ADDITION OF VECTORS

SYNOPSIS

1. **Scalar**: A physical quantity which has only magnitude is called a scalar.

2. **Vector**: A physical quantity which has both magnitude and direction is called a vector. If a vector is represented by a directed line segment $\overline{AB}$, then A and B are called the initial and terminal points. The direction from A to B gives the direction of the vector and the distance from A to B gives its magnitude. A vector is generally denoted by bold face letter. (**Example:** ***a***, ***b***, ***c*** ...et) or the letters with bar: $\bar{a}$, $\bar{b}$, $\bar{c}$ etc).

3. **Null Vector**: A vector whose magnitude is zero is called a null vector. The direction of a null vector is not defined specifically. We denote the null vector by $\bar{0}$.

4. **Negative of a Vector**: The negative vector of $\bar{a}$ is a vector having the same modulus as that of $\bar{a}$, but having opposite direction. It is denoted $-\bar{a}$. If $\overline{AB} = \bar{a}$, then $\overline{BA} = -\bar{a}$.

   Hence $|\bar{a}| = |-\bar{a}|$.

5. **Unit vector**: A vector of length one unit is called unit vector.

   Note 1: Unit vector in the direction of $\bar{a} = \frac{\bar{a}}{|\bar{a}|}$;

   2: Unit vector in the opposite direction of $\bar{a} = -\frac{\bar{a}}{|\bar{a}|}$

   3. Unit vector parallel to $\bar{a} = \pm\frac{\bar{a}}{|\bar{a}|}$.

6. **Equal Vectors**: Two vectors are said to be equal if they have the same magnitude and direction, irrespective of their initial and terminal points.

7. **Like Vectors:** Vectors having the same direction are called a like vectors.

8. **Addition of Vectors** : If $\overline{AB} = \bar{a}$ and $\overline{BC} = \bar{b}$, the vector $\overline{AC}$ defines the sum of $\bar{a}$ and $\bar{b}$ i.e., $\overline{AC} = \overline{AB} + \overline{BC} = \bar{a} + \bar{b}$. This is called the triangle law of vector.

9. Parallelogram Law of Addition of Vectors: If two vectors are represented by two adjacent sides of a parallelogram, their sum with respect to length and direction is represented by the diagonal with the point of intersection of the adjacent sides as the initial point.

If $\overline{OA}$ and $\overline{OC}$ are the adjacent sides of a parallelogram $\overline{OABC}$ and if $\overline{OA} = \bar{a}$, $\overline{OC} = \bar{b}$ then $\overline{OB} = \bar{a} + \bar{b}$ and $\overline{AC} = \bar{b} - \bar{a}$.

10. Position Vector: Let O be the point of reference and P be any point in space. Then $\overline{OP}$ is called the position vector of P relative to Q

If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ then $\overline{AB} = \bar{b} - \bar{a}$.

11. Component of a Vector: Any vector $\bar{a}$ in 3-d space can be represented as an ordered triads (a_1, a_2, a_3) where $a_1, a_2, a_3 \in R$ are called the components of $\bar{a}$. The null vector $\bar{0}$ is denoted by (0, 0, 0).

12. Scalar Multiplication of a Vector: If m is a real number (scalar) and $\bar{a}$ is a non zero vector, then

(i) The modulus of $m\bar{a}$ is $m|\bar{a}|$ when $m > 0$ and the direction of $m\bar{a}$ is in the direction of $\bar{a}$.

(ii) The modulus of $m\bar{a}$ is $-m|\bar{a}|$ when $m < 0$ and the direction of $m\bar{a}$ is opposite to $\bar{a}$.

(iii) $m\bar{a} = 0$ if $m = 0$ or $\bar{a} = \bar{0}$ and the modulus is zero.

If m and n are real numbers and $\bar{a}$ and $\bar{b}$ are vectors, then

(i) $m(n\bar{a}) = (mn)\bar{a}$; **(ii)** $(m + n)\bar{a} = m\bar{a} + n\bar{b}$.

(iii) $m(\bar{a} + \bar{b}) = m\bar{a} + m\bar{b}$.

13. Like vectors and Parallel Vectors: Vectors which have the same direction are said to be like vectors. Two vectors having the same direction or opposite direction are called parallel vectors.

