De Moivre's Theorem

If n is any integer, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

If n is any fraction, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

$(\sin\theta + i\cos\theta)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

If $x = \cos\theta + i\sin\theta$, then $x + \frac{1}{x} = 2\cos\theta$, $x - \frac{1}{x} = 2i\sin\theta$

$x^n + \frac{1}{x^n} = 2\cos n\theta$, $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

1. The n^{th} roots of a complex number form a G.P. with common ratio $\text{cis}\frac{2\pi}{n}$ which is denoted by ω.

2. The points representing n^{th} roots of a complex number in the Argand diagram are concyclic.

3. The points representing n^{th} roots of a complex number in the Argand diagram form a regular polygon of n sides.

4. The points representing the cube roots of a complex number in the Argand diagram form an equilateral triangle.

5. The points representing the fourth roots of complex number in the Argand diagram form a square.

6. The n^{th} roots of unity are $1, w, w^2, \ldots, w^{n-1}$ where $w = \text{cis}\frac{2\pi}{n}$

7. The sum of the n^{th} roots of unity is zero (or) the sum of the n^{th} roots of any complex number is zero.

8. The cube roots of unity are $1, \omega, \omega^2$ where $\omega = \text{cis}\frac{2\pi}{3}$, $\omega^2 = \text{cis}\frac{4\pi}{3}$ or

$\omega = \frac{-1 + i\sqrt{3}}{2}$, $\omega^2 = \frac{-1 - i\sqrt{3}}{2}$.

$1 + \omega + \omega^2 = 0$.

$\omega^3 = 1$

The product of the n^{th} roots of unity is $(-1)^{n-1}$.

The product of the n^{th} roots of a complex number Z is $Z(-1)^{n-1}$.

ω, ω^2 are the roots of the equation $x^2 + x + 1 = 0$

Very Short Answer Questions

1. If n is an integer then show that $(1+i)^{2n}+(1-i)^{2n}=2^{n+1}\cos\frac{n\pi}{2}$

Solution : -

Let $1+i=r\{\cos\theta+i\sin\theta\}$

$(r\cos\theta)^2+(r\sin\theta)^2=2\Rightarrow r=\sqrt{2}$

$\cos\theta=\frac{1}{\sqrt{2}}\ \sin\theta=\frac{1}{\sqrt{2}}\ P.V\ of\ \ \theta=\pi/4$

$\therefore 1+i=\sqrt{2}\left\{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right\}$ similarly $(1-i)=\sqrt{2}\left\{\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right\}$

$$(1+i)^{2n}+(1-i)^{2n}=\left(\sqrt{2}\right)^{2n}\left\{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right\}^{2n}+\left(\sqrt{2}\right)^{2n}\left\{\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right\}^{2n}$$

$=2^n\left\{\cos\frac{2n\pi}{4}+i\not{\sin}\frac{2n\pi}{4}\cos\frac{2n\pi}{4}-i\not{\sin}\frac{2n\pi}{4}\right\}$

$=\ 2^{n+1}\cos\frac{n\pi}{2}$

2. Find the values of the following

(i) $\left(1+i\sqrt{3}\right)^3$

Solution : -

$1+i\sqrt{3}=2\left\{\cos\frac{\pi}{3}+i\sin\pi/3\right\}$ {by mod-amplitude form}

$\left(1+1\sqrt{3}\right)^3=8\left\{\cos\frac{\pi}{3}+i\sin\pi/3\right\}^3$

$=8\{\cos\pi+i\sin\pi\}\quad\left\{\because(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta\right\}$

$=8\{-1+0\}=-8$

(ii) $(1-i)^8$

Solution $\quad(1-i)^8=\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)\right)^8=\left(\sqrt{2}\left\{\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right\}\right)^8=2^4\{\cos 2\pi-i\sin 2\pi\}$

(iii) $(1+i)^{16}$

Solution $(1+i)^{16} = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{16} = 2\{\cos 2\pi + i\sin 2\pi\}$

$= 256$

(iv) $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^3$

Solution. $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

$\left\{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right\}^5 - \left\{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right\}^5$

$\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} - \cos\frac{5\pi}{6} + 1\sin\pi/6$

$2i\sin\frac{5\pi}{6} = (2i)\frac{1}{2} = i$

3. Find all values of $\left(1-i\sqrt{3}\right)^{\frac{1}{3}}$

$\left(1-i\sqrt{3}\right)^{\frac{1}{3}} = \left\{2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right\}^{\frac{1}{3}}$

