PROBABILITY

1. **Random Experiment**: If the result of an experiment is not certain and is any one of the several possible outcomes, then the experiment is called **Random experiment.**

2. **Sample space**: The set of all possible outcomes of an experiment is called the sample space whenever the experiment is conducted and is denoted by S.

3. **Event**: Any subset of the sample space 'S' is called an **Event.**

4. **Equally likely Events**: A set of events is said to be equally likely if there is no reason to expect one of them in preference to the others.

5. **Exhaustive Events**: A set of events is said to be exhaustive of the performance of the experiment always results in the occurrence of at least one of them.

6. **Mutually Exclusive Events**: A set of events is said to the mutually exclusive if happening of one of them prevents the happening of any of the remaining events.

7. **Classical Definition of Probability**: If there are n mutually exclusive equally likely elementary events of an experiment and m of them are favourable to an event A then the probability of A denoted by P(A) is defined as min.

8. **Axiomatic Approach to Probability**: Let S be finite sample space. A real valued function P from power set of S into R is called probability function if

   (1) $P(A) \geq 0 \ \forall \ A \subseteq S$;

   (2) $P(S) = 1, P(\phi) = 0$;

   (3) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \phi$. Here the image of A w.r.t. P denoted by P(A) is called probability of A.

   Note: 1) $P(A) + P(\overline{A}) = 1$

   2) If $A_1 \subseteq A_2$, then $P(A_1) \leq P(A_2)$ where A_1, A_2 are any two events.

9. Odds in favour and odds against an Event: Suppose A is any Event of an experiment. The odds in favour of Event A is $P(A) : P(\overline{A})$. The odds against of A is $P(\overline{A}) : P(A)$.

10. Addition theorem on Probability: If A, B are any two events in a sample space S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

If A and B are exclusive events

$P(A \cup B) = P(A) + P(B)$

* $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)\ P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

11. Conditional Probability: If A and B are two events in sample space and $P(A) \neq 0$. The probability of B after the event A has occurred is called the conditional probability of B given A and is denoted by P(B/A).

$$P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{P(A \cap B)}{P(A)}$$

Similarly

$$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)}.$$

12. Independent Events: The events A and B of an experiment are said to be independent if occurrence of A cannot influence the happening of the event B.

i.e. A, B are independent if $P(A/B) = P(A)$ or $P(B/A) = P(B)$.

i.e. $P(A \cap B) = P(A) . P(B)$.

13. Multiplication Theorem: If A and B are any two events in S then

$P(A \cap B) = P(A)\ P(B/A)$ if $P(A) \neq 0$.

$P(B)\ P(A/B)$ if $P(B) \neq 0$.

The events A and B are independent if

$P(A \cap B) = P(A)\ P(B)$.

A set of events $A_1, A_2, A_3 \dots A_n$ are said to be pair wise independent if

$P(A_i \cap A_j = P(A_i)\ P(A_J)$ for all $i \neq J$.

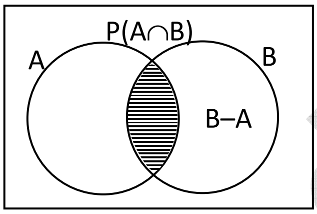
THEOREMS

Theorem: Addition Theorem on Probability. If A, B are two events in a sample space S,

Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Sol. From the figure (Venn diagram) it can be observed that

$(B - A) \cup (A \cap B) = B$, $(B - A) \cap (A \cap B) = \phi$.



$$\therefore P(B) = P[(B - A) \cup (A \cap B)]$$
$$= P(B - A) + P(A \cap B)$$
$$\Rightarrow P(B - A) = P(B) - P(A \cap B) \qquad \dots(1)$$

Again from the figure, it can be observed that

$$A \cup (B - A) = A \cup B, A \cap (B - A) = \phi$$
$$\therefore P(A \cup B) = P[A \cup (B - A)]$$
$$= P(A) + P(B - A)$$
$$= P(A) + P(B) - P(A \cap B) \text{ since from (1)}$$
$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Theorem: Multiplication Theorem on Probability.

Let A, B be two events in a sample space S such that P(A) ≠ 0, P(B) ≠ = 0, then

i) $P(A \cap B) = P(A)P\left(\frac{B}{A}\right)$

ii) $P(A \cap B) = P(B)P\left(\frac{A}{B}\right)$

Sol. Let S be the sample space associated with the random experiment. Let A, B be two events of S show that P(A) ≠ 0 and P(B) ≠ 0. Then by def. of confidential probability.

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$$

$$\therefore P(B \cap A) = P(A)P\left(\frac{B}{A}\right)$$

Again, $\because P(B) \neq 0$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right)$$

$$\therefore P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right) = P(B) \cdot P\left(\frac{A}{B}\right)$$

Baye's Theorem or Inverse probability Theorem:

Statement: If A_1, A_2, ... and A_n are 'n' mutually exclusive and exhaustive events of a random experiment associated with sample space S such that $P(A_i) > 0$ and E is any event which takes place in conjuction with any one of A_i then

$$P(A_k/E) = \frac{P(A_k)P(E/A_k)}{\sum_{i=1}^{n} P(A_i)P(E/A_i)}$$, for any $k = 1,2.....n$;

Proof: Since A_1, A_2,..., A_n are mutually exclusive and exhaustive in sample space S, we have $A_i \cap A_j =$ for i ≠ j, $1 \le i, j \le n$ and $A_1 \cup A_2 \cup .... \cup A_n = S$.

Since E is any event which takes place in conjunction with any one of A_i, we have

$E = (A_1 \cap E) \cup (A_2 \cap E) .... \cup (A_n \cap E).$

We know that $A_1, A_2,$ ...... A_n are mutually exclusive, their subsets $(A_1 \cap E)$, $(A_2 \cap E)$, ... are also mutually exclusive.

Now $P(E)=P(E \cap A_1)+P(E \cap A_2)+.....+ P(E \cap A_n)$ (by axiom of additively)

$= P(A_1)\, P(E/A_1) + P(A_2)\, P(E/A_2) + ... + P(A_n)\, P(E/A_n)$

(By multiplication theorem of probability)

$$= \sum_{i=1}^{n} P(A_i)P(E / A_i) \qquad .... (1)$$

By definition of conditional probability,

$$P(A_k/E) = \frac{P(A_k \cap E)}{P(E)} \quad \text{for}$$

$$= \frac{P(A_k)P(E / A_k)}{P(E)}$$

(By multiplication theorem)

$$= \frac{P(A_k)P(E / A_k)}{\sum_{i=1}^{n} P(A_I)p(E / A_i)} \quad \text{from (1)}$$

Hence the theorem

Very Short Answer Questions

1. a In the experiment of throwing a die, consider the following events:

i. A = {1, 3, 5}, B = {2, 4, 6}, C = {1, 2, 3} are these events equally likely?

Sol. Since the events A, B, C has equal chance to occur hence they are equally likely events.

ii. In the experiment of throwing a die, consider the following events:

A = {1, 3, 5}, B = {2, 4} C = {6}Are these events mutually exclusive ?

Sol. Since the happening of one of the given events A, B, C prevents the happening of other two, hence the given events are mutually exclusive. Otherwise $A \cap B = \phi$, $B \cap C = \phi$, $C \cap A = \phi$ hence they are mutually exclusive events.

ii. In the experiment of throwing a die, consider the events. A = {2, 4, 6}, B = {3, 6}, C= {1, 5, 6} are these events exhaustive?

Sol. A = {2, 4, 6}, B = {3, 6}, C = {1, 5, 6}

Let S be the sample space for the random experiment of throwing a die then S = {1, 2, 3, 4, 5, 6}

$\therefore A \subset S$, $B \subset S$ and $C \subset S$, and $A \cup B \cup C = S$

Hence events A, B, C are exhaustive events.

b. Give two examples of mutually exclusive and exhaustive events.

Sol. Examples of mutually exclusive events:

i) The events {1, 2}, {3, 5} are disjoint in the sample space S = {1, 2, 3, 4, 5, 6}

ii) When two dice are thrown, the probability of getting the sums of 10 or 11.

Examples of exhaustive events:

i) The events {1, 2, 3, 5}, {2, 4, 6} are exhaustive in the sample space

S = {1, 2, 3, 4, 5, 6}

ii) The events {HH, HT}, {TH, TT} are exhaustive in the sample space

S = {HH, HT, TH, TT}

[$\because$ Tossing two coins]

2. Give examples of two events that are neither mutually exclusive nor exhaustive.

Sol. i) Let A be the event of getting an even prime number when tossing a die and let B be the event of getting even number.

$\therefore$ A, B are neither mutually exclusive nor exhaustive.

ii) Let A be the event of getting one head tossing two coins. Let B be the event of getting at least one head tossing two coins.

$\therefore$ A, B are neither mutually exclusive nor exhaustive.

3. Give two examples of events that are neither equally likely nor exhaustive.

Sol. i) Two coins are tossed let A be the event of getting at least one tail.

$\therefore$ A, B are neither equally likely nor exhaustive.

ii) When a die is tossed

Let A be the event of getting an odd prime number and

Let B be the event of getting odd number.

$\therefore$ A, B are neither equally likely nor exhaustive.

4. If 4 fair coins are tossed simultaneously, then find the probability that 2 heads and 2 tails appear.

Sol. 4 coins are tossed simultaneously.

Total number of ways = $2^4 = 16$

$n(S) = 16$

From 4 heads we must get 2 heads.

