# APPLICATIONS OF DERIVATIVES

## SYNOPSIS

### ERRORS AND APPROXIMATIONS

1. If y = f (x) and $\delta$x is a small change in x then the corresponding change in y (approximately) is given by $f^1(x)\,\delta x$. This is called the differential of y is denoted by dy

   $\therefore dy = f^1(x)\,\delta x$

2. The actual change or the actual error in y is denoted by $\delta y = f(x + \delta x) - f(x)$

3. $\delta y \cong dy = f^1(x)\,\delta x$

4. If $\delta$x be the error in x then the approximate value of f (x) is

   $f(x + \delta x) \cong f(x) + f^1(x).\,\delta x$

5. Let $\delta$x be any change in x and $\delta$y be the corresponding change in y. Then

   (i) $\delta$y is called error in y

   (ii) $\frac{\delta y}{y}$ is called relative error in y

   (iii) $\frac{\delta y}{y} \times 100$ is called percentage error in y

6. If $f(x) = ax^2 + bx + c$ then $\delta f - df = a(\delta x)^2$

7. If y = f (x) is a homogeneous function of degree n or $y \alpha x^2$ then

   (i) Relative error in y = n [Relative error in x]

   (ii) % error in y = [% error in x]

# RATE OF CHANGE

1. If x is any variable, $\frac{dx}{dt}$ represents the rate of change of x at time 't'.

2. If s is the displacement of a particle at time 't', then $\frac{ds}{dt}$ represents the velocity of the particle at that instant.

3. If v is the velocity of a particle at time 't', then $\frac{dv}{dt}$ represents the acceleration of the particle at that instant.

4. A particle moving on a straight line comes to rest if $\frac{ds}{dt} = 0$ & $\frac{d^2s}{dt^2} = 0$

5. A particle moving on a straight line is at rest momentarily if $\frac{ds}{dt} = 0$ & $\frac{d^2s}{dt^2} \neq 0$

6. A particle, projected vertically upwards, attains the maximum height when $\frac{ds}{dt} = 0$

7. A particle acquires maximum velocity if $\frac{dv}{dt} = 0$

8. A particle changes it's direction if $\frac{ds}{dt} = 0$ and $\frac{d^2s}{dt^2} \neq 0$

9. If v is velocity of a particle moving along a straight line and v is expressed in terms of displacement 's', then the acceleration of the particle = $v\frac{dv}{ds}$

10. If p (x,y) is a variable point on a curve of $y = f(x)$, then its velocity at time 't' is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

11. The equations of motion of a particle P(x, y) on a plane curve are given by $x = f(t)$, $y = g(t)$. Then the velocity of the particle is given by $\frac{ds}{dt} = \sqrt{[f^1(t)]^2 + [g^1(t)]^2}$

12. If θ is a variable angle associated with a variable point P, then $\frac{d\theta}{dt}$ represents the angular velocity of P at time t.

13. The rate of change in velocity is called the acceleration of the particle at t and is denoted by a

$$\therefore a = \frac{d\upsilon}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = \frac{d\upsilon}{ds}\frac{ds}{dt} = \upsilon \cdot \frac{d\upsilon}{ds}.$$

14. Let O be a fixed point and OX be a fixed ray. Let P be the position of the particle on a curve C at time t such that XOP = θ. Then $\frac{d\theta}{dt}$ is called the angular velocity and is denoted by ω, $\frac{d^2\theta}{dt^2}$ is called the angular acceleration of the particle about 'O' and is denoted by 'α'.

# INCREASING AND DECREASING FUNCTIONS

1.A function f(x) is an increasing function of x if, as x increases, f(x) increases. i.e. f(x) is an increasing function of x if, $x_1 > x_2 \Leftrightarrow f(x_1) > f(x_2)$.

2. A function f(x) is a decreasing function of x, if, as x increases, f(x) decreases i.e. if $x_1 > x_2 \Leftrightarrow f(x_1) < f(x_2)$.

3.A function f(x) is an increasing function of x if $f^1(x) > 0$.

4. A function f(x) is a decreasing function of x if $f^1(x) < 0$.

5. $a^x$ is increasing function of x if a > 1 and decreasing if 0 < a < 1).

6. $\log_a x$ is an increasing function of x if a > 1 and decreasing function if 0<a < 1)

7. If both f and g are either increasing or decreasing, then fog and g of are increasing.

8.If any one of f and g are increasing and an other one is decreasing, then g of and fog are decreasing.

# MAXIMA AND MINIMA

1. Let f be a real function which is differentiable at ‘a’. If $f^1$ (a) = 0 then we say that f(x) is stationary at x = a, (a, f(a)) is called stationary point f(a) is called stationary value.

2. If there exists $\delta > 0$ such that $f(x) \le f(a)$ for every $a - \delta < x < a + \delta$ then f is said to have relative maximum at ‘a’. f(a) is called relative (local) maximum value.
   If there exists $\delta > 0$ such that $f(x) \ge f(a)$ for every $a - \delta < x < a + \delta$ then f is said to have relative minimum at ‘a’. f(a) is called relative (local) minimum value.

3. The points at which a function attains either maximum or minimum are called extreme points or turning points of the function. Maximum or minimum values of a function are called extreme values or turning values of the function.

4. Necessary condition for extreme value of function:
   If a function f(x) has extreme value f(a) then $f^1(a) = 0$, if it exists.

