TANGENTS AND NORMALS

SYNOPSIS

I. Let y = f(x) be a continuous function defined ever an interval I.

1. The slope of the tangent to the curve y = f(x) at the point $P(x_1, y_1)$ is $\left(\frac{dy}{dx}\right)_p$ and is denoted by in.
2. The slope of the tangent to the curve at a point is called the gradient to the curve at that point.
3. Equation of the tangent to the curve y = f(x) at $P(x_1, y_1)$ is $y - y_1 = m(x - x_1)$.
4. A line which is perpendicular to the tangent of the curve at P and passes through P is called the normal to the curve at P.
5. Equation of the normal to the curve y = f(x) at $P(x_1, y_1)$ is $x - x_1 = -m(y - y_1)$.
6. The tangent at any point of the curve $x = at^3$, $y = at^4$ divides the abscissa of the point of contact in the ratio is 1 : 3.
7. Point on the curve $ay^2 = x^3$ the normal at which makes equal intercepts on the axes is $\left(\frac{4a}{9}, \frac{8a}{27}\right)$.
8. The condition for the line y = mx + c to be a tangent to the parabola $y^2 = 4ax$ is $c = \frac{a}{m}$.
9. The condition for the line y = mx + c to be a tangent to the parabola $x^2 = 4ax$ is $c = -am^2$.
10. The equation of common tangent to the parabolas $y^2 = 4ax$, $x^2 = 4by$ is $a^{\frac{1}{3}}x + b^{\frac{1}{3}}y + (ab)^{\frac{2}{3}} = 0$.
11. The condition that the line y = mx + c may be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$.
12. The condition that the line y = mx + c may be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$.

II. Angle of intersection of two plane curve

1. The angle between two curves is defined as the angle between the two tangents at their common point of intersection.

2. Let $y = f_1(x)$ and $y = f_2(x)$ be two plane curves intersect at $P(x_1, y_1)$. If $m_1 = f_1^1(x_1)$ and $m_2 = f_2^1(x_1)$ are the slopes of the tangents at P and θ be the acute angle between them then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, $m_1 m_2 = -1$ and angle is obtuse is "$\pi - \theta$".

3. If $m_1 m_2 = -1$ then the two curves intersect orthogonally at P.

4. If $m_1 = m_2$ then the two curves touch each other at P. Then they will have common tangent.

5. If the curves $y^2 = 4ax$, $x^2 = 4ay$ intersect at (0, 0) and (4a, 4a). At (0, 0) the curves cut each other orthogonally and at (4a, 4a) the angle between the curves is $\mathrm{Tan}^{-1}(3/4)$ the area between the two curves is $16a^2/3$.

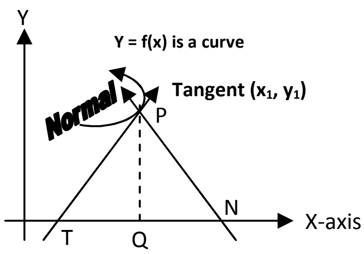
6. If the curves $xy = c^2$, $y^2 = 4ax$ cut each other orthogonally then $c^4 = 32a^4$.

7. The condition that the two curves $x = y^2$, $xy = k$ cut orthogonally is $8k^2 = 1$.

8. If the curves $a_1x^2 + b_1y^2 = 1$, $a_2x^2 + b_2y^2 = 1$ cut each other orthogonally, then $\frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{b_1} - \frac{1}{b_2}$.

9. Angle between the curves $y^2 = 4ax$, $x^2 = 4by$ at their point of intersection $\theta = \mathrm{Tan}^{-1}\left[\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}\right]$.

III. Lengths of the tangent, normal, sub tangent and sub-normal



1. Let the tangent at P to the curve meet x –axis in T and the normal meet OX in N. Draw PQ perpendicular to OX; then

   (i) PT is called the length of the tangent at P.

   (ii) PN is called the length of the normal at P.

   (iii) QT, the projection of PT on OX is called sub-tangent.

   (iv) QN, the projection of PN on OX is called sub-normal.

2. Let $P(x_1, y_1)$ be a point on the curve y = f(x) and $\left(\frac{dy}{dx}\right)_p$ = m. Then

   (i) Length of the tangent at P = $\left|\frac{y_1}{m}\right|\sqrt{1+m^2}$.   (ii) Length of the normal at P = $|y_1|\sqrt{1+m^2}$

   (iii) Length of the sub-tangent at P =.$\left|\frac{y_1}{m}\right|$.   (iv) Length of the sub-normal at P = $|y_1 .m|$.

3. The area of the triangle formed by the tangent at a point $p(x_1, y_1)$, normal at P and x-axis is $y_1^{\,2}\frac{\left(1+m^2\right)}{2|m|}$ sq. units and the y-axis is $\frac{x_1^{\,2}\left(1+m^2\right)}{2|m|}$.

4. Area of the Δle formed by the tangents at (x_1, y_1) on y = f(x) with co-ordinate axes is $\frac{(y_1+mx_1)^2}{2|m|}$.

5. Area of the Δle formed by the normal at $(x_1\ y_1)$ on y-f(x) with co-ordinate axes is $\frac{(x_1+my_1)^2}{2|m|}$.

6. For the curve $x^n + y^n = a^n$ P(x, y) = $(a\cos^{2/n}\theta\ \ a\sin^{2/n}\theta)$ in a point on the curve.

7. The length of the sub-normal at any point on the curve $xy^n = a^{n+1}$ is a constant then n = –2.

8. In the curve $by^2 = (x + a)^3$, $\frac{(\text{L.S.T.})^2}{\text{L.S.N.}} = \frac{8b}{27}$.