$= \left\{2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\right\}^{\frac{1}{3}}$

$= 2^{\frac{1}{3}}\left\{\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right\}^{\frac{1}{3}}$

$= 2^{\frac{1}{3}}\left\{\cos\frac{\left(2k\pi - \frac{\pi}{3}\right)}{3} + isn\left(\frac{2k\pi - \frac{\pi}{3}}{3}\right)\right\} \quad k = 0, 1, 2$

$= 3\sqrt{2}\, cis(6k-1)\frac{\pi}{9} \quad k = 0, 1, 2$

4. Find all values of $(-i)^{\frac{1}{6}}$

Solution : -

$$(-i)^{\frac{1}{6}} = \left\{\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right\}^{\frac{1}{6}}$$

$$= cis\left(\frac{2k\pi - \pi/2}{6}\right) k = 0, 1, 2, 3, 4, 5$$

$$\therefore (-1)^{\frac{1}{6}} = cis(4k-1)\frac{\pi}{12} \quad k = 0, 1, 2, 3$$

5. Find all values of $(1+i)^{2/3}$

$$(1+i)^{2/3} = \left[\left\{\sqrt{2}\left(\cos\frac{\pi}{4} + 1\sin\frac{\pi}{4}\right)\right\}^2\right]^{\frac{1}{3}}$$

$$= \left\{2\left(\cos\frac{\pi}{2} + i\sin\pi/2\right)\right\}^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}}\, cis\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right) k = 0, 1, 2$$

$$= 2^{\frac{1}{3}}\, cis(4k+1)\frac{\pi}{6}\, k = 0, 1, 2$$

6. Find all the values of $(-16)^{\frac{1}{4}}$

$$(-16)^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}}(-1)^{\frac{1}{4}}$$

$$= 2(cis\pi)^{\frac{1}{4}} = 2cis\left(\frac{2k\pi + \pi}{4}\right) k = 0, 1, 2, 3$$

$$= 2cis(2k+1)\frac{\pi}{4} \quad k = 0, 1, 2, 3$$

7. Find all values of $(-32)^{\frac{1}{5}}$

$$(-32)^{\frac{1}{5}} \quad \left(2^5\right)^{\frac{1}{5}}(-1)^{\frac{1}{5}} = 2\{\cos\pi + i\sin\pi\}^{\frac{1}{5}}$$

8. **If** $1, \omega, \omega^2$ **are the cube roots of units then prove that** $\frac{1}{2+\omega}+\frac{1}{1+2\omega}=\frac{1}{1+\omega}$

Solution : -

L.H.S $\frac{1}{2+\omega}+\frac{1}{1+2\omega}$

$$\frac{1+2\omega+2+\omega}{(2+\omega)(1+2\omega)}=\frac{3(1+\omega)}{2+4\omega+\omega+2\omega^2}$$

$$=\frac{3(1+\omega)}{2(1+\omega^2)+5\omega}$$

$$=\frac{3(-\omega^2)}{-2\omega+5\omega} \quad \because 1+\omega=-\omega^2$$

$$1+\omega^2=\omega$$

$$=\frac{-3\omega^2}{3\omega}=-\omega$$

$$=-\frac{1}{\omega^2}=\frac{1}{1+\omega}$$

9. **If 1,** ω, ω^2 **are the cube roots of unity then prove that**

$$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})=49$$

Solution : - $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$ =

$$\{(2-\omega)(2-\omega^2)\}\{(2-\omega)(2-\omega^2)\} \ \{\because \omega^{10}=\omega \qquad \omega^{11}=\omega^2\}$$

$$=\{4-2(\omega+\omega^2+\omega^3)\}\{4-2(\omega+\omega^2)+\omega^3\}$$

$$=(4+2+1)(4+2+1)=49$$

10. **If** $1, \omega, \omega^2$ **are the cube roots of units then prove that**

$$(x+y+z)(x+y\omega+z\omega)(x+y\omega^2+2\omega)=x^3+y^3+z^3-3xyz$$

Solution: -

$$(x+y+z)\{x+y\omega+z\omega^2\}\{x+y\omega^2+z\omega\}$$

$$(x+y+z)\{x^2+xy\omega^2+xz\omega+xy\omega+xy\omega+y^2\omega^3+yz\omega^2+xz\omega^2+yz\omega+z^2\omega^3\}$$

$$(x+y+z)\left\{x^2+y^2+z^2+xy\left(\omega^2+\omega\right)+yz\left(\omega+\omega^2\right)+zx\left(\omega+\omega^2\right)\right\}$$

$$(x+y+z)\left\{x^2+y^2+z^2-xy-yz-zx\right\}$$

$$x^3+y^3+z^3-3xyz$$

11. i) If x = cisθ then find the value of $\left(x^6+\frac{1}{x^6}\right)$.

