Quadratic Expressions

1. The standard form of a quadratic equation is $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$

2. The roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3. For the equation $ax^2+bx+c = 0$, sum of the roots $= -\frac{b}{a}$, product of the roots $=\frac{c}{a}$.

4. If the roots of a quadratic are known, the equation is

x^2 - (sum of the roots)x +(product of the roots)= 0

5. "Irrational roots" of a quadratic equation with "rational coefficients" occur in conjugate pairs. If $p + \sqrt{q}$ is a root of $ax^2 + bx + c = 0$, then $p - \sqrt{q}$ is also a root of the equation.

6. "Imaginary" or "Complex Roots" of a quadratic equation with "real coefficients" occur in conjugate pairs. If p + iq is a root of $ax^2 + bx + c = 0$. Then p - iq is also a root of the equation.

7. Nature of the roots of $ax^2 + bx + c = 0$

Nature of the Roots	Condition
Imaginary	$b^2 - 4ac < 0$
Equal	$b^2 - 4ac = 0$
Real	$b^2 - 4ac \geq 0$
Real and different	$b^2 - 4ac > 0$
Rational	$b^2 - 4ac$ is a perfect square a, b, c being rational
Equal in magnitude and opposite in sign	b = 0
Reciprocal to each other	c = a
both positive	b has a sign opposite to that of a and c
both negative	a, b, c all have same sign
opposite sign	a, c are of opposite sign

8.Two equations $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ have exactly the same roots if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

9. The equations $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ have a common root,

if $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$ and the common root is $\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$ if $a_1b_2 \neq a_2b_1$

10. If $f(x) = 0$ is a quadratic equation, then the equation whose roots are

(i) The reciprocals of the roots of $f(x) = 0$ is $f\left(\frac{1}{x}\right) = 0$

(ii) The roots of $f(x) = 0$, each 'increased' by k is $f(x - k) = 0$

(iii) The roots of $f(x) = 0$, each 'diminished' by k is $f(x + k) = 0$

(iv) The roots of $f(x) = 0$ with sign changed is $f(-x) = 0$

(v) The roots of $f(x) = 0$ each multiplied by $k(\neq 0)$ is $f\left(\frac{x}{k}\right) = 0$

11. Sign of the expression $ax^2 + bx + c$:

The sign of the expression $ax^2 + bx + c$ is same as that of 'a' for all values of x if $b^2 - 4ac \leq 0$ i.e. if the roots of $ax^2 + bx + c = 0$ are imaginary or equal.

If the roots of the equation $ax^2 + bx + c = 0$ are real and different i.e. $b^2-4ac > 0$, the sign of the expression is same as that of 'a' if x does not lie between the two roots of the equation and opposite to that of 'a' if x lies between the roots of the equation.

12. The expression $ax^2 + bx + c$ is positive for all real values of x if $b^2 - 4ac < 0$ and $a > 0$.

13. The expression $ax^2 + bx + c$ has a maximum value when 'a' is negative and $x = -\frac{b}{2a}$.

Maximum value of the expression $= \frac{4ac - b^2}{4a}$.

14. The expression $ax^2 + bx + c$, has a minimum when 'a' is positive and $x = -\frac{b}{2a}$. Minimum value of the expression $= \frac{4ac - b^2}{4a}$.

Theorems

1. **If the roots of $ax^2 + bx + c = 0$ are imaginary, then for $x \in R$, $ax^2 + bx + c$ and a have the same sign.** **[NOV-1998, APR-1999, OCT-1993]**

Proof:

The roots are imaginary

$b^2 - 4ac < 0$ $4ac - b^2 > 0$

$$\frac{ax^2+bx+c}{a} = x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = \left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} > 0$$

$\therefore$ For $x \in R$, $ax^2 + bx + c$ and a have the same sign.

2. **If the roots of $ax^2 + bx + c = 0$ are real and equal to $\alpha = \frac{-b}{2a}$, then $\alpha \neq x \in R$, $ax^2 + bx + c$ and a will have same sign.**

Proof:

The roots of $ax^2 + bx + c = 0$ are real and equal

$$\Rightarrow b^2 = 4ac \Rightarrow 4ac - b^2 = 0$$

$$\frac{ax^2+bx+c}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$= \left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}$$

$$= \left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$$

$$= \left(x+\frac{b}{2a}\right)^2 > 0 \text{ for } x \neq \frac{-b}{2a} = \alpha$$

For $\alpha \neq x \in R, ax^2 + bx + c$ and a have the same sign.

3. Let be the real roots of ax^2 + bx + c = 0 and $\alpha < \beta$. Then

i) $x \in R$, $\alpha < x < \beta \Rightarrow$ ax^2 + bx + c and a have the opposite signs

ii) $x \in R$, $x < \alpha$ or $x > \beta \Rightarrow$ ax^2 + bx + c and a have the same sign.

(APRIL-1993, 1996)

Proof:

α, β are the roots of ax^2 + bx + c = 0

Therefore, ax^2 + bx + c = $a(x-\alpha)(x-\beta)$

$$\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta)$$

i) **Suppose** $x \in R$, $\alpha < x < \beta$

$\Rightarrow x - \alpha > 0$, $x - \beta < 0$

$$\Rightarrow (x-\alpha)(x-\beta) < 0 \Rightarrow \frac{ax^2+bx+c}{a} < 0$$

$\Rightarrow ax^2 + bx + c$, a have opposite sign

ii) **Suppose** $x \in R,\ x < \alpha$

$x < \alpha < \beta$ then $x - \alpha < 0$, $x - \beta < 0$

$$\Rightarrow (x-\alpha)(x-\beta) > 0 \Rightarrow \frac{ax^2+bx+c}{a} > 0$$

$\Rightarrow ax^2 + bx + c$, a have same sign

Suppose $x \in R,\ x > \beta$, $x > \beta > \alpha$ then $x - \alpha > 0, x - \beta > 0$

$\Rightarrow (x-\alpha)(x-\beta) > 0 \quad \Rightarrow \frac{ax^2+bx+c}{a} > 0$

$\Rightarrow ax^2 + bx + c$, a have same sign

$\therefore\ x \in R,\ x < \alpha$ Or $x > \beta$ $\Rightarrow$ $ax^2 + bx + c$ and a have the same sign.