14. Like Parallel and Unlike Parallel Vectors: $\bar{a}$ and $\bar{b}$ are like parallel vectors, if they have the same direction. $\bar{a}$ and $\bar{b}$ are unlike parallel vectors if they have opposite direction.

Let $\lambda \in R$ and $\bar{a}$ be a vector. Then

If $\lambda > 0$ then $\bar{a}$ and $\lambda\bar{a}$ are like parallel vector

If $\lambda < 0$, then $\bar{a}$ and $\lambda\bar{a}$ unlike parallel vectors

If $\lambda = 0$ then $\lambda\bar{a}$ is a null vector.

If $\lambda \neq 0, \bar{a} \neq \bar{0}$, then $\bar{a}$ and $\lambda\bar{a}$ are collinear or parallel vectors.

15. Angle Between Two Vectors: If $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ be two non-zero vectors then $\angle AOB = \theta$, $0 \leq \theta \leq 180^o$ is defined as the angle between $\bar{a}$ and $\bar{b}$ is written as (a, b).

i) $(\bar{a}, \bar{b}) = 0^o \Leftrightarrow \bar{a}$ and $\bar{b}$ are like vectors.

ii) If $(\bar{a}, \bar{b}) = \frac{\pi}{2}$, then $\bar{a}$ and $\bar{b}$ are orthogonal vectors.

iii) $(\bar{a}, \bar{b}) = \pi \Rightarrow \bar{a}$ and $\bar{b}$ are unlike vectors.

iv) If $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$, then angle between $\bar{a}$ and $\bar{b}$ is undefined.

v) $(\bar{a}, \bar{b}) = (\bar{b}, \bar{a})$.

vi) $(\bar{a}, \bar{b}) = -\bar{a}, -\bar{b})$.

vii) $(m\bar{a}, n\bar{b}) = (\bar{a}, \bar{b}) \; \forall$ m, n > 0.

viii) $(m\bar{a}, n\bar{b}) = 180^o - (\bar{a}, \bar{b})$ if m, n are have opposite signs.

16. Collinear Vectors: Vectors lie on same line (or) parallel lines are called Collinear Vectors (or) Parallel Vectors.

i) $\bar{a}, \bar{b}$ are collinear vectors $\Leftrightarrow \bar{a} = \lambda\bar{b}$, λ is a scalar.

ii) $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ are collinear vectors $\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

17. The points $\bar{a}, \bar{b}, \bar{c}$ are collinear if and only if there exist scalars x, y, z, not all zero such that $x\bar{a} + y\bar{b} + z\bar{c} = 0$, x + y + z =0.

If $\overline{AB}$ =K $\overline{AC}$ then the points A,B,C are collinear

18. Coplanar and Non-Coplanar Vectors: Two or more vectors are said to be coplanar if they lie on the same plane.

Vectors equivalent to vectors lying in a plane are also called coplanar vectors.

Vectors that are not coplanar are called non-coplanar vectors.

19. To verify that three given vectors $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ and $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$ are linearly independent or linearly dependent:

Find $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

i) If $\Delta \neq 0$, then they are linearly independent.

ii) If $\Delta = 0$, then they are linearly dependent.

20. To show that 4 points A, B, C, D are coplanar, adopt one of the following:

i) Find $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$. Show that the determinant of their components is zero.

ii) Show that there exist scalars p, q, r, s not all zero such that $p\bar{a} + q\bar{b} + r\bar{c} + s\bar{d} = \bar{0}$ and $p + q + r + s = 0$.

21. Linear dependence and linear independence of vectors.

i) Linear Combination Of Vectors: If $\bar{a}$, $\bar{b}$, $\bar{c}$ are vectors and x, y, z are real numbers, then the vector $r = x\bar{a} + y\bar{b} + z\bar{c}$ is called a linear combination of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

ii) A system of vectors $\overline{a_1}$, $\overline{a_2}$, ….. $\overline{a_n}$ is said to be linearly independent if there exist scalars x_1, x_2, ….., x_n ∍ $x_1\overline{a_1} + x_2\overline{a_2} + \ldots + x_n\overline{a_n} = 0 \Rightarrow x_1 = x_2 = \ldots x_n = 0$.

iii) A system of vectors $\overline{a_1}, \overline{a_2}$, ….. $\overline{a_n}$ is said to be linearly dependent if $x_1\overline{a_1} + x_2\overline{a_2} + \ldots + x_n\overline{a_n} = \bar{0} \Rightarrow$ at least one of the $x_i \neq 0$, i = 1, 2, ….., n..