Sol: **i)** $x=e^{i\theta}$

$$x^6=e^{i6\theta}$$

$$\frac{1}{x^6}=e^{-6i\theta}$$

$$x^6+\frac{1}{x^6}=e^{i6\theta}+e^{-6\theta i}$$

$$=\cos 6\theta+i\sin 6\theta+\cos 6\theta-i\sin 6\theta$$

$$=2\cos 6\theta.$$

ii) If x = cisθ then find cube roots of 8.

$$x=(8)^{1/3}$$

$$x^3-8=0$$

$$(x-2)(x^2+2x+4)=0$$

$$x=2, x=\frac{-2\pm\sqrt{4-16}}{2}$$

$$x=\frac{-2\pm 2\sqrt{3}\,i}{2}$$

$$x=-1\pm\sqrt{3}\,i$$

Roots are 2, 2ω, $2\omega^2$.

12 Prove that $-\omega$ and $-\omega^2$ are roots of $z^2 - z + 1 = 0$ where ω and ω^2 are the complex cube roots of unity.

Sol: $z^2 - z + 1 = 0$

$$z = \frac{1 \pm \sqrt{1-4}}{2}$$

$$z = \frac{1 \pm \sqrt{3}i}{2}$$

$$z = \frac{-[-1 \pm \sqrt{3}i]}{2}$$

$$z = -\omega, -\omega^2$$

13. If 1, ω, ω^2 are the cube roots of unity, then find the values of the following:

i) $(a+b)^3 + (a\omega + b\omega^2)^2 + (a\omega^2 + b\omega)^3$

Sol: **i)** $(a+b)^3 + (a\omega + b\omega^2)^2 + (a\omega^2 + b\omega)^3$

$$= a^3 + b^3 + 3a^2b + 3ab^2 + a^3\omega^3 + b^3\omega^6 + 3a^2\omega^2 \cdot b\omega^2 + 3a\omega \cdot b^2\omega^4 + a^3\omega^6 + b^3\omega^3 + 3a^2b\omega^4 \cdot \omega + 3b^2\omega^2 \cdot a\omega^2$$

$$= a^3 + b^3 + 3a^2b(1 + \omega + \omega^2) + a^3 + b^3 + 3b^2a(\omega^2 + \omega + 1) + a^3 + b^3$$

$$= 3(a^3 + b^3)$$

ii) $(a+2b)^2 + (a\omega^2 + 2b\omega)^2 + (a\omega + 2b\omega^2)^2$

Sol. $(a+2b)^2 + (a\omega^2 + 2b\omega)^2 + (a\omega + 2b\omega^2)^2$

$$= a^2 + 4b^2 + 4ab + a^2\omega^4 + 4b^2\omega^2 + 4ab\omega^3 + a^2\omega^2 + 4b^2\omega^4 + 4ab\omega^3$$

$$= a^2(1 + \omega + \omega^2) + 4b^2(1 + \omega^2 + \omega) + 4ab(1 + \omega^3 + \omega^2)$$

$$= 12ab.$$

iii) $(1 - \omega + \omega^2)^3$

Sol. $(1 - \omega + \omega^2)^3$

Now $1 + \omega + \omega^2 = 0$

$$1 + \omega^2 = -\omega$$

$$= (-\omega - \omega)^3$$
$$= (-2)^3 \omega^3$$
$$= -8$$

iv) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

Sol.

$$= (1-\omega-\omega^2+\omega^3)(1-\omega)(1-\omega^2)$$
$$= (1-\omega-\omega^2+\omega^3)(1-\omega-\omega^2+\omega^3)$$
$$= (1+1+1)(1+1+1)$$
$$= 9$$

v) $\left(\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\right)+\left(\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\right)$