Very Short Answer Questions

1. Find the roots of the following equations.

i) $x^2 - 7x + 12 = 0$

ii) $-x^2 + x + 2 = 0$

iii) $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$

Sol: i)$x^2 - 7x + 12 = 0$

$(x - 4)(x - 3) = 0$

$x = 4, 3.$

ii) $-x^2 + x + 2 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2, -1.$

iii) $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$

$$x = \frac{-10 \pm \sqrt{100 + 32(3)}}{2 \cdot \sqrt{3}}$$

$$x = \frac{-10 \pm 14}{2\sqrt{3}}$$

$$x = \frac{-10+14}{2\sqrt{3}}, \frac{-10-14}{2\sqrt{3}}$$

$$x = \frac{2}{\sqrt{3}}, \frac{-12}{\sqrt{3}}$$

$$x = \frac{2}{\sqrt{3}}, -4\sqrt{3}$$

2. Find the nature of the roots of the following equations, without finding the roots.

i) $2x^2 - 8x + 3 = 0$

ii) $9x^2 - 30x + 25 = 0$

iii) $x^2 - 12x + 32 = 0$

iv) $2x^2 - 7x + 10 = 0$

Sol: i) $2x^2 - 8x + 3 = 0$

Discriminant $= b^2 - 4ac = 64 - 24 > 0$

Roots are real and distinct.

ii) $9x^2 - 30x + 25 = 0$

Discriminant $= b^2 - 4ac = 900 - 4 \times 9 \times 25 = 0$

Roots are real and equal.

iii) $x^2 - 12x + 32 = 0$

Discriminant $= b^2 - 4ac = (12)^2 - 4 \times 32 > 0$

Roots are real and distinct.

iv) $2x^2 - 7x + 10 = 0$

Discriminant $= (-7)^2 - 4 \times 2 \times 10 < 0$

Roots are imaginary.

3. If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the value of following expressions in terms of a, b, c.

i) $\frac{1}{\alpha}+\frac{1}{\beta}$ **ii)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

iii) $\alpha^4\beta^7+\alpha^7\beta^4$ **iv)** $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$, if $c\neq 0$

v) $\frac{\alpha^2+\beta^2}{\alpha^{-2}+\beta^{-2}}$, if $c \neq 0$.

Sol: **i)** $\alpha+\beta=\frac{-b}{a};\alpha\beta=\frac{c}{a}$

$$\frac{\alpha+\beta}{\alpha\beta}=\frac{\frac{-b}{a}}{\frac{c}{a}}=\frac{-b}{c}$$

ii) $\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$

$$=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c^2}{a^2}}$$

$$=\frac{b^2-2ca}{c^2}$$

iii) $\alpha^4\beta^7+\alpha^7\beta^4$

$$=\alpha^4\beta^4(\beta^3+\alpha^3)$$

$$=(\alpha\beta)^4\left[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\right]$$

$$=\frac{c^4}{a^4}\left[\frac{-b^3}{a^3}+\frac{3c}{a}\cdot\frac{b}{a}\right]$$

$$=\frac{c^4}{a^4}\left[\frac{3abc-b^3}{a^3}\right]$$

iv) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$

$$=\frac{(\alpha^2-\beta^2)^2}{\alpha^2\beta^2}$$

$$=\frac{(\alpha+\beta)^2(\alpha-\beta)^2}{\alpha^2\beta^2}$$

$$=\frac{(\alpha+\beta)^2}{\alpha^2\beta^2}\left[(\alpha+\beta)^2-4\alpha\beta\right]$$

$$=\frac{\frac{b^2}{a^2}}{\frac{c^2}{a^2}}\left[\frac{b^2-4ac}{a^2}\right]$$

$$=\frac{b^2(b^2-4ac)}{c^2a^2}$$

v) $=\dfrac{\alpha^2+\beta^2}{\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}}=\alpha^2\beta^2=\dfrac{c^2}{a^2}$.

4. Find the values of m for which the following equations have equal roots.

i) $x^2 - m(2x - 8) - 15 = 0$

ii) $(m + 1)x^2 + 2(m + 3)x + (m + 8) = 0$

iii) $(2m + 1)x^2 + 2(m + 3)x + (m + 5) = 0$

Sol: **i)** $x^2 - m(2x - 8) - 15 = 0$

$\text{Discriminant} = b^2 - 4ac = 0$

$(-2m)^2 - 4(8m-15) = 0$

$4m^2 - 32m + 60 = 0$

$m^2 - 8m + 15 = 0$

$(m-5)(m-3) = 0$

$m = 3, 5$

ii) (m + 1)x² + 2(m + 3)x + (m + 8) = 0

$$\text{Discriminant} = b^2 - 4ac = 0$$
$$4(m+3)^2 - 4(m+8)(m+1) = 0$$
$$(m+3)^2 - (m^2 + 9m + 8) = 0$$
$$6m + 9 - 9m - 8 = 0$$
$$-3m + 1 = 0$$
$$m = \frac{1}{3}$$

iii)(2m + 1)x² +2(m + 3)x + (m + 5) = 0

$$\text{Discriminant } b^2 - 4ac = 0$$
$$4(m+3)^2 - 4(2m+1)(m+5) = 0$$
$$m^2 + 6m + 9 - (2m^2 + 10m + m + 5) = 0$$
$$-m^2 - 5m + 4 = 0$$
$$m^2 + 5m - 4 = 0$$
$$m = \frac{-5 \pm \sqrt{25+16}}{2}$$
$$m = \frac{-5 \pm \sqrt{41}}{2}$$

5. If α and β are the roots of equation $x^2 + px + q = 0$, form a quadratic equation where roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

Sol: $\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

$$x^2 - \left[(\alpha-\beta)^2 + (\alpha+\beta)^2\right]x + \left[(\alpha-\beta)(\alpha+\beta)\right]^2 = 0$$
$$x^2 - \left[2(\alpha^2 + \beta^2)\right]x + (\alpha+\beta)^2$$
$$\left[(\alpha+\beta)^2 - 4\alpha\beta\right] = 0$$
$$x^2 - 2\left[\frac{b^2 - 2ac}{a^2}\right]x + \frac{b^2}{a^2}\left[\frac{b^2 - 4ac}{a^2}\right] = 0$$

b = p, c = q, a = 1

$$x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q) = 0.$$

6. If $x^2 + bx + c = 0$, $x^2 + cx + b = 0$ ($b \neq c$) have a common root, then show that $b + c + 1 = 0$.

Sol: Given equations are $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$

Let α be the common root. Then

$$\alpha^2 + b\alpha + c = 0$$
$$\alpha^2 + c\alpha + b = 0$$
$$\alpha(b-c) + c - b = 0$$
$$\alpha(b-c) = b - c$$
$$\alpha = 1$$
$$\therefore 1 + b + c = 0$$

7. Prove that roots of $(x – a)(x – b) = h^2$ are always real.

Sol: $(x-a)(x-b) = h^2$

$$x^2 - (a+b)x + (ab - h^2) = 0$$

Discriminant $= (a+b)^2 - 4(ab - h^2) = 0$

$$= (a+b)^2 - 4ab + 4h^2$$
$$= (a-b)^2 + 4h^2$$
$$= (a-b)^2 + (2h)^2 > 0$$

$\therefore$ Roots are real.

8). Find two consecutive positive even integers, the sum of whose squares in 340.

Sol: **Let** $2n$, $2n + 2$ be the two positive even integers. Then

$$(2n)^2 + (2n+2)^2 = 340$$
$$4n^2 + 4n^2 + 8n + 4 = 340$$
$$8n^2 + 8n + 4 = 340$$
$$2n^2 + 2n + 1 = 85$$
$$2n^2 + 2n - 84 = 0$$
$$n^2 + n - 42 = 0$$
$$(n+7)(n-6) = 0$$
$$n = 6$$

12, 14 are two numbers.

