DIFFERENTIATION

SYNOPSIS

1. Let y f(x) be a real function defined in (x – δ, x + δ) which is a neighbourhood of real number x.

a) $\underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h}$ If it exists, is called the derivative of f at 'x' and is denoted by $f^1(x)$ or dy/dx.

b) $\underset{h\to 0+}{Lt}\dfrac{f(x+h)-f(x)}{h}$ If it exists, is called the right hand derivative of f at 'x and is denoted by $Rf^1(x)$ or $f^1(x+)$.

c) $\underset{h\to 0+}{Lt}\dfrac{f(x-h)-f(x)}{-h}$ If it exists, is called the left hand derivative of f at 'x' and is denoted by $Lf^1(x)$ or $f^1(x-)$

A function f is differentiable at x or $f^1(x)$ exists $f^1(x+)$ and $f^1(x-)$ both exist and are equal.

2. **Differentiability on an Interval**: A function f(x), defined on [a, b] is said to be differentiable on [a, b] if it is differentiable at every c ∈ (a, b) and both

$\underset{h\to 0+}{Lt}\dfrac{f(a+h)-f(a)}{h}$ and $\underset{h\to 0+}{Lt}\dfrac{f(b-h)-f(b)}{-h}$ exist.

Remark: If a function has a derivative at 'c', then f is continuous at c. The converse need not be true. For example the function |x|, x sin (1/x) are continuous at x = 0 but none of these has a derivative at x = 0.

3. **Fundamental theorems**: Let u, v be the functions of x whose derivatives exist.

a) $\dfrac{d}{dx}(k) = 0$ where k is a constant and $\dfrac{d}{dx}(ku) = k.\dfrac{du}{dx}$.

b) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$, (sum or difference rule).

c) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, (product rule).

d) $\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, (quotient rule)

Note: $\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right) = \frac{ad-bc}{(cx+d)^2}$.

e) If y = f(t) and t = g(x), then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$ (function of a function rule)

f) $\frac{dy}{dx}.\frac{dx}{dy} = 1$ (or) $\frac{dx}{dy} = 1/(dy/dx)$. (Derivative of inverse of a function)

4. **Logarithmic Differentiation**: If $y = [f(x)]^{g(x)}$ where f(x) > 0 and f(x), g(x) are the differentiable functions at x, then $\frac{dy}{dx} = [f(x)]^{g(x)}\left[g(x)\frac{f^1(x)}{f(x)} + g^1(x)\log f(x)\right]$.

5. **Parametric Equations**: If x = f(t), y = g(t), then dy/dx = (dy/dt) ÷ (dx/dt)

6. Derivative of one function w.r.t. another function. Let u = f(x) and v = g(x) be differentiable at x and g(x) ≠ 0. Then $\frac{du}{dv} = \frac{f^1(x)}{g^1(x)}$.

7. If f(x, y) = c is a function of x and y then $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$.

i) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then $\frac{dy}{dx} = \frac{-(ax+hy+g)}{hx+by+f}$.

ii) If $x^m y^n = a^{m+n}$ then $\frac{dy}{dx} = \frac{-my}{nx}$.

8. If f(x, y) = 0 is a homogeneous function then $\frac{dy}{dx} = \frac{y}{x}$.

9. If f(x) = |x| then |f′(O)| does not exist.

10. f(x) = |x| is continuous at x = 0 but not differentiable at x = 0.

11. $x \sin\frac{1}{x}$, $x \cos\frac{1}{x}$ are continuous at x = 0 but not differentiable at x = 0.

12. If f(x + y) = f(x) + f(y) for all x, y ∈ R then f′(x) = f(x) . f′(x).

13. Derivatives of some important function.

S. No.	f(x)	f′(x)
1.	R	R
2.	x	1
3.	x^n, n ∈ N	nx^{n-1}
4.	x^n, n ∈ z	nx^{n-1}
5.	x^n, n ∈ R	nx^{n-1}
6.	e^x	e^x
7.	a^x, a ∈ R^+	$a^x \log a$
8.	log x	1/x
9.	log |x|	1/x
10.	|x|	sgn(x) or $\frac{\|x\|}{x}$

11.	x^x	$x^x(1 + \log x)$
12.	sin x	cos x
13.	cos x	– sin x
14.	tan x	$\sec^2 x$
15.	cot x	$-\operatorname{cosec}^2 x$
16.	sec x	sec x tan x
17.	cosec x	– cosec x cot x
18.	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
19.	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
20.	$\tan^{-1} x$	$\frac{1}{1+x^2}$
21.	$\cot^{-1} x$	$\frac{-1}{1+x^2}$
22.	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
23.	$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
24.	sinhx	coshx
25.	coshx	sinhx
26.	tanhx	$\operatorname{sech}^2 hx$

27.	cothx	$-\text{cosech}^2x$
28.	sechx	– sechx tanhx
29.	cosechx	– cosechx cothx
30.	$\sinh^{-1}x$	$\frac{1}{\sqrt{1+x^2}}$
31.	$\cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
32.	$\tanh^{-1}x$	$\frac{1}{1-x^2}$
33.	$\coth^{-1}x$	$\frac{1}{1-x^2}$
34.	$\text{sech}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1-x^2}}$
35.	$\text{cosech}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Note: i) sgn (x) = $\begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is called signum function its domain = R and its range {–1 , 0, 1}.

ii) If f(x) = $\frac{|x|}{x}$, x ≠ 0 then its domain = R – {0} and its range = {–1, 1}.

14. If y is a function of u where u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

15. If x = a $\cos^3\theta$, y = a $\sin^3\theta$ then $\frac{dy}{dx} = -\tan\theta$

16. If x = a $\cos^4\theta$, y = a $\sin^4\theta$ then $\frac{dy}{dx} = -\tan^2\theta$

17. If x = a(θ + sinθ), y = a (1-cosθ) then $\frac{dy}{dx} = \tan\frac{\theta}{2}$

18. If x = a (θ-sinθ), y = a(1-cosθ) then $\frac{dy}{dx} = \cot\frac{\theta}{2}$

19. If x = a (cost + t sint), y = a(sint - t cost) then $\frac{dy}{dx} = \tan t$.

20. If x = a $\left[\cos t + \log\tan\frac{t}{2}\right]$, $y = a\left[\ \sin t\right]$ then $\frac{dy}{dx} = \tan t$.

21. $\frac{d}{dx}(\sin x^0) = \frac{\pi}{180}\cos x^0$.

22. While differentiating the given function using trigonometric transformation, observe the following points

a) If the function involve the term $\sqrt{a^2 - x^2}$, then put x = asinθ or x = acosθ

b) If the function involve the term $\sqrt{a^2 + x^2}$, then put x = atanθ or x = a cotθ

c) If the function involve the term $\sqrt{x^2 - a^2}$, then put x = asecθ or x=a cosecθ

d) If the function involve the term $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, then put x = acosθ or x = acos2θ