Sol. $\left(\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\right)+\left(\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\right)$

$$= \frac{\omega^2(a+b\omega+c\omega^2)}{c\omega^2+a\omega^3+b\omega^4} + \frac{a\omega^2+b\omega^3+c\omega^4}{\omega^2(b+c\omega+a\omega^2)}$$
$$= \omega^2 + \frac{1}{\omega^2}$$
$$= \omega^2 + \frac{\omega}{\omega^3}$$
$$\Rightarrow \omega^2 + \omega = -1$$

vi) $(1-\omega)^3 + (1+\omega^2)^3$

Sol. $(1-\omega)^3 + (1+\omega^2)^3$

$$= (-\omega^2)^3 + (-\omega)^3$$
$$= -1 + (-1)$$
$$= -2.$$

vii) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5$ $\left(\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\right)+\left(\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\right)$

$(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5$

$$1+\omega^2=-\omega$$

$$=(-2\omega)^5+(-2\omega^2)^5$$

$$=(-2)^5(\omega^2+\omega)$$

$$=(-2)^5(-1)=32.$$

Short Answer Questions

1. α,β **are the roots of the equation** $x^2-2x+4=0$ **then for any** $n\in N$ **show that**

$$\alpha^n+\beta^n=2^{n+1}\cos\frac{n\pi}{3}$$

Solution: -

$$x^2-2x+4=0\Rightarrow x=\frac{2\pm\sqrt{4-16}}{2}=\frac{2\pm 2i\sqrt{3}}{2}$$

$$\alpha=2\left\{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right\}\ \beta=2\left\{\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right\}$$

$$\alpha^n+\beta^n=\left\{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right\}^n+\left\{2\left(\cos\frac{\pi}{3}-\sin\frac{\pi}{3}\right)\right\}^n$$

$$=2^n\left\{\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}+\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right\}$$

$$=2^n\left\{2\cos\frac{n\pi}{3}\right\}=2^{n+1}\cos\frac{n\pi}{3}$$

2. $\cos\alpha+\cos\beta+\cos\vartheta=0=\sin\alpha+\sin\beta+\sin\vartheta=0$ **then show that**

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\vartheta=3\cos(\alpha+\beta+\vartheta)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\vartheta=3\sin(\alpha+\beta+\vartheta)$

(iii) $\cos(2\alpha-\beta-\vartheta)+\cos\{2\beta-\vartheta-\alpha\}+\sin(2\vartheta-\alpha-\beta)=3$

(iv) $\sin(2\alpha-\beta-\vartheta)+\sin(2\beta-\vartheta-\alpha)+\sin(2\vartheta-\alpha-\beta)=0$

(v) $\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$

(vi) $\sin 2\alpha + \sin 2\beta + \sin 2\vartheta = 0$

(vii) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \vartheta = 0$

(viii) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \vartheta = 3/2$

(ix) $\cos(\alpha + \beta) + \cos(\beta + \vartheta) + \cos(\vartheta + \alpha) = 0$

(x) $\sin(\alpha + \beta) + \sin(\beta + \vartheta) + \sin(\vartheta + \alpha) = 0$

Solution : -

Let $x = \cos\alpha + i\sin\alpha \quad y = \cos\beta + i\sin\beta : z = \cos\vartheta + i\sin\vartheta$

$x + y + z = (\cos\alpha + \cos\beta + \cos\vartheta) + i(\sin\alpha + \sin\beta + \sin\vartheta)$

$x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 = 3xyz$

Proof of (i) & (ii)

$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\vartheta + i\sin\vartheta)^3 = 3cis\alpha\, cis\beta\, cis\vartheta$

$cis3\alpha + cis3\beta + cis3\vartheta = 3cis(\alpha + \beta + \vartheta)$

$(\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\vartheta + i\sin 3\vartheta) = 3\cos(\alpha + \beta + \vartheta) + 3i\sin(\alpha + \beta + \vartheta)$

By comparing real and imaginary parts on both sides

$\cos 3\alpha + \cos 3\beta + \cos 3\vartheta \;\; + 3\cos(\alpha + \beta + \vartheta)$

$\sin 3\alpha + \sin 3\beta + \sin 3\vartheta = 3\sin(\alpha + \beta + \vartheta)$

Proof of (iii) & (iv)

We know that $\sin 3\alpha + \sin 3\beta + \sin 3\vartheta = 3\sin(\alpha + \beta + \vartheta)$