9. Find the roots of the equation
$4x^2 - 4x + 17 = 3x^2 - 10x - 17$.

Sol. Given equation can be rewritten as

$x^2 + 6x + 34 = 0$.

The roots of the quadratic equation

$ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here a = 1, b = 6 and c = 34.

Therefore the roots of the given equation are

$$\frac{-6 \pm \sqrt{(6)^2 - 4(1)(34)}}{2(1)} = \frac{-6 \pm \sqrt{-100}}{2}$$

$$= \frac{-6 \pm 10i}{2} \text{(since } i^2 = -1)$$

$$= -3 + 5i,\ -3 - 5i$$

Hence the roots of the given equation are: $-3 + 5i$ and $-3 - 5i$.

10. For what values of m, the equation $x^2 - 2(1 + 3m)x + 7(3 + 2m) = 0$ will have equal roots?

Sol. The given equation has equal roots $\Rightarrow$ discriminant is 0.

$$\Rightarrow \Delta = \{-2(1+3m)\}^2 - 4(1)7(3+2m) = 0$$

$$4(1+3m)^2 - 28(3+2m) = 0$$

$$4(9m^2 - 8m - 20) = 0$$

$$4(m-2)(9m+10) = 0$$

$$36(m-2)\left(m + \frac{10}{9}\right) = 0$$

$$\Rightarrow m = 2 \text{ or } m = -\frac{10}{9}.$$

Therefore the roots of the given equation are equal iff $m \in \left\{-\frac{10}{9}, 2\right\}$.

11).If α and β are the roots of $ax^2 + bx + c = 0$, find the values of $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ in terms of a, b, c.

Sol. From the hypothesis

$\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

$\therefore \alpha^2+\beta^2=(\alpha+\beta)^2-2(\alpha\beta)$

$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)$

$=\frac{b^2}{a^2}-\frac{2ac}{a^2}=\frac{b^2-2ac}{a^2}$

And $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$

$=(\alpha+\beta)[(\alpha+\beta)^2-2\alpha\beta-\alpha\beta]$

$=(\alpha+\beta)\left[(\alpha+\beta)^2-3(\alpha\beta)\right]$

$=\left(-\frac{b}{a}\right)\left[\left(\frac{-b}{a}\right)^2-3\left(\frac{c}{a}\right)\right]$

$=-\frac{b}{a}\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)=\frac{3abc-b^3}{a^3}$.

12. Form a quadratic equation whose roots are $2\sqrt{3}-5$ and $-2\sqrt{3}-5$.

Sol. Let $\alpha=2\sqrt{3}-5$ and $\beta=-2\sqrt{3}-5$.

Then $\alpha+\beta=(2\sqrt{3}-5)+(-2\sqrt{3}-5)=-10$ and $\alpha\beta=(2\sqrt{3}-5)(-2\sqrt{3}-5)$

$=(2\sqrt{3})(-2\sqrt{3})-(2\sqrt{3})(5)-5(-2\sqrt{3})-5(-5) \quad =-12-10\sqrt{3}+10\sqrt{3}+25=13.$

The required equation is $x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-(-10)x+13=0$.

i.e., $x^2 + 10x + 13 = 0$

13. Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$. If $c \neq 0$, then form the quadratic equation whose roots are $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$.

Sol. From the hypothesis we have

$\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Since, $c\neq 0, \alpha\neq 0$ and $\beta\neq 0$

$$\frac{1-\alpha}{\alpha}+\frac{1-\beta}{\beta}=\frac{\beta(1-\alpha)+\alpha(1-\beta)}{\alpha\beta}$$

$$=\frac{\alpha+\beta-2\alpha\beta}{\alpha\beta}$$

$$=\frac{\alpha+\beta}{\alpha\beta}-2=\frac{\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)}-2$$

$$=-\frac{b}{a}-2=-\left(2+\frac{b}{c}\right)$$

And $\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1-\beta}{\beta}\right)=\frac{1-(\alpha+\beta)+\alpha\beta}{\alpha\beta}$

$$=\frac{1-\left(-\frac{b}{a}\right)+\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)}=\frac{a+b+c}{c}.$$

Therefore $x^2-\left\{-\left(2+\frac{b}{c}\right)\right\}x+\frac{a+b+c}{c}=0$ is the required equation.

14. Find a quadratic equation, the sum of whose roots is 1 and the sum of the squares of the roots is 13.

Sol. Let α and β be the roots of a required equation.

Then $\alpha+\beta=1$ and $\alpha^2+\beta^2=13$.

Since $\alpha\beta=\frac{1}{2}\left[(\alpha+\beta)^2-(\alpha^2+\beta^2)\right]$,

$$\alpha\beta=\frac{1}{2}\left[(1)^2-13\right]=-6.$$

Therefore $x^2-(\alpha+\beta)x+\alpha\beta=0$ becomes $x^2-x-6=0$.

This is a required equation.

15. Find all numbers which exceed their square root by 12.

Sol. Let x be the number satisfying the given condition.

Then $x=\sqrt{x}+12 \Rightarrow x-12=\sqrt{x}$ ... (1)

On squaring both sides and simplifying we obtain $(x-12)^2=x$

$\Rightarrow x^2-24x+144=x$

$\Rightarrow x^2-25x+144=0$

$\Rightarrow x(x-16)-9(x-16)=0$

$\Rightarrow (x-16)(x-9)=0$

The roots of the equation (x – 16)(x – 9) = 0 are 9 and 16.

But x = 9 does not satisfy equation (1), while x = 16 satisfies (1). Therefore the required number is 16.

16. Prove that there is a unique pair of consecutive positive odd integers such that the sum of their squares is 290 and find it.

Sol. Let x and x+2 be the two consecutive positive odd integers.

$x^2+(x+2)^2=290$ ...(1)

$\Leftrightarrow x^2+x^2+4x+4=290$

$\Leftrightarrow 2x^2+4x-286=0$

$\Leftrightarrow x^2+2x-143=0$

$\Leftrightarrow x^2+13x-11x-143=0$

$\Leftrightarrow x(x+13)-11(x+13)=0$

$\Leftrightarrow (x+13)(x-11)=0$

$\Leftrightarrow x\in\{-13,11\}$

Hence 11 is the only positive odd integer satisfying equation (1).

Therefore (11, 13) is the unique pair of integers which satisfies the given condition.