Number of ways of getting 2 heads.

$$= {}^4C_2 = \frac{4.3}{1.2} = 6$$

$\therefore n(E) = 6$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{16} = \frac{3}{8}$$

$\therefore$ Probability of getting 2 heads and 2 tails

$$= \frac{3}{8}$$

5. Find the probability that a non-leap year contains i) 53 Sundays ii) 52 Sundays only

Sol. A non - leap year contains 365 days 52 weeks and 1 day more.

i) We get 53 Sundays when the remaining day is Sunday.

Number of days in week = 7

$\therefore n(S) = 7$

Number of ways getting 53 Sundays.

$n(E) = 1$

$\therefore$ p(E) = $\frac{n(E)}{n(S)} = \frac{1}{7}$

$\therefore$ Probability of getting 53 Sundays = $\frac{1}{7}$

ii) Probability of getting 52 Sundays

p(E) = 1– p(E) $= 1 - \frac{1}{7} = \frac{6}{7}$

6. Two dice are rolled. What is the probability that none of the dice shows the number 2 ?

Sol. Random experiment is rolling 2 dice.

$n(S) = 6^2 = 36$

Let E be the event of not getting 2

$n(E) = 5 \times 5 = 25$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{25}{36}$$

7. In an experiment of drawing a card at random from a pack, the event of getting a spade is denoted by A and getting a pictured card (King, Queen or Jack) is denoted by B. Find the probability of A, B, $A \cap B$ and $A \cup B$.

Sol. A is the event of getting a spade from the pack

$$\therefore P(A) = \frac{13}{52} = \frac{1}{4}$$

B is the event of getting a picture card

$$P(B) = \frac{4 \times 3}{52} = \frac{3}{13}$$

$A \cap B$ is the event of getting a picture card in spades.

$n(A \cap B) = 3$, $n(s)$ 52

$P(A \cap B) = \frac{3}{52}$

$A \cup B$ is the event of getting spade or a picture card.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{4} + \frac{3}{13} - \frac{3}{52} = \frac{13+12-3}{52} = \frac{22}{52} = \frac{11}{26}$$

8. In a class of 60 boys and 20 girls, half of the boys and half of the girls know cricket. Find the probability of the event that a person selected from the class is either a boy or a girl who knows cricket.

Sol. Let A be the event that the selected person is a boy and B be the event that the selected person knows a cricket. When a person is selected from the class and S be the sample space.

Now, $n(S) = {}^{80}C_1 = 80$, $n(A) = {}^{60}C_1 = 60$

$n(B) = {}^{40}C_1 = 40$: $n(A \cap B) = {}^{30}C_1 = 30$

$$\therefore P(A) = \frac{60}{80}, P(B) = \frac{40}{80}, P(A \cap B) = \frac{30}{80}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{60}{80} + \frac{40}{80} - \frac{30}{80} = \frac{70}{80} = \frac{7}{8}$$

9. For any two events A and B, show that $P(A^C \cap B^C) = 1 + P(A \cap B) - P(A) - P(B)$.

Sol. $A^C \cap B^C = \overline{A \cup B}$

$$P\left(A^C \cap B^C\right) = P\left(\overline{A \cup B}\right)$$

$$= 1 - P(A \cup B)$$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 + P(A \cap B) - P(A) - P(B)$

10. Two persons A and B are rolling two dice on the condition that the person who gets 3 will win the game. If A starts the game, then find the probabilities of A and B respectively to win the game.

Sol. $P = \text{Probability of getting } 3 = \frac{1}{6}$

$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Probability success (p) = $\frac{1}{6}$

Probability of failure (q) = $\frac{5}{6}$

A may win the game either in I trail or in III trial or in V trial etc.

Probability of A win

$= p + q.\ q.\ p + q.\ q.\ q.\ q.\ p + \ldots$

$$\therefore P(A) = \frac{p}{1-q^2} = \frac{\left(\frac{1}{6}\right)}{1-\left(\frac{5}{6}\right)^2} = \frac{1}{6} \times \frac{36}{11} = \frac{6}{11}$$

$\therefore$ Probability of B's win = 1 – p(A)

$p(B) = 1 - \frac{6}{11} = \frac{5}{11}$

Hence $p(A) = \frac{6}{11}$ and $p(B) = \frac{5}{11}$

11. A, B, C are 3 newspaper from a city. 20% of the population read A, 16% read B, 14% read C, 8% both B and C, 2% all the three. Find the percentage of the population who read at least one newspaper.

Sol. Given $p(A)=\frac{20}{100}=0.2$

$$p(B)=\frac{16}{100}=0.16$$

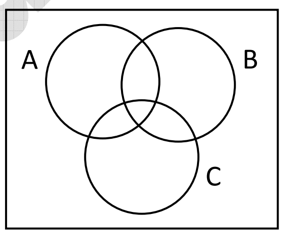
$$p(C)=\frac{14}{100}=0.14$$

$$p(A\cap B)=\frac{8}{100}=0.08$$

$$p(B\cap C)=\frac{4}{100}=0.04$$

$$p(A\cap C)=\frac{5}{100}=0.05$$

$$p(A\cap B\cap C)=\frac{2}{100}=0.02$$



$$p(A\cup B\cup C)=p(A)+p(B)+p(C)-p(A\cap B)-p(B\cap C)-p(C\cap A)+p(A\cap B\cap C)$$

$$=0.2+0.16+0.14-0.08-0.04-0.05+0.02$$

$$=0.52-0.17=0.35$$

Percentage of population who read at least one newspaper = $0.35\times 100 = 35\%$

12. If one ticket is randomly selected from tickets numbered 1 to 30. Then find the probability that the number on the ticket is

i) A multiple of 5 or 7

ii) A multiple of 3 or 5

Sol. i) number of ways drawing one ticket

$= n(S) = 30C_1 = 30$

Suppose A is the event of getting a multiple of 5 and B is the event of getting a multiple of 7

$A = \{5, 10, 15, 20, 25, 30\}$

$B = \{7, 14, 21, 28\}$

$A \cap B = \phi \Rightarrow$ A and B are mutually exclusive

$P(A \cup B) = P(A) + P(B)$

$$= \frac{6}{30} + \frac{4}{30} = \frac{10}{30} = \frac{1}{3}$$

Probability of getting a multiple of 5 or 7 $= \frac{1}{3}$

ii) Suppose A is the event of getting a multiple of 3 and B is the event of getting a multiple of 5.

$A = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$

$B = \{5, 10, 15, 20, 25, 30\}$

$A \cap B = \{15, 30\}$

$$P(A) = \frac{10}{30}; P(B) = \frac{6}{30}; P(A \cap B) = \frac{2}{30}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{10}{30} + \frac{6}{30} - \frac{2}{30}$$

$$= \frac{10+6-2}{30} = \frac{14}{30} = \frac{7}{15}$$

Probability of getting a multiple of 3 or 5 = $\frac{7}{15}$

13. If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.

Sol. i) Let A be the event that the sum of the numbers is even when two numbers are selected out of 20 consecutive natural numbers.

In 20 consecutive natural numbers, we have 10 odd and 10 even natural number.

$\because$ The sum of two odd natural numbers is an even number and the sum of two even natural numbers is also an even number

$$n(A) = {}^{10}C_2 + {}^{10}C_2 = \frac{2(10\times 9)}{1\times 2} = 190 \quad n(S) = {}^{20}C_2 = \frac{20\times 19}{1\times 2} = 190 \quad P(A) = \frac{n(A)}{n(S)} = \frac{90}{190} = \frac{9}{19}$$

ii) Probability that the sum of two numbers is an odd number

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{9}{19} = \frac{10}{19}$$

14. A game consists of tossing a coin 3 times and noting its outcome. A boy wins if all losses give the same outcome and loses otherwise. Find the probability that the boy loses the game.

Sol. If a coin is tossed 3 times then the total number of sample points n(S) = 2^3 = 8.

For winning the game all tosses gives the same outcome.

Let E be such that event then E = {HHH, TTT}

$\therefore$ n(E) = 2

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{8}$$

$\therefore$ The probability for the boy to loose the game.

$$P(\bar{E}) = 1 - P(E)$$

$$= 1 - \frac{2}{8} = \frac{6}{8}.$$

15. If E_1, E_2 are two events with $E_1 \cap E_2 = \phi$, then show that $P(E_1^C \cap E_2^C) = P(E_1^C) - P(E_2)$.

Sol. $P(E_1^C \cap E_2^C) = P\left[(E_1 \cup E_2)^C\right]$

$$= 1 - [P(E_1 \cup E_2)]$$
$$= 1 - [P(E_1) + P(E_2)]$$
$$[\because P(E_1 \cap E_2) = P(\phi) = 0]$$
$(\because$ from addtion theorem)
$$= 1 - P(E_1) - P(E_2)$$
$$= P(E_1^C) - P(E_2)$$

16. Three screws are drawn at random from a lot of 50 screws, 5 of which are defective. Find the probability of the event that all 3 screws are non-defective assuming that the drawing is (a) with replacement, (b) without replacement.

Sol. Let S be the sample space

$\therefore$ The total number of screws = 50

The number of defective screws is 5 and the remaining 45 screws are non-defective.