$\frac{x^3 + y^3 + z^3}{xyz} = 3 \Rightarrow \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3$

$\frac{cis2\alpha}{cis\beta cis\vartheta} + \frac{cis2\beta}{cis\vartheta . cis\alpha} + \frac{cis2\vartheta}{cis\alpha\, cis\beta} = 3$

$cis(2\alpha - \beta - \vartheta) + cis(2\beta - \vartheta - \alpha) + \cos(2\vartheta - \alpha - \beta) = 3$

$\{\cos(2\alpha - \beta - \vartheta) + i\sin(2\alpha - \beta - \vartheta)\} + \cos(2\beta - \vartheta - \alpha) + 1\sin(2\beta - \vartheta - \alpha)$

$+ \cos(2\vartheta - \alpha - \beta) + 1\sin(2\vartheta - \alpha - \beta) = 3$

Comparing real and imaginary parts on both sides

$\cos(2\alpha - \beta - \vartheta) + \cos(2\beta - \vartheta - \alpha) + \cos(2\vartheta - \alpha - \beta) = 3$

$\sin(2\alpha - \beta - \vartheta) + \sin(2\beta - \vartheta - \alpha) + \sin(2\vartheta - \alpha - \beta) = 0$

Proof of V & VI

We know that $x + y + z = 0$

$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{\cos\alpha + 1\sin\alpha} + \frac{1}{\cos\beta + i\sin\beta} + \frac{1}{\cos\vartheta + i\sin\vartheta}$

$= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\vartheta - i\sin\vartheta$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

$x + y + z = 0 \Rightarrow (x + y + z)^2 = 0 \Rightarrow x^2 + y^2 + z^2 + 2xy + 2y + 2zx = 0$

$x^2 + y^2 + z^2 + 2xyz\left\{\frac{1}{z} + \frac{1}{x} + \frac{1}{y}\right\} = 0$

$(cis\alpha)^2 + (cis\beta)^2 + (cis\vartheta)^2 + 2(cis\alpha\, cis\beta\, cis\vartheta)(0)$

$\left\{\because \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0\right\}$

$cis2\alpha + cis2\beta + cis2\vartheta = 0 \Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\vartheta) + i(\sin 2\alpha + \sin 2\beta + \sin 2\vartheta) = 0$

By comparing real and imaginary parts on both sides

$\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$

$\sin 2\alpha + \sin 2\beta + \sin 2\vartheta = 0$

Proof of (vii)

From the above problem we can prove

$\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$

$2\cos^2\alpha + 2\cos^2\beta - 1 + 2\cos^2\vartheta - 1 = 0$

$2\{\cos^2\alpha + \cos^2\beta + \cos^2\vartheta\} = 3$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\vartheta = \frac{3}{2}$

Proof viii

$$\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$$

$$1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\vartheta = 0$$

$$3 = 2\left\{\sin^2\alpha + \sin^2\beta + \sin^2\vartheta\right\}$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\vartheta = \frac{3}{2}$$

Proof of (ix) and (x)

In proving iv & v we proved

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

$$\therefore yz + zx + xy = 0$$

$$\therefore cis\alpha\, cis\beta + cis\beta\, cis\vartheta + cis\vartheta\, cis\alpha = 0$$

$$= cis(\alpha+\beta) + \cos(\beta+\vartheta) + cis(\vartheta+\alpha) = 0$$

$$\left\{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right\} + \left\{\cos(\beta+\vartheta)\right\} + \left\{\cos(\vartheta+\alpha) + i\sin(\vartheta+\alpha)\right\} = 0$$

By comparing real and imaginary parts on both sides

$$\cos(\alpha+\beta) + \cos(\beta+\vartheta) + \cos(\vartheta+\alpha) = 0$$

$$\sin(\alpha+\beta) + \sin(\beta+\vartheta) + \sin(\vartheta+\alpha) = 0$$

3. If n is an integer and $z = cis\theta$ then show that $\frac{z^{2n}-1}{z^{2n}+1} = i\tan n\theta$

Solution : -

$$\frac{z^{2n}-1}{z^{2n}+1} = \frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta + i\sin\theta)^{2n} + 1}$$

$$= \frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta + i\sin 2n\theta + 1}$$

$$= \frac{-(1-\cos 2n\theta) + i\sin 2n\theta}{(1+\cos 2n\theta) + i\sin 2n\theta}$$

$$= \frac{i^2\left(2\sin^2 n\theta\right) + 2i\sin n\theta\cos n\theta}{2\cos^2 n\theta + 2i\sin n\theta\cos n\theta}\left\{\because -1 = i^2\right\}$$

$$=\frac{2i\sin n\theta\left\{\cos n\theta+i\sin n\theta\right\}}{2\cos n\theta\left\{\cos n\theta+i\sin n\theta\right\}}=i\tan n\theta$$