17. **Find the quadratic equation whose roots are 2, 5.**

Sol: The quadratic equation whore roots are 2,5 is x^2-(sum of the roots) x+ product of the roots =0

$x^2 - (2 + 5)x + 2.5 = 0$ i.e., $x^2 - 7x + 10 = 0$.

ii. **Find the quadratic equation whose roots are $3+\sqrt{2}, 3-\sqrt{2}$.**

Sol: The quadratic equation is x^2-(sum of the roots) x+ product of the roots =0

$$x^2-(3+\sqrt{2}+3-\sqrt{2})x+(3+\sqrt{2})(3-\sqrt{2})=0$$

i.e., $x^2 - 6x + 7 = 0$.

iii. **Find the quadratic equation whose roots are 2 + 3i, 2 – 3i .**

Sol: The quadratic equation is x^2-(sum of the roots) x+ product of the roots =0

$x^2 - (2 + 3i + 2 - 3i)x + (2 + 3i)(2 - 3i) = 0$

i.e., $x^2 - 4x + 13 = 0$.

iv. **Find the quadratic equation whose roots are –1, $\frac{-7}{3}$.**

Sol: $x^2-\left(\frac{-7}{3}-1\right)x+\left(\frac{-7}{3}\right)(-1)=0$

i.e., $3x^2+10x+7=0$

v. **Find the quadratic equation whose roots are $\pm\frac{3}{\sqrt{2}}$.**

Sol: $x^2-\left(\frac{-3}{\sqrt{2}}+\frac{3}{\sqrt{2}}\right)x+\frac{3}{\sqrt{2}}\times\left(\frac{-3}{\sqrt{2}}\right)=0$

$x^2-\frac{9}{2}=0 \Rightarrow 2x^2-9=0$

vi. Find the quadratic equation whose roots are $\frac{3 \pm i\sqrt{5}}{2}$.

Sol: $x^2 - \left(\frac{3+i\sqrt{5}}{2} + \frac{3-i\sqrt{5}}{2}\right)x + \left(\frac{3+i\sqrt{5}}{2}\right)\left(\frac{3-i\sqrt{5}}{2}\right) = 0$

$$x^2 - 3x + \frac{7}{2} = 0$$

$2x^2 - 6x + 7 = 0$

vii. Find the quadratic equation whose roots are $\frac{r}{q}, -\frac{q}{p}$.

Sol: $x^2 - \left(\frac{r}{q} - \frac{q}{p}\right)x + \frac{r}{q} \times \frac{-q}{p} = 0 \Rightarrow pqx^2 + (q^2 - rp)x - rq = 0$

18. If the quadratic equations $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$, $(b \neq c)$ have a common root then show that $a + 4b + 4c = 0$.

Sol: $ax^2 + 2bx + c = 0$

$ax^2 + 2cx + b = 0$

Let α be a common root.

$a\alpha^2 + 2b\alpha + c = 0$

$a\alpha^2 + 2c\alpha + b = 0$

$2\alpha(b - c) + c - b = 0$

$(2\alpha - 1)(b - c) = 0$

$\alpha = \frac{1}{2}$

$\frac{a}{4} + \frac{2b}{2} + c$

$a + 4b + 4c = 0$

19. If $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$ have a common root, then find p.

Sol: $x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 1, 5$

Now $x^2 - 12x + p = 0$ can have 1 or 5 as root then $1 - 12 + p = 0$

$p = 11$ (or)

$25 - 60 + p = 0$

$p = 35$.

20. If $x^2 - 6x + 5 = 0$ and $x^2 - 3ax + 35 = 0$ have a common root, then find a.

Sol: $x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 5, 1$

Now, 5, 1 satisfy

$x^2 - 3ax + 35 = 0$

$25 - 15a + 35 = 0$

$60 = 15a$

$a = 4$

$1 - 3a + 35 = 0$

$3a = 36$

$a = 12$.

21. If the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ have a common root and the first equation have equal roots, then prove that $2(b + d) = ac$.

Sol: $x^2 + ax + b = 0$

$x^2 + cx + d = 0$

$x^2 + ax + b = 0$ have equal roots.

$\Rightarrow a^2 - 4b = 0$

$a^2 = 4b$

$x^2 + ax + b = 0$

$x^2 + cx + d = 0$

$\alpha(a - c) + b - d = 0$

as $2\alpha = -a$

$$\alpha = -\frac{a}{2}$$

$$\frac{a^2}{4} + c\left(\frac{-a}{2}\right) + d = 0$$

$$a^2 - 2ac + 4d = 0$$

$$4b + 4d = 2ac$$

$$\text{or } 2(b + d) = ac.$$

22. Discuss the signs of the following quadratic expressions when x is real.

i) $x^2 - 5x + 4$

ii) $x^2 - x + 3$

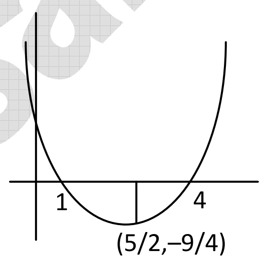
Sol: i)$f(x) = x^2 - 5x + 4$

$$f(x) = (x - 4)(x - 1)$$

$$y = \left(x - \frac{5}{2}\right)^2 + 4 - \frac{25}{4}$$

$$y = \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$$

$$y + \frac{9}{4} = \left(x - \frac{5}{2}\right)^2$$



$f(x) < 0$

$1 < x < 4$

f(x) > 0

x > 4 and x < 1

$(-\infty < x < 1) \cup (4 < x < \infty)$

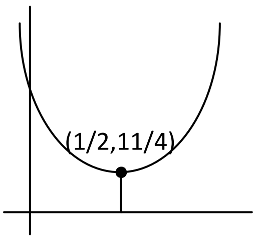
ii) $x^2 - x + 3$

$f(x) = x^2 - x + 3$

$$f(x) = \left(x - \frac{1}{2}\right)^2 + 3 - \frac{1}{4}$$

$$y = \left(x - \frac{1}{2}\right)^2 + \frac{11}{4}$$

$$y - \frac{11}{4} = \left(x - \frac{1}{2}\right)^2$$



$f(x) > 0 \ \forall \ x \in$ Real numbers.