Let A be the event of getting drawing of the 3 screws are non-defective.

a) With replacement

$$P(A) = \frac{^{45}C_1}{^{50}C_1} \times \frac{^{45}C_1}{^{50}C_1} \times \frac{^{45}C_1}{^{50}C_1}$$

$$= \frac{45}{50} \times \frac{45}{50} \times \frac{45}{50} = \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} = \left(\frac{9}{10}\right)^3$$

$$P(A) = \left(\frac{9}{10}\right)^3$$

b) Without replacement

$$P(A)=\frac{^{45}C_1}{^{50}C_1}\times\frac{^{44}C_1}{^{49}C_1}\times\frac{^{43}C_1}{^{48}C_1}$$

$$=\frac{45}{50}\times\frac{44}{49}\times\frac{43}{48}\Rightarrow P(A)=\frac{1419}{1960}$$

17. If A, B, C are three independent events such that $P(A\cap B^C\cap C^C)=\frac{1}{4}$

$P(A^C\cap B\cap C^C)=\frac{1}{8}, P(A^C\cap B^C\cap C^C)=\frac{1}{4}$ **then find P(A), P(B) and P(C).**

Sol. Since A, B, C are independent events.

$$P(A\cap B^C\cap C^C)=\frac{1}{4}$$

$$\Rightarrow P(A)\cdot P(B^C)\cdot P(C^C)=\frac{1}{4} \quad ...(1)$$

$$P(A^C\cap B\cap C^C)=\frac{1}{8}$$

$$\Rightarrow P(A^C)\cdot P(B)\cdot P(C^C)=\frac{1}{8} \quad ...(2)$$

$$P(A^C\cap B^C\cap C^C)=\frac{1}{4}$$

$$\Rightarrow P(A^C)\cdot P(B^C)\cdot P(C^C)=\frac{1}{4} \quad ...(3)$$

$$\frac{(1)}{(3)}\Rightarrow\frac{P(A)}{P(A^C)}=\frac{1/4}{1/4}=1$$

$$\Rightarrow\frac{P(A)}{1-P(A)}=1\Rightarrow P(A)=1-P(A)$$

$$\Rightarrow 2P(A)=1\Rightarrow P(A)=\frac{1}{2}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{P(B)}{P(B^C)} = \frac{1/8}{1/4} \Rightarrow \frac{P(B)}{1-P(B)} = \frac{1}{2}$$

$$\Rightarrow 2P(B) = 1 - P(B) \Rightarrow 3P(B) = 1$$

$$\therefore P(B) = \frac{1}{3}$$

From (1) $\Rightarrow P(A)\cdot P(B^C)\cdot P(C^C) = \frac{1}{4}$

$$\Rightarrow \left(\frac{1}{2}\right)\left(1-\frac{1}{3}\right)P(C^C) = \frac{1}{4}$$

$$\Rightarrow P(C^C) = \frac{1}{4}\times 2\times\frac{3}{2} = \frac{3}{4}$$

$$\therefore P(C) = 1 - P(C^C) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(C) = \frac{1}{4}$$

18. There are 3 black and 4 white balls in one bag. 4 black and 3 white balls in the second bag. A die is rolled and the first bag is selected if it is 1 or 3 and the second bag for the rest. Find the probability of drawing a black ball from the bag thus selected.

Sol. Probability of selecting first bag = $\frac{2}{6} = \frac{1}{3}$

Probability of selecting second bag = $1 - \frac{1}{3} = \frac{2}{3}$

Probability of getting a black ball from first bag = $\frac{3}{7}$

Probability of getting a black ball from the second bag = $\frac{4}{7}$

Probability of drawing a black ball $= \frac{1}{3}\cdot\frac{3}{7} + \frac{2}{3}\cdot\frac{4}{7} = \frac{3+8}{21} = \frac{11}{21}$

19. A, B, C are aiming to shoot a balloon. A will succeed 4 times out of 5 attempts. The chance of B to shoot the balloon is 3 out of 4 and that of C is 2 out of 3. If three aim the balloon simultaneously, then find the probability that at least two of them hit the balloon.

Sol. Given $P(A) = \frac{4}{5}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{2}{3}$

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{4}{5} = \frac{1}{5}$$

$$P(\bar{B}) = 1 - P(B) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(\bar{C}) = 1 - P(C) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P(A \cap B \cap \bar{C}) = P(A)P(B)P(\bar{C})$$

$$= \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{5}$$

$$P(A \cap \bar{B} \cap C) = P(A)P(\bar{B})P(C)$$

$$= \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{2}{15}$$

$$P(\bar{A} \cap B \cap C) = P(\bar{A})P(B)P(C)$$

$$= \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{10}$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

$$= \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{2}{5}$$

Probability that at least two of them hit the balloon = $\frac{1}{5} + \frac{2}{15} + \frac{1}{10} + \frac{2}{3} = \frac{25}{30} = \frac{5}{6}$

20. If A, B are two events, then show that $P\left(\frac{A}{B}\right)P(B) + P\left(\frac{A}{B^C}\right)P(B^C) = P(A)$**.**

Sol. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

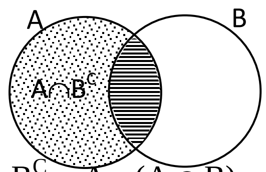
Hint: $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow P\left(\frac{A}{B}\right)P(B) = P(A\cap B) \qquad ...(1)$$

$$P\left(\frac{A}{B^C}\right) = \frac{P(A\cap B^C)}{P(B^C)} \qquad ...(2)$$

$$(1)+(2) \Rightarrow P\left(\frac{A}{B}\right)P(B) + P\left(\frac{A}{B^C}\right)P(B^C)$$

$$= P(A\cap B) + P(A\cap B^C)$$



Hint: $A\cap B^C = A-(A\cap B)$

$$= P(A\cap B) + P(A) - P(A\cap B) = P(A)$$

21. A pair of dice are rolled. What is the probability that they sum to 7, given that neither die shows a 2?

Sol.Let A be the event that the sum of the two dice is 7, then

$$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

Let B be the event that neither die shows a 2

$$\begin{aligned}B = \{&(1,1),(1,3)(1,4),(1,5),(1,6),\\ &(3,1),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

$$n(B) = 25$$
$$A\cap B = \{(1,6),(3,4),(4,3),(6,1)\}$$
$$n(A\cap B) = 4$$

Required probability

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{n(A\cap B)}{n(B)} = \frac{4}{25}$$

22. A pair of dice are rolled. What is the probability that neither die shows a 2, given that they sum to 7?

Sol. Let A be the event that the sum on two dice is 7.

$$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$
$$\therefore n(A) = 6$$

Let B be the event that neither die shows a 2

$$\begin{aligned} B = \{&(1,1),(1,3)(1,4),(1,5),(1,6),\\ &(3,1),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

$$n(B) = 25$$
$$A \cap B = \{(1,6),(3,4),(4,3),(6,1)\}$$
$$n(A \cap B) = 4$$

Required probability

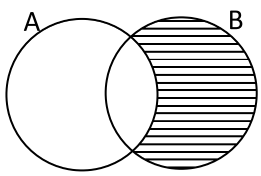
$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)} = \frac{4}{6} = \frac{2}{3}$$

23. If A, B are any two events, in an experiment, and P(B) ≠ 1. Show that

$$P\left(\frac{A}{B^C}\right) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

Hint: $P(A \cap B^C) = P(A) - P(A \cap B)$

Sol. By definition of condition probability



$$P\left(\frac{A}{B^C}\right)=\frac{P(A\cap B^C)}{P(B^C)}=\frac{P(A)-P(A\cap B)}{1-P(B)}$$

$$\because P(B^C)=P(\overline{B})=1-P(B)$$

24. An urn contains 12 red balls and 12 green balls. Suppose two balls are drawn one after another without replacement. Find the probability that the second ball drawn is green given that the first ball drawn is red.

Sol. Total number of balls in an urn n(S) = 24

Let E_1 be the event of drawing a red ball in first draw $P(E_1)=\frac{{}^{12}C_1}{24}=\frac{1}{2}$

Now the number of balls remaining are 23

Let $\frac{E_2}{E_1}$ be the events of drawing a green ball in the second drawn $P\left(\frac{E_2}{E_1}\right)=\frac{12}{23}$.

$\therefore$ Required probability

$$P(E_1\cap E_2)=P(E_1)\cdot P\left(\frac{E_2}{E_1}\right)$$

$$=\frac{1}{2}\times\frac{12}{23}=\frac{6}{23}$$

$\therefore$ The probability true the second ball drawn is green given that the first ball drawn is red

$$=\frac{6}{23}$$

25. A single die is rolled twice in succession. What is the probability that the number showing on the second toss is greater than that on the first rolling?

Sol. A single die is rolled twice. Let S be the sample space n(S) = 6^2 = 36.

Let A be the event of getting the required event.

$$A = \{(1,2),(1,3)(1,4),(1,5),(1,6),\\ (2,3),(2,4),(2,5),(2,6),(3,4),\\ (3,5),(3,6),(4,5),(4,6),(5,6)\}$$

$$n(A) = 15$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

26. If one card is drawn at random from a pack of cards then show that event of getting an ace and getting heart are independent events.

Sol. Suppose A is the event of getting an ace and B is the event of getting a heart.

$$\therefore P(A) = \frac{4}{52} = \frac{1}{13},\ P(B) = \frac{13}{52} = \frac{1}{14}$$

$A \cap B$ is the event of getting a Heart's ace

$$\therefore P(A \cap B) = \frac{1}{52} = \frac{1}{13} \cdot \frac{1}{4} = P(A) \cdot P(B)$$

$\therefore$ A and B are independent events.

27. The probability that a boy A will get a scholarship is 0.9 and that another boy B will get is 0.8. What is the probability that at least one of them will get the scholarship?

Sol. Suppose E_1 is the event of boy A getting scholarship and E_2 is the event of another boy B getting the scholarship.