iv) Any two collinear vectors are linearly dependent.

v) Any set of vectors containing the null vector is linearly dependent.

vi) Any three coplanar vectors are linearly dependent. Any three non-coplanar vectors are linearly independent.

vii) If $\bar{a}$ and $\bar{b}$ are two non-zero, non-collinear vectors such that $x\bar{a} + y\bar{b} = 0$, then x = 0 and y = 0.

viii) Any vector $\bar{r}$ in the plane generated by two non-collinear vectors $\bar{a}$ and $\bar{b}$ can be expressed in the form $r = x\bar{a} + y\bar{b}$; $x, y \neq 0$.

22. Direction Cosines of a Vector: Direction Ratios:

Let $\bar{i}, \bar{j}, \bar{k}$ be an orthogonal unit vector triad in the right handed system and $\bar{r}$ a vector. If $\alpha = (\bar{r}, \bar{i})$, $\beta = (\bar{r}, \bar{j})$ and $\gamma = (\bar{r}, \bar{k})$, then cos α . cos β . cos γ are called the direction cosines of $\bar{r}$. We denote them by *l*, *m*, *n* respectively. The numbers proportional to direction cosines of a given vector, then *kl*, *km*, *kn* are called the direction ratios of that vector, $k \in$ R.

Some Important Results:

i) If *l, m, n* are the direction cosines of a line, then $l^2, m^2, n^2 = 1$.

ii) If $\overline{OP} = \bar{r}$ and P is the ordered triad (x, y, z) then x = r cos α = lr, y = r cos β = mr and z = r cos γ = nr.

iii) The direction cosines of the vectors $\bar{i}, \bar{j}, \bar{k}$ are respectively (1, 0, 0), (0, 1, 0), (0, 0, 1).

iv) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = lr\bar{i} + mr\bar{j} + nr\bar{k}$, then $\bar{r} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{r}}{r} = l\bar{i} + m\bar{j} + n\bar{k}$; if $|\bar{r}| = r$.

Hence the direction cosines of $\bar{r}$ are coefficients of $\bar{i}, \bar{j}, \bar{k}$ in the unit vector of $\bar{r}$.

If $\bar{i}, \bar{j}, \bar{k}$ are three non-coplanar unit vectors along the axes OX, OY, OZ such that $\bar{i}, \bar{j}, \bar{k}$ form a vector triad in the right handed system, then $((\bar{i}, \bar{j}) = (\bar{j}, \bar{k}) = (\bar{k}, \bar{i}) = 90^o$. If $\bar{r}$ is any vector, there exist a unique triad of real numbers x, y, z such that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Then $r = |\bar{r}| = \overline{OP} = \sqrt{x^2 + y^2 + z^2}$.

23. Division Formulae:

i) If the position vectors of the points A, B w.r.t. O are $\bar{a}$ and $\bar{b}$ if the point C divides the line segment $\overline{AB}$ in the ratio m: n internally (m > 0, n > 0), then the position vector of C is $\overline{OC} = \frac{m\bar{b}+n\bar{a}}{m+n}$.

ii) If C is an externally point that divides A($\bar{a}$), B($\bar{b}$) in the ratio m : n, then $\overline{OC} = \frac{m\bar{b}-n\bar{a}}{m-n}$.

iii) If the point C divides A ($\bar{a}$), B ($\bar{b}$) in the ratio 1 : 1 (mid point), then $\overline{OC}\ \frac{\bar{a}+\bar{b}}{2}$.

iv) Points of trisection: Two points which divide a line segment into 3 equal parts are called trisecting points in ratio 1 : 2 or 2 : 1.

v) The position vector of the centroid G of the triangle ABC with vertices $\bar{a}$, $\bar{b}$, $\bar{c}$ is $\frac{\bar{a}+\bar{b}+\bar{c}}{3}$

and the in-centre I = $\frac{a\bar{a}+b\bar{b}+c\bar{c}}{a+b+c}$, where a = BC, b = CA and c = AB.

vi) If G is the centroid of ΔABC, then $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$.

vii) If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are the position vectors of the vertices A, B, C and D respectively of a tetrahedron ABCD, the position vector of its centroid is $\frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$.