23. For what values of x the following expressions are positive?

i) $x^2 - 5x + 6$

ii) $3x^2 + 4x + 4$

iii) $4x - 5x^2 + 2$

Sol: i) $f(x) = x^2 - 5x + 6$

f(x) > 0

$x^2 - 5x + 6 > 0$

$(x - 3)(x - 1) > 0$

x > 3 or x < 1

$(-\infty < x < 1) \cup (3 < x < \infty)$

ii) $3x^2 + 4x + 4$

$f(x) = 3x^2 + 4x + 4$

$f(x) > 0$

$3x^2 + 4x + 4 > 0$

$$x^2 + \frac{4}{3}x + \frac{4}{3} > 0$$

$$\left(x + \frac{2}{3}\right)^2 + \frac{4}{3} - \frac{4}{9} > 0$$

$$\left(x + \frac{2}{3}\right)^2 + \frac{8}{9} > 0 \ \forall x \in R.$$

iii) $f(x) = 4x - 5x^2 + 2$

$= 5x^2 - 4x - 2$

$$= -5\left[\left(x^2 - \frac{4}{5}x - \frac{2}{5}\right)\right]$$

$$= -5\left[\left(x - \frac{2}{5}\right)^2 - \frac{4}{25} - \frac{2}{5}\right]$$

$$= -5\left[\left(x - \frac{2}{5}\right)^2 - \frac{14}{25}\right]$$

$4x - 5x^2 + 2 > 0$

$5x^2 - 4x - 2 < 0$

$$x = \frac{4 \pm \sqrt{16 + 40}}{10}$$

$$x = \frac{4 \pm \sqrt{14}}{10}$$

$$x = \frac{2 \pm \sqrt{14}}{5}$$

$$\frac{2 - \sqrt{14}}{5} < x < \frac{2 + \sqrt{14}}{5}$$

24. For what values of x, the following expressions are negative?

i) $x^2 - 7x + 10$

ii) $15 + 4x - 3x^2$

iii) $2x^2 + 5x - 3$

Sol: **i) $f(x) = x^2 - 7x + 10$**

$f(x) < 0$

$(x - 5)(x - 2) < 0$

$2 < x < 5$

ii) $f(x) = 15 + 4x - 3x^2$

$f(x) < 0$

$15 + 4x - 3x^2 < 0$

$3x^2 - 4x - 15 > 0$

$3x^2 - 9x + 5x - 15 > 0$

$3x(x - 3) + 5(x - 3) > 0$

$(x - 3)(3x + 5) > 0$

$x > 0, x < \frac{-5}{3}$

$\left(-\infty < x < \frac{-5}{3}\right) \cup (3 < x < \infty)$

iii) $f(x) = 2x^2 + 5x - 3$

$f(x) < 0$

$2x^2 + 5x - 3 < 0$

$2x^2 + 6x - x - 3 < 0$

$2x(x + 3) - 1(x + 3) < 0$

$(2x - 1)(x + 3) < 0$

$-3 < x < \frac{1}{2}.$

25. Find the changes in the sign of the following expressions and find their extreme values.

i) $x^2 - 5x + 6$

ii) $15 + 4x - 3x^2$

Sol: i)$f(x) = x^2 - 5x + 6$

$f(x) = (x - 3)(x - 2)$

$f(x) < 0$

$2 < x < 3$

And $f(x) > 0$

$(-\infty < x < 2) \cup (3 < x < \infty)$

$$y = \left(x - \frac{5}{2}\right)^2 + \frac{6-25}{4}$$

$$y_{min} = -\frac{1}{4}$$

ii) $f(x) = 15 + 4x - 3x^2$

$f(x) < 0$

$15 + 4x - 3x^2 < 0$

$3x^2 - 4x - 15 < 0$

$3x^2 - 9x + 5x - 15 < 0$

$3x(x - 3) + 5(x - 3) < 0$

$(x - 3)(3x + 5) < 0$

$$-\frac{3}{5} < x < 3$$

$f(x) > 0$

$$\left(-\infty < x < \frac{-3}{5}\right) \cup (3 < x < \infty)$$

$y = 15 + 4x - 3x^2$

$$y = -3\left(x^2 - \frac{4}{3}x - 5\right)$$

$$y = -3\left[\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - 5\right]$$

$$y = -3\left(x - \frac{2}{3}\right)^2 + \frac{49}{9} \cdot 3$$

$$y = -3\left(x - \frac{2}{3}\right)^2 + \frac{49}{3}$$

$$y_{min} = \frac{49}{3}$$

26. Find the maximum or minimum of the following expression as x varies over R.

i) $x^2 - x + 7$

iii)$2x + 5 - 3x^2$

iv)$ax^2 + bx + a$ (a, b $\in$ R and a $\neq$ 0)

Sol: i)$y = x^2 - x + 7$

$$y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 7$$

$$y = \left(x - \frac{1}{2}\right)^2 + \frac{27}{4}$$

$$y_{min} = \frac{27}{4}$$

ii) $f(x) = 2x + 5 - 3x^2$

$$y = -3\left[x^2 - \frac{2}{3}x - \frac{5}{3}\right]$$

$$y = -3\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9} - \frac{15}{9}\right]$$

$$y = -3\left(x - \frac{1}{3}\right)^2 + \frac{16}{9}$$

$$y_{max} = \frac{16}{3}$$

iii)f(x) = ax^2 + bx + a

$$y = a\left(x^2 + \frac{b}{a}x + 1\right)$$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 + 1 - \frac{b^2}{4a^2}\right]$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4a^2 - b^2}{4a}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4a^2}{4a}\right)$$

$$y_{max} = \frac{-b^2 + 4a^2}{4a}, a < 0$$

y_{min} when $a > 0$.

27. 1. Solve the following in equations by algebraic method.

i) $15x^2 + 4x - 4 \leq 0$

ii) $x^2 - 2x + 1 < 0$

iii)$2 - 3x - 2x^2 \geq 0$

Sol: i)$15x^2 + 4x - 4 \leq 0$

$15x^2 + 10x - 6x - 4 \leq 0$

$(3x + 2)(5x - 2) \leq 0$

$$\left(x + \frac{2}{3}\right)\left(x - \frac{2}{5}\right) \leq 0$$

$$\frac{-2}{3} \leq x \leq \frac{2}{5}$$

ii) $x^2 - 2x + 1 < 0$

$(x - 1)^2 < 0$

Not possible.

$\because (x - 1)^2 \geq 0$

No solution.

iii) $2 - 3x - 2x^2 \geq 0$

$2x^2 + 3x - 2 \leq 0$

$2x^2 + 4x - x - 2 \leq 0$

$2x(x + 2) - 1(x + 2) \leq 0$

$(2x - 1)(x + 2) \leq 0$

$-2 \leq x \leq \frac{1}{2}$.

Short Answer Questions

1. If x_1, x_2 are the roots of equation $ax^2 + bx + c = 0$ and $c \neq 0$. Find the value of $(ax_1+b)^{-2}+(ax_2+b)^{-2}$ in terms of a, b, c.

Sol: $ax^2 + bx + c = 0$

$$x_1 + x_2 = \frac{-b}{a} \;\&\; x_1.x_2 = \frac{c}{a}$$

$$ax_1^2 + bx_1 + c = 0$$

$$x_1\left(ax_1 + b\right) = -c$$

$$ax_1 + b = \frac{-c}{x_1}$$

Similarly $ax_2 + b = \frac{-c}{x_2}$

$$\therefore (ax_1+b)^{-2}+(ax_2+b)^{-2} = \frac{x_1^2 + x_2^2}{c^2}$$

$$= \frac{1}{c^2}\left[(x_1+x_2)^2 - 2x_1x_2\right]$$

$$= \frac{1}{c^2}\left[\frac{b^2 - 2ac}{a^2}\right]$$

$$= \frac{b^2 - 2ac}{a^2c^2}$$

2. If α, β are the roots of equation $ax^2 + bx + c = 0$, find a quadratic equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$.