Given $P(E_1) = 0.9$, $P(E_2) = 0.8$

E_1 and E_2 are independent events.

$$P(E_1 \cap E_2) = P(E_1),\ P(E_2) = (0.9)(0.8) = 0.72$$

Probability that at least one of them will give a scholarship

$$\begin{aligned}&= P(E_1 \cup E_2)\\&= P(E_1) + P(E_2) - P(E_1 \cap E_2)\\&= (0.9) + (0.8) - (0.72) = 0.98\end{aligned}$$

28. If A, B are two events with $P(A \cup B) = 0.65$ and $P(A \cap B) = 0.15$, then find the value of $P(A^C) + P(B^C)$.

Sol. By addition theorem on probability

$$\begin{aligned}&P(A \cup B) = P(A) + P(B) - P(A \cap B)\\&P(A) + P(B) = P(A \cup B) + P(A \cap B)\\&\quad = 0.65 + 0.15 = 0.8\end{aligned}$$

$$\begin{aligned}&P(A^C) + P(B^C) = 1 - P(A) + 1 - P(B)\\&\quad = 2 - [P(A) + P(B)] = 2 - 0.8 = 1.2\end{aligned}$$

29. If A, B, C are independent events, show that $A \cup B$ and C are independent events.

Sol. $\because$ A, B, C are independent events.

$\Rightarrow$ A, B ; B, C ; C, A are also independent events.

$$\begin{aligned}&P(A \cap B \cap C) = P(A)P(B)P(C)\\&P(A \cap C) = P(A)P(C)\\&P(B \cap C) = P(B)P(C)\\&P(A \cap B) = P(A)P(B)\\&P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]\\&= P(A \cap C) + P(B \cap C) - P[(A \cap C) \cap (B \cap C)]\\&= P(A)P(C) + P(B)P(C) - P(A \cap B \cap C)\\&= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)\\&= [P(A) + P(B) - P(A)P(B)]P(C)\\&= P(A \cup B) \cdot P(C)\end{aligned}$$

$\therefore$ $A \cup B$ and C are independent events.

30. A and B are two independent events such that the probability of the both the events to occur is 1/6 and the probability of both the events do not occur is 1/3. Find the probability of A.

Sol. A and B are independent events.

$$P(A \cap B) = P(A)P(B) = \frac{1}{6} \qquad ...(1)$$

$$P(\bar{A} \cap \bar{B}) = \frac{1}{3} \Rightarrow P(\overline{A \cup B}) = \frac{1}{3}$$

$$P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(A) + P(B) - P(A \cap B) = \frac{2}{3}$$

$$P(A) + P(B) = P(A \cap B) + \frac{2}{3}$$

$$= \frac{2}{3} + \frac{1}{6} = \frac{4+1}{6} = \frac{5}{3} \qquad ...(2)$$

Suppose P(A) = x and P(B) = y so that

$$x + y = \frac{5}{6}, xy = \frac{1}{6} \text{ [From (1) and (2)]}$$

$$(x-y)^2 = (x+y)^2 - 4xy$$

$$= \left(\frac{5}{6}\right)^2 - 4 \cdot \frac{1}{6} = \frac{25}{36} - \frac{4}{6} = \frac{25-24}{36} = \frac{1}{36}$$

$$x - y = \pm\frac{1}{6}$$

Case(i) : $x - y = \frac{1}{6}, x + y = \frac{5}{6}$

Adding $\Rightarrow$ 2x = 1 $\Rightarrow$ x = $\frac{1}{2}$

$\therefore\ P(A) = \frac{1}{2}$

Case (ii): $x - y = -\frac{1}{6} \Rightarrow 2x = \frac{4}{6} \Rightarrow x = \frac{1}{3}$

$\therefore P(A) = \frac{1}{3}$

$P(A) = \frac{1}{2}$ (or) $\frac{1}{3}$

31. A fair die is rolled. Consider the events. A = {1, 3, 5}, B = {2, 3} and C = {2, 3, 4, 5}. Find

i) $P(A \cap B), P(A \cup B)$

ii) $P\left(\frac{A}{B}\right), P\left(\frac{B}{A}\right)$

iii) $P\left(\frac{A}{C}\right), P\left(\frac{C}{A}\right)$

iv) $P\left(\frac{B}{C}\right), P\left(\frac{C}{B}\right)$

Sol. A fair die is rolled

$P(A) = \frac{3}{6} = \frac{1}{2}$

$P(B) = \frac{2}{6} = \frac{1}{3}$

$P(C) = \frac{4}{6} = \frac{2}{3}$

$n(S) = 6^1 = 6$

Given A = {1, 3, 5}, B = {2, 3}, C = {2, 3,4,5}

i) $A \cap B = \{3\} P(A \cap B) = P\{3\} = \frac{1}{6}$

$\therefore P(A \cap B) = \frac{1}{6}$

$P(A \cup B) = \{1, 2, 3, 5\}$

$n(A \cup B) = 4$

$n(S) = 6$

$P(A \cup B) = \frac{4}{6} = \frac{2}{3}$

ii) $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/3} = \frac{1}{2}$

$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$

iii) $P\left(\frac{A}{C}\right) = \frac{P(A \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}$

$\therefore A \cap C = \{3, 5\}$

$P\left(\frac{C}{A}\right) = \frac{P(A \cap C)}{P(A)} = \frac{2/6}{3/6} = \frac{2}{3}$

iv) $P\left(\frac{B}{C}\right) = \frac{P(B \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}$

$\therefore B \cap C = \{2, 3\}$

$P\left(\frac{C}{B}\right) = \frac{P(C \cap B)}{P(B)} = \frac{2/6}{2/6} = 1$

32. If A, B, C are three events in a random experiment, prove the following.

i) $P\left(\frac{A}{A}\right) = 1$

Sol. $P\left(\frac{A}{A}\right) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$

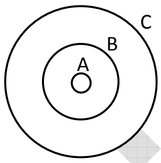
ii) $P\left(\frac{\phi}{A}\right)=0$

Sol. $P\left(\frac{\phi}{A}\right)=\frac{P(\phi\cap A)}{P(A)}=\frac{0}{P(A)}=0$

iii) $A\subset B\Rightarrow P\left(\frac{A}{C}\right)\le P\left(\frac{B}{C}\right)$

Sol. We know that

$$P\left(\frac{A}{C}\right)=\frac{P(A\cap C)}{P(C)}$$



$$P\left(\frac{B}{C}\right)=\frac{P(B\cap C)}{P(C)}$$

$A\subseteq B$

$A\cap C\le B\cap C$

$P(A\cap C)\le P(B\cap C)$

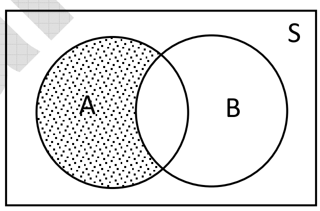
$$\frac{P(A\cap C)}{P(C)}\le\frac{P(B\cap C)}{P(C)}$$

$$\therefore P\left(\frac{A}{C}\right)\le P\left(\frac{B}{C}\right)$$

iv) P(A – B) = P(A) – P(A ∩ B)

Sol. $A-B=\{x/x\in A \ni x\notin B\}$

$A-B=A-(A\cap B)$



$P(A-B)=P[A-(A\cap B)]=P(A)-P(A\cap B)$

v) If A, B are mutually exclusive and P(B) > 0 then $P\left(\frac{A}{B}\right)=0$.

Sol. We know $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{0}{P(B)}=0$

[$\because$ A, B are mutually exclusive events]

Hint: A, B are mutually exclusive than $A \cap B = \phi$, $\Rightarrow P(A \cap B) = 0$.

vi) If A, B are mutually exclusive then $P\left(\frac{A}{B^C}\right)=\frac{P(A)}{1-P(B)}$ **; when P(B) ≠ 1.**

Sol. Given $P(A \cap B) = 0$ ($\because$ A and B are mutually exchange)

$$P\left(\frac{A}{B^C}\right)=\frac{P(A\cap B^C)}{P(B^C)}=\frac{P[A-(A\cap B)]}{P(B^C)}$$

$$=\frac{P(A)-P(A\cap B)}{1-P(B)}=\frac{P(A)-0}{1-P(B)}$$

$$\therefore P\left(\frac{A}{B}\right)=\frac{P(A)}{1-P(B)}$$

$\because P(B)\neq 1$

vii) If A, B are mutually exclusive and P(A∪B)≠0 then $P\left(\frac{A}{A\cup B}\right)=\frac{P(A)}{P(A)+P(B)}$.

Hint: $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Sol. $P\left(\frac{A}{A\cup B}\right)=\frac{P[A\cap(A\cup B)]}{P(A\cup B)}$

$$=\frac{P[(A\cap A)\cup(A\cap B)]}{P(A\cup B)}$$

$$=\frac{[P(A\cap A)+P(A\cap B)]}{P(A)+P(B)-P(A\cap B)}=\frac{P(A)+0}{P(A)+P(B)-0}$$

$$\therefore P\left(\frac{A}{A\cup B}\right)=\frac{P(A)}{P(A)+P(B)}$$

33. Suppose that a coin is tossed three times. Let event A be 'getting three heads' and B be the event of 'getting a head on the first toss'. Show that A and B are dependent events.

Sol. When a coin is tossed 3 times,

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$\Rightarrow n(S) = 6, n(A) = 1, P(A) = \frac{1}{6}$

When event B occurred, the possibilities are HHH, HHT, HTH, HTT in which HHH is favorable for the event $\frac{A}{B}$.

$P\left(\frac{A}{B}\right) = \frac{1}{4}$

Hence $P(A) \neq P\left(\frac{A}{B}\right)$

$\therefore$ A, B are dependent events.