24. Rotation of a Vector about an Axis:

Let $\bar{a}$ = (a_1, a_2, a_3). If the system is rotated about

i) X-axis through an angle α, then the new coordinates of $\bar{a}$ are (a_1, a_2 cos α + sin α, –a_2 sin α + a_3 cos α)

ii) Y-axis through an angle α, then the new coordinates of $\bar{a}$ are (–a_3 sin α + a_1 cos α, a_2, a_3 cos α + a_1 sin α).

iii) Z-axis through an angle α, then the new coordinates of $\bar{a}$ are (a_1 cos α + a_2 sin α, –a_1 sin α + a_2 cos α, a_3).

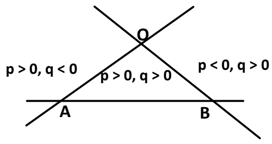
25. Let O be the origin, $\overline{OA} = \overline{a}$, $\overline{OB} = \overline{b}$ be two vectors. Then the point $\overline{OC} = p\overline{a} + q\overline{b}$ lies

i) Inside ΔABC, if $p > 0$, $q > 0$ and $p + q < 1$.

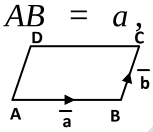
ii) Outside ΔOAB but inside ∠AOB if $p > 0$, $q > 0$ and $p + q > 1$.

iii) Outside ΔOAB but inside ∠AOB if $p < 0$, $q > 0$ and $p + q > 1$.

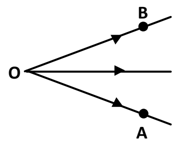
iv) Outside ΔOBA but inside ∠OBA if $p > 0$, $q < 0$ and $p + q < 1$.



26. If ABCD is a parallel such that $\overline{AB} = \overline{a}$, $\overline{BC} = \overline{b}$ then $\overline{AC} = \overline{b} + \overline{a}$, $\overline{BD} = \overline{b} - \overline{a}$.



27. Unit vector bisecting the angle between $\overline{OA} = \overline{a}$, $\overline{OB} = \overline{b}$ is $\dfrac{\hat{a}+\hat{b}}{\left|\hat{a}+\hat{b}\right|}$. The vector bisecting angle between $\overline{OA} = \overline{a}$; $\overline{OB} = \overline{b}$ is of the form $\lambda\left(\hat{a}+\hat{b}\right)$.



28. Vector Equation of a Line:

i) Vector equation of a line parallel to the vector $\overline{b}$ and passing through the point A with position vector $\overline{a}$ is $\overline{r} = \overline{a} + t\,\overline{b}$.

ii) Vector equation of a straight line parallel to $\overline{b}$ and passing through the origin is $\overline{r} = t\overline{b}$; $t \in R$.

iii) Vector equation of a straight line passing through A ($\overline{a}$), B ($\overline{b}$) is $\overline{r} = (1 - t)\,\overline{a} + t\overline{b}$, $t \in R$.

29. Cartesian Equation of a Line:

i) Cartesian equation to the straight line passing through the point (a_1, a_2, a_3) and parallel to the vector $b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$ is $\dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2} = \dfrac{z-a_3}{b_3}$.

ii) Cartesian equation of the straight line passing through two given points (a_1, a_2, a_3) and (b_1, a_2, b_3) is

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3} \text{ or}$$

$$= \frac{y-b_2}{b_2-a_2} = \frac{z-b_3}{b_3-a_3}$$

30. Vector Equation of Plane:

i) Vector equation of a plane passing through a point A($\bar{a}$) and parallel to the non-collinear vectors $\bar{b}$ and $\bar{c}$ is $\bar{r} = \bar{a} + s\bar{b} + t\bar{c}$; s, t ∈ R.

ii) Vector equation of plane passing through the origin and parallel to $\bar{b}$, $\bar{c}$ is $\bar{r} = s\bar{b} + t\bar{c}$; s, t ∈ R.

iii) Vector equation of a plane passing through three non collinear points A ($\bar{a}$), B ($\bar{b}$) and C ($\bar{c}$) is $\bar{r} = (1 - s - t)\bar{a} + s\bar{b} + t\bar{c}$; s, t ∈ R.

iv) Vector equation of a plane passing through the points A($\bar{a}$) . B($\bar{b}$) and parallel to C($\bar{c}$) is $\bar{r} = (1 - s)\ \bar{a} + s\bar{b} + t\bar{c}$; s, t ∈ R.