Sol: $$x^2 - x\left[\alpha^2+\beta^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right] + (\alpha^2+\beta^2)\left[\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right]=0$$

$$x^2 - x\left[(\alpha+\beta)^2-2\alpha\beta+\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}\right]+\left[\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}\right]=0$$

$$x^2 - x\left[\frac{b^2-2ac}{a^2}+\frac{(b^2-2ac)a^2}{c^2}\right]+\left[\frac{b^2-2ac}{c^2}\right]^2 a^2=0$$

$$a^2c^2x^2-(b^2-2ac)(a^2+c^2)x+(b^2-2ac)^2=0$$

3. Solve the following equation $2x^4+x^3-11x^2+x+2=0$.

Sol: $2x^4+x^3-11x^2+x+2=0$

$$2x^4+x^3-10x^2-x^2+x+2=0$$

$$x^2(2x^2+x-10)-(x^2-x-2)=0$$

$$x^2[(2x+5)(x-2)]-[(x-2)(x+1)]=0$$

$$(x-2)\left[x^2(2x+5)-x-1\right]=0$$

$$(x-2)[2x^3+5x^2-x-1]=0$$

$$(x-2)[(2x-1)(x^2+3x+1)]=0$$

$$x=2,\frac{1}{2};x^2+3x+1=0$$

$$x=\frac{-3\pm\sqrt{9-4}}{2}$$

$$x=\frac{-3\pm\sqrt{5}}{2}$$

4. $3^{1+x}+3^{1-x}=10$

Sol: Let $3^x = t$

$$3\cdot t+\frac{3}{t}=10$$

$$3t^2+3-10t=0$$

$$3t^2-9t-t+3=0$$

$3t(t-3)-1(t-3)=0$

$(3t-1)(t-3)=0$

$t=\frac{1}{3}, t=3$

$3^x=3^{-1}, 3^x=3^1$

$x=-1,1$

5. $\sqrt{\frac{x}{x-3}}+\sqrt{\frac{x-3}{x}}=\frac{5}{2}$ **When $x \neq 0, x \neq 3$.**

Sol: $\sqrt{\frac{x}{x-3}}=t$

$t+\frac{1}{t}=\frac{5}{2}$

$2t^2-5t+2=0$

$2t^2-4t-t+2=0$

$2t(t-2)-1(t-2)=0$

$(2t-1)(t-2)=0$

$t=\frac{1}{2}; t=2$

$\frac{x}{x-3}=4$

$x=4x-12$

$3x=12$

$x=4$

(Or)

$\frac{x}{x-3}=\frac{1}{4}$

$4x=x-3$

$3x=-3$

$x=-1$

6. $2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+5=0$**, when $x \neq 0$.**

Sol: $x+\frac{1}{x}=t$

$2t^2-7t+5=0$

$2t^2-5t-2t+5=0$

$2t(t-1)-5(t-1)=0$

$t=\frac{5}{2}, t=1$ but $x+\frac{1}{x}\geq 2\ \forall x\in R^+$

$x+\frac{1}{x}=\frac{5}{2}$ only possible

$2x^2-5x+2=0$

$(2x-1)(x-2)=0$

$x=\frac{1}{2},2$

Now if $x+\frac{1}{x}=1$

$x^2-x+1=0$

$x=\frac{1\pm\sqrt{1-4}}{2}$

$x=\frac{1\pm\sqrt{3}i}{2}$

7. $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0$**, when $x\neq 0$.**

Sol: $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0$

$x+\frac{1}{x}=t$

$t^2-5t+4=0$

$(t-4)(t-1)=0$

$t=4,1$

$x+\frac{1}{x}=4$

$x^2-4x+1=0$

$x=\frac{4\pm\sqrt{16-4}}{2}$

$x=2\pm\sqrt{3}$

(Or)

$x+\frac{1}{x}=1$

$x^2-x+1=0$

$x=\frac{1\pm\sqrt{1-4}}{2}$

$x=\frac{1\pm\sqrt{3}i}{2}$

8. Find a quadratic equation for which the sum of the roots is 7 and the sum of the squares of the roots is 25.

Sol: $\alpha+\beta=7, \alpha^2+\beta^2=25$

$(\alpha+\beta)^2-2\alpha\beta=25$

$49-2\alpha\beta=25$

$24=2\alpha\beta$

$\alpha\beta=12$

$x^2-7x+12=0.$

9. Determine the range of the following expressions.

i) $\dfrac{x^2+x+1}{x^2-x+1}$

ii) $\dfrac{x+2}{2x^2+3x+6}$

iii) $\dfrac{(x-1)(x+2)}{x+3}$

iv) $\dfrac{2x^2-6x+5}{x^2-3x+2}$

Sol: **i) y =** $\dfrac{x^2+x+1}{x^2-x+1}$

$yx^2-yx+y=x^2+x+1$

$x^2(y^2-1)+x(-y-1)+y-1=0$

Discriminant ≥ 0

$(-y-1)^2-4(y-1)(y-1)\geq 0$

$(y^2+2y+1)-4(y^2-2y+1)\geq 0$

$-3y^2+10y-3\leq 0$

$3y^2-10y+3\leq 0$

$3y^2-9y-y+3\leq 0$

$3y(y-3)-1(y-3)\le 0$

$(3y-1)(y-3)\le 0$

$\frac{1}{3}\le y\le 3$

ii) y = $\frac{x+2}{2x^2+3x+6}$

$2x^2y+3xy+6y-x-2=0$

$x^2(2y)+x(3y-1)+6y-2=0$

Discriminant ≥ 0

$(3y-1)^2-4(2y)(6y-2)\ge 0$

$9y^2-6y+1-48y^2+16y\ge 0$

$-39y^2+10y+1\ge 0$

$39y^2-10y-1\le 0$

$39y^2-13y+3y-1\le 0$

$13y(3y-1)+(3y-1)\le 0$

$(13y+1)(3y-1)\le 0$

$\frac{-1}{13}\le y\le \frac{1}{3}$.