34. Suppose that an unbiased pair of dice is rolled. Let A denote the event that the same number shows on each die. Let B denote the event that the sum is greater than 7. Find

(i) $P\left(\frac{A}{B}\right)$**, (ii)** $P\left(\frac{B}{A}\right)$**.**

Sol. n(S) = 36

Let A be the event of getting the same number on two dice.

n(A) =

{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}=6

$P(A) = \frac{6}{36}$

Let B be the event at getting the sum greater than 7.

n(B) = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)} = 15

$P(B) = \frac{15}{36}$

$n(A \cap B) = \{(4,4),(5,5),(6,6)\} = 3$

$P(A \cap B) = \frac{3}{36}$

i) $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{36}}{\frac{15}{36}} = \frac{3}{15} = \frac{1}{5}$

ii) $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{3}{36}}{\frac{1}{6}} = \frac{3}{6} = \frac{1}{2}$

35. Prove that A and B independent events if and only if $P\left(\frac{A}{B}\right) = P\left(\frac{A}{B^C}\right)$.

Sol. Let A and B are independent.

L.H.S. : $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$= \frac{P(A)P(B)}{P(B)} = P(A)$

R.H.S.: $P\left(\frac{A}{B^C}\right) = \frac{P(A \cap B^C)}{P(B^C)}$

$= \frac{P(A)P(B^C)}{P(B^C)} = P(A)$

$\therefore$ L.H.S. = R.H.S.

36. A number x is drawn arbitrarily from the set {1, 2, 3, … 100}. What is the probability that $\left(x+\frac{100}{x}\right)>29$.

Sol. Here the total number of cases is 100.

Let A be the event that x selected from the set {1, 2, 3, …100} has the property $x+\frac{100}{x}>29$

Now $x+\frac{100}{x}>29$

$$\Leftrightarrow x^2-29x+100>0$$
$$\Leftrightarrow (x-4)(x-25)>0$$
$$\Leftrightarrow x>25 \text{ or } x>4$$
$$\Leftrightarrow x\in\{1,2,3,26,27,...100\}=A(\text{say})$$

so that the number of cases favorable to A is 78.

$\therefore$ The required probability $P(A)=\frac{78}{100}$

37. Two squares are chosen at random on a chess board. Show that the probability that they have a side in common is 1/18.

Sol. The number of ways of choosing the first square is 64 and that of the second is 63.

$\therefore$ The number of ways of choosing the first and second squares = 64×63

$\therefore$ n(S) = 64×63

Let E be the event that these squares have a side in common.

If the first square happens to be one of the squares in the four corners of the chess board, the second square (with common side) can be choosen in 2 ways.

If the first square happens to be any one of the remaining 24 squares along the four sides of the chess board other than the corner, the second square can be choosen in 3 ways.

If the first square happens to be any one of the remaining 36 inner squares, then the second square can be choosen in 4 ways.

Hence the number of cases favourable to E is $(4 \times 2) + (24 \times 3) + (36 \times 4) = 224$

$\therefore$ The required probability =

$$\frac{n(E)}{n(S)} = \frac{224}{64 \times 63} = \frac{1}{18}$$

38. A fair coin is tossed 200 times. Find the probability of getting a head an odd number of times.

Sol. The total number of cases is 2^{200}

The number of favorable cases is

$$= {}^{200}C_1 + {}^{200}C_3 + {}^{200}C_5 + \ldots + {}^{200}C_{199}$$

$$= \frac{2^{200}}{2} = 2^{199}$$

$\therefore$ Probability $\frac{2^{191}}{2^{200}} = \frac{1}{2}$.

39. A and B are among 20 persons sit at random along a round table. Find the probability that there are any 6 persons between A and B.

Sol. Let A occupy any seat around the table. Then there are 19 seats available for B. But if there are to be six persons between A and B, then B has only two ways to sit.

$\therefore$ Probability = $\frac{2}{19}$

40. Out of 30 consecutive integers two are drawn at random. Then what is the probability that their sum is odd.

Sol. The total number of ways of choosing 2 out of 30 numbers = ${}^{30}C_2$

Out of these 30 numbers, 15 are even and 15 are odd.

For the sum of the choosen two numbers to be odd, one should be odd and the other even.

$\therefore$ The number of cases favorable

$$= {}^{15}C_1 \times {}^{15}C_1$$

$$\therefore \text{ Probability } = \frac{{}^{15}C_1 \times {}^{15}C_1}{{}^{30}C_2} = 2 \times \frac{15 \times 15}{30 \times 29} = \frac{15}{29}$$

41. What is the probability of throwing a total score of 7 with two dice?

Sol. Let S be the sample space and A be the event of getting a total score of 7 when two dice are thrown

S = {(1, 1),(1, 2)…(1,6),(2,1)…(2,6)…

(6,1),(6,2)…(6,6)}

n(S) = 36

Hint: $6^2 = 36$

A = {(1, 6) (6, 1) (2, 5) (5, 2) (3, 4) (4, 3)}

n(A) = 6

$$\therefore \; P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

42. What is the probability of obtaining two tails and one head when three coins are tossed?

Sol. Let S be the sample space and A be the event of getting two tails and one head when three coins are tossed.

$$n(S) = 2^3 = 8$$
$$A = [HTT, THT, TTH]$$
$$n(A) = 3$$
$$P(A) = \frac{3}{8}$$

43. A page is opened at random from a book containing 200 pages. What is the probability that the number of the page is a perfect square?

Sol. Let S be the sample space. Let A be the event of getting on the page is perfect square.

n(S) = 200

Let A be the event of drawing a page whose number is perfect square.

A = {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196}

n(A) = 14

$$P(A) = \frac{14}{200} = \frac{7}{100} = 0.07$$

44. Find the probability of drawing on ace or a spade from a well shuffled pack of 52 cards?

Sol.**Hint:** A pack of cards means of pack containing 52 cards, 26 of them are red and 26 of them are black coloured. These 52 cards are divided into 4 sets namely Hearts, Spades, Diamonds and Clubs. Each set contains of 13 cards names A, 2, 3, 4, 5, 6, 7, 8, 9, 10, K, Q,J.

Let E_1 be the event of drawing a space and E_2 be the event of drawing an ace. E_1, E_2 are not mutually exclusive.

$n(A) = 13,\ n(B) = 4,\ n(A \cap B) = 1$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

45. If A and B are two events, show that

i) $P(A \cap B^C) = P(A) - P(A \cap B)$ and

ii) The probability that one of them occurs is given by $P(A) + P(B) - 2P(A \cap B)$.

Sol. i) We have $A = (A \cap B) \cup (A \cap B^C)$

and $(A \cap B) \cap (A \cap B^C) = \phi$

$\therefore P(A) = (A \cap B) + P(A \cap B^C)$

$\therefore P(A \cap B^C) = P(A) - P(A \cap B)$ ...(1)

ii) Let E be the event that exactly one of them, (i.e.) either A or B occurs. Given

$E = (A - B) \cup (B - A) = (A \cap B^C) \cup (B \cap A^C)$

So $P(E) = P(A \cap B^C) + P(B \cap A^C)$

$\because (A \cap B^C) \cap (B \cap A^C) = \phi$

$\Rightarrow P(E) = P(A) - P(A \cap B) + P(B) - P(A \cap B)$

by (1)

$\therefore P(E) = P(A) + P(B) - 2P(A \cap B)$

46. Suppose A and B are events with P(A) = 0.5, P(B) = 0.4 and P(A $\cap$ B) = 0.3. Find the probability that (i) A does not occur, (ii) neither A nor B occurs.

Sol. We have A^C = the event "A does not occur"

$(A \cup B)^C$ = neither A nor B occurs

$\therefore P(A^C) = 1 - P(A) = 1 - 0.5 = 0.5$

Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.5 + 0.4 - 0.3 = 0.6$

$P[(A \cup B^C)] = 1 - P(A \cup B) = 1 - 0.6 = 0.4$

Short Answer Questions

1. A pair of dice rolled 24 times. A person wins by not getting a pair of 6's on any of the 24 rolls. What is the probability of his winning?

Sol. Random experiment is tossing two dice 24 times

$= 36\times 36\times \ldots\ldots\ldots 36 = (36)^{24}$

$\therefore$ n(S) = $(36)^{24}$

Let A be the event of not getting a pair of 6's on any of the 24 rolls.

$\therefore$ number of ways favorable to an event A

$= 35\times 35\times \ldots\ldots\ldots \times 35 = (35)^{24}$

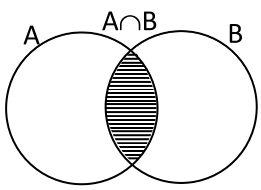
$n(A) = (35)^{24}$

$$\therefore P(A) = \frac{(35)^{24}}{(36)^{24}} = \left(\frac{35}{36}\right)^{24}$$

2. If P is a probability function, then show that for any two events A and B.

$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$

Sol. For any sets A, B we have



$A \cap B \le A \le A \cup B$

$P(A \cap B) \le P(A) \le P(A \cup B)$ ...(1)

By addition theorem of probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\leq P(A) + P(B)$ ...(2)

From (1), (2) we get

$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

3. In a box containing 15 bulbs, 5 are defective. If 5 bulbs are selected at random from the box, find the probability of the event, that

i) None of them is defective

ii) Only one of them is defective.

iii) At least one of them is defective.

Sol. Out of 15 bulbs, 5 are defective probability of selecting a defective bulb $= P = \frac{5}{15} = \frac{1}{3}$

We are selecting 5 bulbs $n(S) = {}^{15}C_5$

i) None of them is defective. All the 5 bulbs must be selected from 10 good bulbs. This can be done in $10C_5$ ways.

$$= P(A) = \frac{10C_5}{15C_5} = \frac{10.9.8.7.6}{15.14.13.12.11} = \frac{12}{143}$$

ii) Only one of them is defective in 4 good and 1 defective balls.