5. Sufficient conditions for extreme values:
   Let f(x) be derivable at x = a
   a) x = a is point of relative maximum of f(x) if $f^1(a)$ changes sign from +ve to –ve as x passes through the point x = a from left nbd to right nbd.
   b) x = a is a point of relative minimum of f(x) if $f^1(a)$ changes sign from –ve to +ve as x passes through the point x = a from left nbd to right nbd.

   **Note:** If $f^1(a)$ has the same sign in the entire neighbourhood of x = a then x = a is not extremum.

6. Sufficient conditions for extreme values.

Let f(x) be derivable at x = a and $f^{11}(a)$ exists and is non-zero.

a) $f^{1}(a) = 0$ and $f^{11}(a) < 0 \Rightarrow x = a$ is a point of relative maximum.

b) $f^{1}(a) = 0$ and $f^{11}(a) > 0 \Rightarrow x = a$ is a point of relative minimum.

7. If f(x) is increasing in [a, b] then f(a) = Minimum value and f(b) = Maximum value of f(x) in [a, b].

If f(x) is decreasing in [a, b] then f(a) = Maximum value and f(b) = Minimum value of f(x) in [a, b].

8. 1. The maximum value of $a \cos^2 x + b \sin^2 x$ is 'a' and minimum value = b (If a > b)

2. The minimum value of f(x) = a tan x + b cot x is $2\sqrt{ab}$ and attains at tan x = $\sqrt{b/a}$ .

3. The minimum value of $f(x) = a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$ is $(a + b)^2$ and attained at tan x = $\sqrt{b/a}$

4. The minimum value of f(x) = a sec x + b cosec x is $(a^{2/3} + b^{2/3})^{3/2}$ and it is attained at tan x = $(a/b)^{1/3}$.

5. The maximum value of $f(x) = \sin^m x . \cos^n x$ is $\frac{m^{m/2}.n^{n/2}}{(m+n)^{\frac{m+n}{2}}} = \sqrt{\frac{m^m.n^n}{(m+n)^{m+n}}}$ and attained at tan x = $\sqrt{\frac{m}{n}}$

9. 1. The sum of two numbers is k. If the sum of their squares is minimum. Then the numbers are K/2, K/2.

2. The sum of two numbers is k and the least sum of their squares is $K^2/2$.

3. The sum of two numbers is K. If their product is maximum, then the numbers are k/2, k/2.

4. The product of two positive numbers is K. If the sum of their squares is minimum, then the numbers ae $\sqrt{K}$ , $\sqrt{K}$ .

5. Sum of two numbers is k. If the product of the square of the first and cube of the second is maximum then the numbers 2k/5, 3k/5.

10. 1. If a > 0, b > 0, x > 0, the least value of f(x) = ax + $\frac{b}{x}$ is $2\sqrt{ab}$.

2. If x > 0, the least value of f(x) = x + $\frac{1}{x}$ is 2.

11. 1. The maximum rectangle inscribed in a circle is square.

2. The maximum area of rectangle in a cube of radius r is $2r^2$.

12. 1. The maximum triangle inscribed in a circle is equilateral triangle.

2. The maximum area of a triangle in a circle of radius r is $\frac{3\sqrt{3}}{4}$ $r^2$ sq. units.

13. 1. The perimeter of a sector is 'C' cms. Then maximum area of sector is $\frac{C^2}{16}$ sq. cm.

2. Perimeter of sector is given. The area of sector is maximum. Then the angle of sector is 2 radians.

3. The area of sector is 'a' sq. cm. Then the least perimeter of sector is $4\sqrt{a}$ cm.

14. 1. The hypotenuse of a right angled triangle is 'a'. If the area of triangle is maximum. Then the sides are $\frac{a}{\sqrt{2}}$, $\frac{a}{\sqrt{2}}$.

2. Two sides of a triangle are given. The area of the triangle is maximum. Then the angle between sides is $\pi/2$.

3. The sum of hypotenuse and one side of right angled triangle is given. The area is maximum. Then the angle between the sides is $\pi/3$.

15. 1. An open box of maximum volume is made from a square piece of tin of side 'a' by cutting for four equal square pieces from four corners and folding up the tin then the length of square cut is $\frac{'a'}{6}$.

2. An open box of maximum volume is made from a rectangular piece of tin of length ‘a’ and breath ‘b’ by cutting four equal square pieces from four corners and folding up the tin. Then the length of box is $1/6\left\{(a+b)-\sqrt{a^2+b^2-ab}\right\}$.

16. 1. The area of the greatest rectangle inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is 2ab and the sides are $a\sqrt{2}$, $b\sqrt{2}$.

2. Maximum area of Δ formed by a line through $(x_1, y_1)$ and coordinate axes is $2|x_1, y_1|$.

# MEANVALUE THEOREMS

**Rolle'sTheorem:** If a function $f$ : $[a, b]R$ is such that

i) It is continuous on $[a, b]$

ii) It is derivable on $(a, b)$ and

iii) $f(a) = f(b)$ then there exists at least one $c \in (a,b)$ such that $f'(c) = 0$.

**Lagrange's mean -value theorem or first mean - value theorem:**

If a function $f$ : $[a, b]$ $R$ is such that

i) It is continuous on $[a, b]$.

ii) It is derivable on $(a, b)$ then there exists at least one $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$