iii) y = $\frac{(x-1)(x+2)}{x+3}$

$y=\frac{x^2+x-2}{x+3}$

$yx + 3y = x^2 + x - 2$

$x^2 + x(1-y) - 2 - 3y = 0$

Discriminant ≥ 0

$(1-y)^2 + 4(2+3y) \ge 0$

$y^2 - 2y + 1 + 8 + 12y \ge 0$

$y^2 + 10y + 9 \ge 0$

$(y + 1)(y + 9) \geq 0$

$y \geq -1$ or $y \leq -9$

$(-\infty < y \leq -9) \cup (-1 \leq y < \infty)$

iv)y = $\dfrac{2x^2 - 6x + 5}{x^2 - 3x + 2}$

$y(x^2 - 3x + 2) = 2x^2 - 6x + 5$

$x^2(y - 2) + x(-3y + 6) + 2y - 5 = 0$

Discriminant ≥ 0

$(6 - 3y)^2 - 4(y - 2)(2y - 5) \geq 0$

$9y^2 - 36y + 36 - 8y^2 + 36y - 40 \geq 0$

$y^2 - 4 \geq 0$

$(y - 2)(y + 2) \geq 0$

$y \geq 2$ or $y \leq -2$

$(-\infty < y \leq -2) \cup (2 \leq y < \infty)$

10. Prove that $\dfrac{1}{3x+1} + \dfrac{1}{x+1} - \dfrac{1}{(3x+1)(x+1)}$ **does not lie between 1 and 4 if x ∈ R.**

Sol: $y = \dfrac{x + 1 + 3x + 1 - 1}{(3x+1)(x+1)}$

$y = \dfrac{4x + 1}{3x^2 + 4x + 1}$

$3yx^2 + x(4y - 4) + y - 1 = 0$

Discriminant ≥ 0

$4(y - 1)^2 - 4 \cdot 3y(y - 1) \geq 0$

$16(y - 1)^2 - 12y(y - 1) \geq 0$

$4(y - 1)[4(y - 1) - 3y)] \geq 0$

$4(y-1)(y-4) \geq 0$

$(y-1)(y-4) \geq 0$

$\Rightarrow y \geq 4$ or $y \leq 1.$

11. If x is real, prove that $\frac{x}{x^2-5x+9}$ lies between $\frac{-1}{11}$ and 1.

Sol: $y = \frac{x}{x^2-5x+9}$

$y(x^2-5x+9) = x$

$x^2y + x(-5y-1) + 9y = 0$

Discriminant ≥ 0

$(-5y-1)^2 - 4 \cdot 9y^2 \geq 0$

$25y^2 + 10y + 1 - 36y^2 \geq 0$

$-11y^2 + 10y + 1 \geq 0$

$11y^2 - 11y + y - 1 \leq 0$

$11y(y-1) + (y-1) \leq 0$

$(y-1)(11y+1) \leq 0$

$\frac{-1}{11} \leq y \leq 1$

12. In the expression $\frac{x-p}{x^2-3x+2}$ takes all values of $x \in$ R, then find the bounds for p.

Sol: $y = \frac{x-p}{x^2-3x+2}$

$y(x^2-3x+2) = x-p$

$x^2y + x(-3y-1) + 2y + p = 0$

Discriminant ≥ 0

$(-3y-1)^2 - 4y(2y+p) \geq 0$

$9y^2 + 6y + 1 - 8y^2 - 4p \geq 0$

$y^2 + y(6-4p) + 1 \geq 0$

Discriminant < 0

$(6-4p)^2-4<0$

$16p^2-48p+36-4<0$

$16p^2-48p+32<0$

$p^2-48p+32<0$

$p^2 - 3p + 2 < 0$

$(p - 2)(p - 1) < 0$

$1 < p < 2.$

13. If $c^2 \neq ab$ and the roots of $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then show that $a^3 + b^3 + c^3 = 3abc$ or $a = 0$.

Sol: Roots are equal $\Rightarrow$ Discriminant = 0

$4(a^2-bc)^2-4(c^2-ab)(b^2-ac)=0$

$a^4+b^2c^2-2a^2bc-b^2c^2+c^3a+b^3a-a^2bc=0$

$a^4-2a^2bc+c^3a+b^3a-a^2bc=0$

$a[a^3+b^3+c^3-3abc]=0$

$\therefore a^3+b^3+c^3-3abc=0 \text{ or } a=0.$

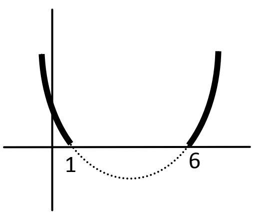
14. Solve the following in equations by graphical method.

i) $x^2 - 7x + 6 > 0$

ii) $15x^2 + 4x - 4 \leq 0$.

Sol: i) $x^2 - 7x + 6 > 0$

$(x - 6)(x - 1) > 0$

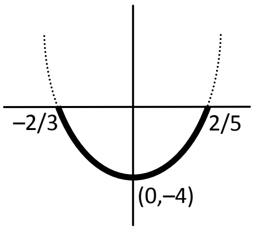


ii) $\mathbf{15x^2 + 4x - 4 \leq 0}$

$15x^2 + 10x - 6x - 4 \leq 0$

$5x(3x + 2) - 2(3x + 2) \leq 0$

$\frac{-2}{3} \leq x \leq \frac{2}{5}$



15. Solve the following in equations.

i) $\sqrt{3x-8} < -2$

ii) $\sqrt{-x^2+6x-5} > 8-2x$.

Sol: **i)** $\sqrt{3x-8} < -2$

Possible when $3x - 8 > 0$

$x > \frac{8}{3}$

Also $\sqrt{3x-8} \geq 0$

$\therefore$ Solution does not exist.

ii) $\sqrt{-x^2+6x-5} > 8-2x$

Possible

$-x^2 + 6x - 5 \geq 0$

$x^2 - 6x + 5 \leq 0$

$(x - 5)(x - 1) \leq 0$

$1 \leq x \leq 5 \quad \ldots (1)$

Squaring on both sides we get

$-x^2+6x-5>64+4x^2-32x$

$0>5x^2-38x+64+5$

or $5x^2-38x+69<0$

$5x^2-23x-15x+69<0$

$5x(x-3)-23(x-3)<0$

$(x-3)(5x-23)<0$

$3<x<\frac{23}{5}$ … (2)

Intersection of (1) and (2)

$3 < x \le 5$.

16. The cost of a piece of cable wire is Rs.35/-. If the length of the piece of wire is 4 meters more and each meter costs Rs.1/- less, the cost would remain unchanged. What is the length of the wire?

Sol. Let the length of the piece of wire be l meters and the cost of each meter br Rs.x/-.

By the given conditions lx = 35

Also, (l + 4)(x – 1) = 35 ...(1)

i.e., lx – l + 4x – 4 = 35...(2)

From (1) and (2), 35 – l + 4x – 4 = 35

i.e., 4x = l + 4

Therefore $x=\frac{l+4}{4}$.

On substituting this value of ‘x’ in (1) and simplifying, we get

$$l\left(\frac{l+4}{4}\right)=35$$

$\Rightarrow l^2+4l-140=0$

$\Rightarrow l^2+14l-10l-140=0$

$\Rightarrow l(l+14)-10(l+14)=0$

$\Rightarrow (l+14)(l-10)=0$

The roots of the equation $(l+14)(l-10)=0$ are –14 and 10.

Since the length cannot be negative, l = 10.

Therefore the length of the piece of wire is 10 meters.

17. Suppose that the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Then show that $a^3 + b^3 + c^3 = 3abc$.

Sol. The condition for two quadratic equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ to have a common root is

$$(c_1a_2-c_2a_1)^2=(a_1b_2-a_2b_1)(b_1c_2-b_2c_1)$$

Here $a_1=a, b_1=b, c_1=c, a_2=b, b_2=c$ and $c_2=a$.

Therefore $(cb-a^2)^2=(ac-b^2)(ba-c^2)$

i.e., $b^2c^2-2a^2bc+a^4=$

$a^2bc-ac^3-b^3a+b^2c^2$

i.e., $a^4+ab^3+ac^3=3a^2bc$

Hence $a^3+b^3+c^3=3abc$ (since $a \neq 0$).

18. Find the maximum or minimum value of the quadratic expression

(i) $2x - 7 - 5x^2$ (ii) $3x^2 + 2x + 11$

Sol.i) Comparing the given expression with $ax^2 + bx + c$,

We have a = –5, b = 2 and c = –7.

So, $\dfrac{4ac-b^2}{4a}=\dfrac{4(-5)(-7)-(2)^2}{4(-5)}=-\dfrac{34}{5}$

And $\dfrac{-b}{2a}=\dfrac{-2}{2(-5)}=\dfrac{1}{5}$.