This can be done in $10C_4\ 5C_1$

$$= \frac{10.9.8.7}{1.2.3.4}.5 = 210 \times 5 = 1050.$$

Probability of selecting one defective

$$= \frac{1050}{15C_5}$$

$$= (1050)\frac{1.2.3.4.5}{15.14.13.12.11} = \frac{50}{143}$$

iii)Probability at least one of them is defective

$$=P(A)=1-P(A)=1-\frac{12}{143}=\frac{131}{143}$$

4. A and B are seeking admission into I.I.T. the probability for A to be selected is 0.5 and that both to be selected is 0.3. is it possible that the probability of B to be selected is 0.9 ?

Sol. Given $P(A) = 0.5$; $P(A \cap B) = 0.3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.5 + P(B) - 0.3$

$= 0.2 + P(B)$

$P(A \cup B) \leq 1$

$0.2 + P(B) \leq 1$

$P(B) \leq 0.8$

$\therefore$ It is not possible to have $P(B) = 0.9$

5. The probability for a contractor to get a road contract is 2/3 and to get a building contract is 5/9 the probability to get at least on contract is 4/5. Find the probability to gets both the contracts.

Sol. Suppose A is the event of getting a road contract. B is the event of getting a building contract

$$\text{Given } P(A)=\frac{2}{3};P(B)=\frac{5}{9};P(A\cup B)=\frac{4}{5}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$=\frac{2}{3}+\frac{5}{9}-\frac{4}{5}=\frac{30+25-36}{45}=\frac{19}{45}$$

Probability to get both contracts $= \frac{19}{45}$

6. In a committee of 25 members, each member is proficient either in mathematics or in statistics or in both. If 19 of these are proficient in mathematics, 16 in statistics, find the probability that a person selected from the committee is proficient in both.

Sol. When a person is chosen at random from the academy consisting of 25 members, let A be the event that the person is proficient in mathematics, B be the event that the person is proficient in statistics and S be the sample space. Since 19 members are proficient in mathematics and 16 members are proficient in statistics.

$$P(A) = \frac{19}{25}, P(B) = \frac{16}{25}$$

Since everyone is either proficient in mathematics or statistics or in both

$A \cup B = S$

$\Rightarrow P(A \cap B) = P(S)$

$\Rightarrow P(A) + P(B) - P(A \cap B) = P(S)$

$$\Rightarrow \frac{19}{25} + \frac{16}{25} - P(A \cap B) = 1$$

$$= P(A \cap B) = \frac{19}{25} + \frac{16}{25} - 1 = \frac{19+16-25}{25} = \frac{10}{25}$$

$$\therefore P(A \cap B) = \frac{2}{5}$$

7. A, B, C are three horse in a race. The probability of A to win the race is twice that of B and probability of B is twice that of C. what are the probabilities of A , B and C to win the race ?

Sol. Let A, B, C be the events that the horses A, B, C win the race respectively.

Given P(A) = 2P(B), P(B) = 2P(C)

$\therefore$ P(A) = 2P(B) 2[2P(C)] = 4P(C)

Since the horses A, B and C run the race,

$A \cup B \cup C = S$ and A, B, C are mutually disjoint

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$\Rightarrow P(S) = 4p(C) + 2p(C) + P(C)$

$\Rightarrow 1 = 7P(C)$

$$\therefore P(C) = \frac{1}{7}$$

$$P(A) = 4P(C) = 4 \times \frac{1}{7} = \frac{4}{7}$$

$$P(B) = 2P(C) = 2 \times \frac{1}{7} = \frac{2}{7}$$

$$\therefore P(A) = \frac{4}{7}, P(B) = \frac{2}{7}, P(C) = \frac{1}{7}$$

8. A bag contains 12 two rupee coins, 7 one rupee coins and 4 half rupee coins. If 3 coins are selected at random find the probability that

i) The sum of the 3 coins is maximum

ii) The sum of the 3 is minimum

iii)Each coin is of different value.

Sol. In the bag, there are 12 two rupee, 7 one rupee and 4 half rupee coins.

Total number of coins = 12 + 7 + 4 = 23

Number of ways drawing 3 coins $23C_3$

$n(S) = 23C_3$

i) We get maximum amount, if the coins are 2 rupee coins.

Number of drawing 3 two rupee coins

$= 12C_3$

$n(E_1) = 12C_3$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{12C_3}{23C_3}$$

ii) We get minimum amount if 3 coins are taken from 4 half rupee coins. Number of ways of drawing 3 half rupee coins $= 4C_3$

$n(E_2) = 4C_3$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4C_3}{23C_3}$$

iii)Each coin is of different value is we must draw one coin each.

This can be done in $12C_1$, $7C_1$, $4C_1$ ways

$n(E_3) = 12C_1 \times 7C_1 \times 4C_1 = 12 \times 7 \times 4$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{12\times7\times4}{23C_3}$$

9. The probabilities of three events A, B, C are such that P(A) = 0.3, P(B) = 0.4, P(C) = 0.8, P(A ∩ B) = 0.08, P(A ∩C) = 0.28, P(A ∩ B ∩ C) = 0.09 and P(A ∪ B ∪ C) ≥ 0.75. Show that P(B ∩ C) lies in the interval [0.23, 0.48].

Sol. $P(A \cup B \cup C) \geq 0.75$

$0.75 \leq P(A \cup B \cup C) \leq 1$

$$\Rightarrow 0.75 \leq P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \leq 1$$

$$\Rightarrow 0.75 \leq 0.3 + 0.4 + 0.8 - 0.08 - 0.28 - P(B \cap C) + 0.09 \leq 1$$

$$\Rightarrow 0.75 \leq 1.23 - P(B \cap C) \leq 1$$

$$\Rightarrow -0.75 \geq P(B \cap C) - 1.23 \geq -1$$

$$\Rightarrow 0.48 \geq P(B \cap C) \geq 0.23$$

$$\Rightarrow 0.23 \leq P(B \cap C) \leq 0.48$$

∴ P(B ∩ C) lies in the interval [0.23, 0.48].

10. The probabilities of three mutually exclusive events are respectively given as $\frac{1+3p}{3}, \frac{1-p}{4}, \frac{1-2p}{2}$. Prove that $\frac{1}{3} \leq p \leq \frac{1}{2}$.

Sol. Suppose A, B, C are exclusive events such that

$$P(A) = \frac{1+3p}{3}$$

$$P(B) = \frac{1-p}{4}$$

$$P(C) = \frac{1-2p}{2}$$

We know that

$0 \le P(A) \le 1$

$0 \le \frac{1+3p}{3} \le 1$

$0 \le 1+3p \le 3$

$-1 \le 3p \le 3-1$

$\frac{-1}{3} \le p \le \frac{2}{3}$ ...(1)

$0 \le P(B) \le 1$

$0 \le \frac{1-p}{4} \le 1$

$0 \le 1-p \le 4$

$-1 \le -p \le 4-1$

$1 \ge p \ge -3$

$-3 \le p \le 1$ ...(2)

$0 \le P(C) \le 1$

$0 \le \frac{1-2p}{2} \le 1$

$0 \le 1-2p \le 2$

$-1 \le -2p \le 2-1$

$1 \ge 2p \ge -1$

$\frac{1}{2} \ge p \ge -\frac{1}{2}$

$\frac{-1}{2} \le p \le \frac{1}{2}$ ...(3)

Since A, B, C are exclusive events,

$0 \le P(A \cup B \cup C) \le 1$

$\Rightarrow 0 \le P(A) + P(B) + P(C) \le 1$

$\Rightarrow 0 \le \frac{4+12P+3-3P+6-12P}{12} \le 1$

$\Rightarrow 0 \le \frac{13-3P}{12} \le 1$

$\Rightarrow 0 \le 13-3P \le 12$

$\Rightarrow -13 \le -3P \le 12-13$

$\Rightarrow 13 \ge 3P \ge 1$

$$\Rightarrow \frac{13}{3} \geq P \geq \frac{1}{3}$$

$$\Rightarrow \frac{1}{3} \leq P \leq \frac{13}{3} \quad ...(4)$$

Max. of $\left\{\frac{-1}{3}, -3, \frac{-1}{2}, \frac{1}{3}\right\} = \frac{1}{3}$

Min. of $\left\{\frac{2}{3}, 1, \frac{1}{2}, \frac{13}{3}\right\} = \frac{1}{2}$

(1), (2), (3) and (4) holds if $\frac{1}{3} \leq p \leq \frac{1}{2}$.

11. On a festival day, a man plans to visit 4 holy temples A, B, C, D in a random order. Find the probability that he visits (i) A before B (ii) A before B and B before C.

Sol. Given that 4 holy temples are A, B, C and D.

Number of ways to visit 4 holy temples in ${}^4P_4$ ways.

