. The condition that the roots of $ax^3 + bx^2 + cx + d = 0$ may be in G.P. is

$ac^3 = b^3d$.

28. The condition that the roots of $ax^3+bx^2+cx + d = 0$ may be in H.P. is $2c^3+27ad^2= 9bcd$.

29. **i)** If the reciprocal of every root of an equation is also a root of it, then the equation is said to be a reciprocal equation.

ii) If the coefficients from one end of an equation are equal in magnitude and sign to the coefficients from the other end, then the equation is said to be the reciprocal equation of First Type.

iii) If the coefficients from one end of an equation are equal in magnitude and opposite in sign to the coefficients from the other end, then the equation is said to be the reciprocal equation of Second Type.

30. **i)** $x = -1$ is a root of the reciprocal equation of first type and of odd degree

ii) $x = 1$ is a root of the reciprocal equation of second type and of odd degree.

iii) $x = \pm 1$ are two roots of reciprocal equation of second type and of even degree.