Since a < 0, $2x - 7 - 5x^2$ has absolute maximum at x = $\dfrac{1}{5}$ and the maximum value is $-\dfrac{34}{3}$.

ii) Here a = 3, b = 2 and c = 11.

So $\dfrac{4ac-b^2}{4a}=\dfrac{4(3)(11)-(2)^2}{4(3)}=\dfrac{32}{3}$ and $\dfrac{-b}{2a}=\dfrac{-2}{6}=-\dfrac{1}{3}$.

Since a > 0, $3x^2 + 2x + 11$ has absolute minimum at $x = -\frac{1}{3}$ and the minimum value is $\frac{32}{3}$.

19. Find the changes in the sign of $4x - 5x^2 + 2$ for $x \in R$ and find the extreme value.

Sol. Comparing the given expression with

$ax^2 + bx + c$, we have $a = -5 < 0$.

The roots of the equation $5x^2 - 4x - 2 = 0$ are $\frac{2 \pm \sqrt{14}}{5}$.

Therefore, when $\frac{2-\sqrt{14}}{5} < x < \frac{2+\sqrt{14}}{5}$, the sign of $4x - 5x^2 + 2$ is positive and when $x < \frac{2-\sqrt{14}}{5}$ or $x > \frac{2+\sqrt{14}}{5}$, the sign of $4x - 5x^2 + 2$ is negative.

Since a < 0, the maximum value of the expression $4x - 5x^2 + 2$ is

$$\frac{4ac - b^2}{4a} = \frac{4(-5)(2) - (4)^2}{4(-5)} = \frac{-56}{-20} = \frac{14}{5}.$$

Hence the extreme value of the expression $4x - 5x^2 + 2$ is $\frac{14}{5}$.

20. Show that none of the values of the function $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ over R lies between 5 and 9.

Sol. Let y_0 be a value of the given function.

Then $\exists\, x_0 \in R$ such that

$$y_0 = \frac{x_0^2 + 34x_0 - 71}{x_0^2 + 2x_0 - 7}.$$

If $y_0 = 1$, then clearly $y_0 \notin (5,9)$.

Suppose that $y_0 \neq 1$. Then the equation $y_0(x^2 + 2x - 7) = x^2 + 34x - 71$ is a quadratic equation and x_0 is a real root of it.

Therefore

$(y_0 - 1)x^2 + (2y_0 - 34)x - (7y_0 - 71) = 0$ is a quadratic equation having a real root x_0.

Since all the coefficients of this quadratic equation are real, the other root of the equation is also real.

Therefore

$$\Delta = (2y_0 - 34)^2 + 4(y_0 - 1)(7y_0 - 71) \geq 0.$$

On simplifying this we get

$$y_0^2 - 14y_0 + 45 \geq 0$$

i.e., $(y_0 - 5)(y_0 - 9) \geq 0$

Therefore $y_0 \leq 5$ or $y_0 \geq 9$.

Hence y_0 does not lie in (5, 9).

Hence none of the values of the given function over R lies between 5 and 9.

21. Find the set of values of x for which the inequalities $x^2 - 3x - 10 < 0$, $10x - x^2 - 16 > 0$ hold simultaneously.

Sol. We have $x^2 - 3x - 10 = (x + 2)(x - 5)$.

Hence –2 and 5 are the roots of the equation $x^2 - 3x - 10 = 0$.

Since the coefficient of x^2 in the quadratic expression $x^2 - 3x - 10$ is positive.

$x^2 - 3x - 10 < 0 \Leftrightarrow -2 < x < 5$.

We have $10x - x^2 - 16 = -(x - 2)(x - 8)$.

Hence 2 and 8 are the roots of the equation $10x - x^2 - 16 = 0$.

Since the coefficient of x^2 in the quadratic expression $10x - x^2 - 16$ is negative.

$10x - x^2 - 16 > 0 \Leftrightarrow 2 < x < 8$.

Hence $x^2 - 3x - 10 < 0$ and $10x - x^2 - 16 > 0$

$\Leftrightarrow x \in (-2, 5) \cap (2, 8)$

Therefore the solution set is

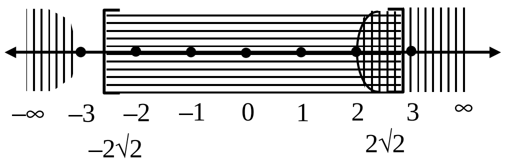
$\{x \in R : 2 < x < 5\}$.

22. Solve the in equation $\sqrt{x+2} > \sqrt{8-x^2}$.

Sol. When a, b ∈ R and $a \geq 0, b \geq 0$ then $\sqrt{a} > \sqrt{b} \Leftrightarrow a > b \geq 0$.

Therefore

$\sqrt{x+2} > \sqrt{8-x^2} \Leftrightarrow (x+2) > (8-x^2) \geq 0$ and $x > -2 \, | \, x \, | < 2\sqrt{2}$.



We have

$$(x+2) > (8-x^2) \Leftrightarrow x^2 + x - 6 > 0 .$$

Now $x^2 + x - 6 = (x+3)(x-2)$

Since –3 and 2 are the roots of the equation $x^2 + x - 6 = 0$ and the coefficient of x^2 in $x^2 + x - 6$ is positive,

$$x^2 + x - 6 > 0 \Leftrightarrow x \in (-\infty, -3) \cup (2, \infty)$$

We have

$$8 - x^2 \geq 0 \Leftrightarrow x^2 \leq 8 \Leftrightarrow | \, x \, | \leq 2\sqrt{2}$$

$$\Leftrightarrow x \in \left[-2\sqrt{2}, 2\sqrt{2}\right]$$

Also $x + 2 > 0 \Leftrightarrow x > -2$

Hence $x + 2 > 8 - x^2 \geq 0$

$$\Leftrightarrow x \in \left((-\infty, -3) \cup (2, \infty)\right) \cap \left[-2\sqrt{2}, 2\sqrt{2}\right] \text{ and } x > -2$$

$$\Leftrightarrow x \in (2, 2\sqrt{2})$$

Therefore the solution set is

$$\left\{x \in \mathbf{R} : 2 < x \leq 2\sqrt{2}\right\}.$$

23. Solve the in equation $\sqrt{(x-3)(2-x)} < \sqrt{4x^2+12x+11}$.

Sol. The given in equation is equivalent to the following two inequalities.

$(x-3)(2-x) \geq 0$ and

$(x-3)(2-x) < 4x^2 + 12x + 11.$

$(x-3)(2-x) \geq 0$

$\Leftrightarrow (x-2)(x-3) \leq 0$

$\Leftrightarrow 2 \leq x \leq 3$

$-x^2 + 5x - 6 < 4x^2 + 12x + 11$

$\Leftrightarrow 5x^2 + 7x + 17 = 0$

The discriminate of the quadratic expression $5x^2 + 7x + 17$ is negative.

Hence $5x^2 + 7x + 17 > 0 \ \forall \ x \in \mathbf{R}$.

Hence the solution set of the given in equation is $\{x \in \mathbf{R} : 2 \leq x \leq 3\}$.

∴