$\therefore n(S) = {}^4P_4 = 4! = 24$

ii) A before B :

Case-1: | A | | | |

$\rightarrow {}^3P_1 \times {}^2P_2 = 3 \times 2 = 6$

Case-2: | | a | | |

$\rightarrow {}^2P_1 \times {}^2P_2 = 2 \times 2 = 4$

Case-3: | | | a | |

$\rightarrow {}^1P_1 \times {}^2P_2 = 1 \times 2 = 2$

12

n(A) = 12

$$P(A) = \frac{n(A)}{n(S)} = \frac{12}{24} = \frac{1}{2}.$$

ii) A before B and B before C :

Case-1: $\boxed{A}\boxed{B}\boxed{}\boxed{}$

$\rightarrow {}^{2}P_{1} \times {}^{1}P_{1} = 2 \times 1 = 2$

Case-2: $\boxed{A}\boxed{}\boxed{B}\boxed{}$

$\rightarrow {}^{1}P_{1} \times {}^{1}P_{1} = 1 \times 1 = 1$

Case-3: $\boxed{}\boxed{A}\boxed{B}\boxed{C}$

$\rightarrow {}^{1}P_{1} = 1$

$\underline{\overline{\quad 4 \quad}}$

n(B) = 4

$$\therefore P(A) = \frac{n(B)}{n(S)} = \frac{4}{24} = \frac{1}{6}$$

12. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. The particulars of 5 persons are as follows:

S.No.	Name	Sex	Age in years
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetala	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. Find the probability that the spokesperson will be either male or above 35 years.

Sol.Let A be the event of selecting a male n(A) = 3.

Let B be the event of selecting a person whose age is above 35.

N(B) = 2

n(S) = 5, $n(A \cap B) = 1$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$= \frac{3}{5} + \frac{2}{5} - \frac{1}{5} = \frac{4}{5}$$

13. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, find the probability that (i) you both enter the same section (ii) you both enter the different sections.

Sol. $n(S) = {}^{100}C_{40}$

i) You both enter the same section:

$$n(A) = {}^{98}C_{38} + {}^{98}C_{58}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}^{98}C_{38} + {}^{98}C_{58}}{{}^{100}C_{40}} = \frac{17}{33}$$

ii) You both enter the different sections:

$$n(A) = {}^{98}C_{39} + {}^{98}C_{59}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}^{98}C_{39} + {}^{98}C_{59}}{{}^{100}C_{40}} = \frac{16}{33}$$

14. Suppose A and B are independent events with P(A) = 0.6, P(B) = 0.7 compute

(i) $P(A \cap B)$ **(ii) $P(A \cup B)$**

(iii) $P\left(\frac{B}{A}\right)$ **(iv)** $P(A^C \cap B^C)$

Sol. Given A, B are independent events and

P(A) = 0.7, P(B) = 0.7

i) $P(A \cap B) = P(A)P(B) = 0.6 \times 0.7 = 0.42$

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.7 - 0.42 = 0.88$

iii) $P\left(\frac{B}{A}\right) = P(B) = 0.7$

iv) $P(A^C \cap B^C) = P(A^C) \cdot P(B^C)$

(A^C and B^C are also independent events)

$= [1 - P(A)][(1 - P(B)]$

$= (1 - 0.6)(1 - 0.7) = 0.4 \times 0.3 = 0.12$

15. The probability that Australia wins a match against India in a cricket game is given to be 1/3. If India and Australia play 3 matches, what is the probability that,

i) Australia will loose all the three matches?

ii) Australia will win at least one match?

Sol. Suppose A is the event of Australia winning the match.

Given $P(A) = \frac{1}{3}$

$\therefore P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$

i) Probability that Australia will loose the all three matches.

$= [P(\bar{A})]^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

ii) Probability that Australia will win at least one match

$= 1 - [P(\bar{A})]^3 = 1 - \frac{8}{27} = \frac{19}{27}$

16. Three boxes numbered I, II, III contains balls as follows

	White	Black	Red
I	1	2	3
II	2	1	1
III	4	5	3

One box is randomly selected and a ball is drawn from it. If the ball is red, then find the probability that it is from box II.

Sol. Let B_1, B_2, B_3 be the events of selecting 1st, 2nd and 3rd box respectively.

Then $P(B_1) = P(B_2) = \frac{1}{3}$

Probability of selecting a red ball from the first box = $\frac{3}{6} = P\left(\frac{R}{B_1}\right)$

Probability of selecting a red ball from the second box = $\frac{1}{4} = P\left(\frac{R}{B_2}\right)$

Probability of selecting a red ball from the third box = $\frac{3}{12} = P\left(\frac{R}{B_3}\right)$

Assuming that the ball is red, probability it is from box II,

$$P\left(\frac{B_2}{R}\right) = \frac{P(B_2)P(R/B_2)}{P(B_1)P(R/B_1) + P(B_2)P(R/B_2) + P(B_3)P(R/B_3)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{4}}{\frac{1}{3}\left(\frac{3}{6} + \frac{1}{4} + \frac{3}{12}\right)} = \frac{\frac{1}{4}}{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = \frac{1}{4}$$

17. A person secures a job in a construction company in which the probability that the workers go on strike is 0.65 and the probability that the construction job will be completed on time if there is no strike is 0.80. If the probability that the construction job will completed on time even if there is a strike is 0.32, determine the probability that the construction job will be completed on time.

Sol. Given that the probability that the workers go on strike = 0.65

$$P(S) = 0.65$$

$$P(\overline{S}) = 1 - P(S) = 1 - 0.65 = 0.35$$

The probability that the construction job will be completed on time, if there is no strike = 0.80.

$$P\left(\frac{A}{\overline{S}}\right) = 0.80$$

The probability that the construction job will be completed on time even if there is a strike = 0.32.

$$P\left(\frac{A}{S}\right) = 0.32$$

$$\therefore P(A) = P(A \cap S) + P(A \cap \overline{S})$$

$$= P(S) \cdot P\left(\frac{A}{S}\right) + P(\overline{S}) \cdot P\left(\frac{A}{\overline{S}}\right)$$

$$= 0.65 \times 0.32 \times 0.35 \times 0.80 = 0.488.$$

18. For any two events A, B show that

$$P(A \cap B) - P(A)P(B) = P(A^C)P(B) - P(A^C \cap B) = P(A)P(B^C) - P(A \cap B^C)$$

Sol. $P(A^C)P(B) - P(A^C \cap B)$

$$= [1 - P(A)]P(B) - P[(S - A) \cap B]$$

$$= P(B) - P(A)P(B) - P[B - (A \cap B)]$$

$$= P(B) - P(A)P(B) - P(B) + P(A \cap B)$$

$$= P(A \cap B) - P(A)P(B) \quad \ldots(1)$$

$P(A)P(B^C) - P(A \cap B^C)$

$$= P(A)[1 - P(B)] - P(A \cap (S - B))$$

$$= P(A) - P(A)P(B) - P[A - (A \cap B)]$$

$$= P(A) - P(A)P(B) - P(A) + P(A \cap B)$$

$$= P(A \cap B) - P(A)P(B) \quad \ldots(2)$$

From (1) and (2), we get

$$P(A\cap B)-P(A)P(B)$$

$$=P(A^C)P(B)-P(A^C\cap B)$$

$$=P(A)P(B^C)-P(A\cap B^C)$$

19. If A, B, C are three events. Show that

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

Sol. Write $B \cup C = D$ then

$$P(A \cup B \cup C) = P(A \cup D)$$

$$\therefore P(A\cup D)=P(A)+P(D)-P(A\cap D)$$
$$=[P(A)+P(B\cup C)-P(A\cap(B\cup C)]$$
$$=P(A)+P(B)+P(C)-P(B\cap C)-[P(A\cap B)\cup(A\cap C)]$$
$$=P(A)+P(B)+P(C)-P(B\cap C)-[P(A\cap B)+P(A\cap C)-P(A\cap B\cap D\cap C)]$$
$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

20. Suppose there are 12 boys and 4 girls in a class. If we choose three children one after another in succession, what is the probability that all the three are boys?

Sol. Let E_i be the event of choosing a boy child in i^{th} trial (i = 1, 2, 3). We have to find

Here

$$P(E_1) = \frac{12}{16},\ P\left(\frac{E_2}{E_1}\right)=\frac{11}{15},\ P\left(\frac{E_3}{E_2}\cap E_1\right)=\frac{5}{7}$$

$P(E_1\cap E_2\cap E_3)$ by the multiplication theorem

$$P(E_1\cap E_2\cap E_3)=P(E_1)P\left(\frac{E_2}{E_1}\right)P\left(\frac{E_3}{E_1\cap E_2}\right)$$

$$=\frac{12}{16}\times\frac{11}{15}\times\frac{10}{14}=\frac{11}{28}$$

21. A speaks the truth in 75% of the cases, B is 80% cases. What is the probability that their statements about an incident do not match?

Sol. Let E_1, E_2 be the events that A and B respectively speak truth about an incident.

Then $P(E_1) = \frac{75}{100} = \frac{3}{4}$, $P(E_2) = \frac{80}{100} = \frac{4}{5}$

So that $P\left(E_1^C\right) = \frac{1}{4}$, $P\left(E_2^C\right) = \frac{1}{5}$

Let E be the event that their statements do not match about the incident. Then this happens in two mutually exclusive ways.

i) A speaks truth, B tells lie

ii) A tells lie, B speaks truth. These two events are represented by $E_1 \cap E_2^C$, $E_1^C \cap E_2$.

$$\therefore P(E) = P(E_1 \cap E_2^C) + P(E_1^C \cap E_2)$$

$$= P(E_1)P(E_2^C) + P(E_1^C)P(E_2)$$

$(\because E_1, E_2$ are independent)

$$= \frac{3}{4} \times \frac{1}{5} + \frac{1}{4} \times \frac{4}{5} = \frac{7}{20}$$

22. A problem in Calculus is given to two students A and B whose chances of solving it are 1/3 and 1/4. What is the probability that the problem will be solved if both of them try independently?

Sol. Let E_1 and E_2 denote the events that the problems is solved by A and B respectively.

Given that $P(E_1) = \frac{1}{3}$ and $P(E_2)\frac{1}{4}$

Note that these two are independent events.

Therefore the required probability

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= P(E_1) + P(E_2) - P(E_1)P(E_2)$$

$(\because E_1, E_2$ are independent)

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

23. If A and B toss a fair coin 50 times each simultaneously. Then show that the probability that both of them will not get tails at the same toss is $(3/4)^{50}$.

Sol. In each toss there are four choices

i) A gets H, B gets H

ii) A gets T, B gets H

iii) A gets H, B gets T

iv) A gets T, B gets T

Therefore the total number of choices is 4^{50}

Out of the four cases listed above, (i), (ii) and (iii) are favourable.

(iv) is not favourable to the occurrence of the required event, say E.

$$\therefore P(E) = \frac{3^{50}}{4^{50}} = \left(\frac{3}{4}\right)^{50}$$

24. Let A and B be two events of an experiment with P(A) = 0.2, P(A ∪ B) = 0.8 and P(B) = x. Find x so that A and B are independent.

Sol. Given A and B are independent. Then

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$
$$\text{i.e. } 0.8 = 0.2 + x - 0.2 \times x$$
$$\therefore x = \frac{0.6}{0.8} = \frac{3}{4}$$

25. If A and B are independent events of a random experiment show that A^C and B^C are also independent.

Sol. If A and B are independent then

$$P(A \cup B) = P(A) + P(B)$$
$$\text{Now } P(A^C \cap B^C) = P[(A \cup B)^C]$$
$$= 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - [P(A) + P(B) - P(A)P(B)]$$
$$= [1 - P(A)][1 - P(B)] = P(A^C)P(B^C)$$
$$\therefore A^C, B^C \text{ are indenpendent.}$$

Long Answer Questions

1. Three Urns have the following composition of balls.

Urn I: 1 white, 2 black

Urn II: 2 white, 1 black

Urn III: 2 white, 2 black

One of the Urn is selected at random and a ball is drawn. It turns out to be white. Find the probability that it come from Urn III.

Sol. Let E_i be the event of choosing the Urn i = 1, 2, 3 and $P(E_i)$ be the probability of choosing the Urn i = 1, 2, 3. Then $P(E_1) = P(E_2) = P(E_3) = 1/3$.

Having chosen the Urn i, the probability of drawing a white ball, $P(W/E_i)$, is given by

$$P\left(\frac{W}{E_1}\right)=\frac{1}{3}\ ;\ P\left(\frac{W}{E_2}\right)=\frac{2}{3};\ P\left(\frac{W}{E_3}\right)=\frac{2}{4}$$

We have to find the probability $P\left(\frac{E_3}{W}\right)$ by Baye's theorem.

$$P\left(\frac{E_3}{W}\right)=$$

$$\frac{P(E_3)P\left(\frac{W}{E_3}\right)}{P(E_1)P\left(\frac{W}{E_1}\right)+P(E_2)P\left(\frac{W}{E_2}\right)+P(E_3)P\left(\frac{W}{E_3}\right)}$$

$$=\frac{\frac{1}{3}\cdot\frac{2}{4}}{\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{2}{4}}=\frac{\frac{1}{2}}{\frac{1}{3}+\frac{2}{3}+\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$$

2. In a shooting test the probability of A, B, C hitting the targets are $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ respectively. It all of them fire at the same target. Find the probability that

i) Only one of them hits the target.

ii) At least one of them hits the target.

Sol. The probabilities that A, B, C hitting the targets are denoted by

$$P(A) = \frac{1}{2},\ P(B) = \frac{2}{3} \text{ and } P(C) = \frac{3}{4}$$

$$\therefore P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$
$$P(\bar{B}) = 1 - P(B) = 1 - \frac{2}{3} = \frac{1}{3}$$
$$P(\bar{C}) = 1 - P(C) = 1 - \frac{3}{4} = \frac{1}{4}$$

i) Probability that only one of them hits the target

$$= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A}\bar{B}C)$$

$$= P(A) \cdot P(\bar{B}) \cdot P(\bar{C}) + P(\bar{A}) \cdot P(B) \cdot P(\bar{C})$$
$$+ P(\bar{A}) \cdot P(\bar{B}) \cdot P(C)$$

($\because$ A, B, C are independent events)

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4}$$
$$= \frac{1+2+3}{24} = \frac{6}{24} = \frac{1}{4}$$

ii) Probability that at least one of them hits the target = $P(A \cup B \cup C)$

= 1 – Probability that none of them hits the target.

$$= 1 - P(\bar{A}\ \bar{B}\ \bar{C})$$
$$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$
$$= 1 - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = 1 - \frac{1}{24} = \frac{23}{24}$$

3. In a certain college, 25% of the boys and 10% of the girls are studying mathematics. The girls constitute 60% of the student strength. If a student selected at random is found studying mathematics, find the probability that the student is a girl.

Sol. Probability that a student select to be a girl $P(G) = \frac{60}{100} = \frac{6}{10}$

$\Rightarrow$ Probability that a student select to be a boy

$= P(B) = \frac{40}{100} = \frac{4}{10}$

Probability that a boy studying mathematics

$$P\left(\frac{M}{B}\right) = \frac{25}{100} = \frac{1}{4}$$

Similarly probability that a girl studying mathematics : $P\left(\frac{M}{G}\right) = \frac{10}{100} = \frac{1}{10}$

We have to find P(G/M) By Baye's theorem.

$$P\left(\frac{G}{M}\right) = \frac{P(G)\cdot P\left(\frac{M}{G}\right)}{P(B)\cdot P\left(\frac{M}{B}\right) + P(G)\cdot P\left(\frac{M}{G}\right)}$$

$$= \frac{\frac{6}{10}\cdot\frac{1}{10}}{\frac{4}{10}\cdot\frac{1}{4} + \frac{6}{10}\cdot\frac{1}{10}} = \frac{\frac{6}{100}}{\frac{1}{10} + \frac{6}{100}} = \frac{6}{10+6} = \frac{3}{8}$$

4. A person is known to speak truth 2 out of 3 times. He throws a die and reports that it is 1. Find the probability that it is actually 1.

Sol. Given that the probability of a person to speak truth = 2/3.

$$P(T) = \frac{2}{3}$$

$$P(\overline{T}) = \frac{1}{3}$$

The probability of a die to get $1 = \frac{1}{6}$

$$P\left(\frac{A}{T}\right) = \frac{1}{6}$$

$$P\left(\frac{A}{\overline{T}}\right) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(A) = P(A \cap T) + P(A \cap \overline{T})$$

$$= P\left(\frac{A}{T}\right) + P(\overline{T}) \cdot P\left(\frac{A}{T}\right)$$

$$= \frac{2}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{5}{6}$$

$$= \frac{2+5}{18} = \frac{7}{18}$$

$$P\left(\frac{T}{A}\right) = \frac{P(T \cap A)}{P(A)}$$

$$= \frac{P(T) \cdot P(A/T)}{P(A)}$$

$$= \frac{\frac{2}{3} \times \frac{1}{6}}{\frac{7}{18}} = \frac{2}{18} \times \frac{18}{7} = \frac{2}{7}.$$

5. Three boxes B_1, B_2 and B_3 contain balls detailed below.

	White	Black	Red
B_1	2	1	2
B_2	3	2	4
B_3	4	3	2

A die is thrown, B_1 is chosen if either 1 or 2 turns up, B_2 is chosen if 3 or 4 turns up and B_3 is chosen if 5 or 6 turns up. Having chosen a box in this way, a ball is chosen at random from this box. If the ball drawn is of red colour, what is the probability that it comes from box B_2?

Sol. Let $P(E_i)$ be the probability of choosing the box B_i ($i = 1, 2, 3$).

Then P(E_i) = $\frac{2}{6}=\frac{1}{3}$; for i = 1, 2, 3

Having chosen the box B_i, the probability of drawing a red ball, say, P(R/E_i) is given by

$$P\left(\frac{R}{E_1}\right)=\frac{2}{5},\ P\left(\frac{R}{E_2}\right)=\frac{4}{9}\text{ and }P\left(\frac{R}{E_3}\right)=\frac{2}{9}$$

We have to find the probability P(E_2/R)

By Bayer's theorem, we get

$$P\left(\frac{E_2}{R}\right)=$$

$$\frac{P(E_2)P(R/E_2)}{P(E_1)P\left(\frac{R}{E_1}\right)+P(E_2)P\left(\frac{R}{E_2}\right)+P(E_3)P\left(\frac{R}{E_3}\right)}$$

$$=\frac{\frac{1}{3}\times\frac{4}{9}}{\frac{1}{3}\left(\frac{2}{5}+\frac{4}{9}+\frac{2}{9}\right)}=\frac{\frac{4}{18}}{\frac{18+20+10}{5\times9\times3}}=\frac{5}{12}$$

20. An urn contain w white balls and b black balls. Two players Q and R alternately draw a with replacement from the urn. The player that draws a white ball first wins the game. If Q begins the game, find the probability that Q wins the game.

Sol. Let W denote the event of drawing a white ball at any draw and B that of a black ball. Then

$$P(W)=\frac{W}{w+b},\ P(B)=\frac{b}{w+b}$$

Let E be the event that Q wins the game

$$=P(W\ P(E)\ BBW\ P(E)\ BBBBW\ P(E)...)$$
$$=P(W)+P(BBW)+P(BBBBW)+...$$
$$=P(W)+P(B)P(B)P(W)+P(B)P(B)P(B)P(B)P(W)+...$$
$$=P(W)(1+P(B)^2+P(B)^4+...)$$
$$=\frac{P(W)}{1-P(B)^2}=\frac{\frac{W}{w+b}}{1-\left(\frac{b}{w+b}\right)^2}=\frac{w+b}{2+2b}$$