4. If $(1+x)^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n$ **, then show that**

i) $a_0-a_2+a_4-a_6+\ldots=2^{n/2}\cos\frac{n\pi}{4}$

ii) $a_1-a_3+a_5-a_7+\ldots=2^{n/2}\sin\frac{n\pi}{4}$ **.**

Sol: $(1+x)^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n$

Put x = i

$(1+i)^n=a_0+a_1i+a_2i^2+\ldots+a_ni^n$

$$\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n=(a_0-a_2+a_4\ldots)+i(a_1-a_3+a_5\ldots)$$

$$2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)=(a_0-a_2+a_4\ldots)+i(a_1-a_3+a_5\ldots)$$

Equating Real parts both sides

$$a_0-a_2+a_4\ldots=2^{n/2}\cos\frac{n\pi}{4}$$

Equating imaginary parts

$$a_1-a_3+a_5\ldots=2^{n/2}\sin\frac{n\pi}{4}.$$

5. Solve the following equations

(i) $x^4-1=0$

Solution : -

(i) $x^4-1=0\Rightarrow x^4=1$

$$\therefore x=(1)^{\frac{1}{4}}=(\cos 0+i\sin 0)^{\frac{1}{4}}$$

$$=\left\{\cos\frac{2k\pi}{4}+\frac{i\sin 2k\pi}{4}\right\}k=0,1,2,3$$

$$=\cos 0+1\sin 0\quad cis\frac{\pi}{2}\quad cis\pi\ cis\frac{3\pi}{2}$$

$= 1, \; I, -1, -i$

$= \pm 1, \pm i$

(ii) $x^5 + 1 = 0$

Solution : -

$$x^5 + 1 = 0 \Rightarrow (\cos\pi + i\sin\pi)^{\frac{1}{5}}$$

$$x = \cos\left(\frac{2k\,\pi + \pi}{5}\right) k = 0, 1, 2, 3, 4$$

$$\therefore x = cis\,\frac{\pi}{5}, cis\,\frac{3\pi}{5}\; cis\pi, cis\,\frac{7\pi}{5}, \cos\frac{9\pi}{5}$$

(iii) $x^9 - x^5 + x^4 - 1 = 0$

Solution : -

$$x^9 - x^5 + x^4 - 1 = 0$$

$$x^5\left(x^4 - 1\right) + 1\left(x^4 - 1\right) = 0$$

$$\left(x^4 - 1\right) = 0 : \left(x^5 + 1\right) = 0$$

Do (i) , (ii) to get the solution of (iii)

(iv) $x^4 + 1 = 0$

Solution : -

$$x^4 + 1 = 0 \Rightarrow x = (-1)^{\frac{1}{4}} = (cis\pi)^{\frac{1}{4}}$$

$$x = cis\left(\frac{2k\pi + \pi}{4}\right) k = 0, 1, 2, 3$$

$$x = cis\,\frac{\pi}{4}, cis\,\frac{3\pi}{4}\; cis\,\frac{5\pi}{4}\; cis\,\frac{7\pi}{4}$$

6. If n is a positive integer then show that

$$(p+iq)^{\frac{1}{n}}+(p-iq)^{\frac{1}{n}}=2\left(p^2+q^2\right)^{\frac{1}{2n}}\cos\left\{\frac{1}{n}arc.\tan\frac{q}{p}\right\}$$

Solution : -

Let $p+iq=r\{\cos\theta+i\sin\theta\}$

$r\cos\theta=p \quad r\sin\theta=q \quad \Rightarrow r^2=p^2+q^2$

$\therefore r=\sqrt{p^2+q^2}$

$\cos\theta=\dfrac{p}{\sqrt{p^2+q^2}}\ \sin\theta=\dfrac{q}{\sqrt{p^2+q^2}}$

$\tan\theta=\dfrac{q}{p} \Rightarrow \theta=\tan^{-1}\left(\dfrac{p}{q}\right)$

$(p+iq)^{\frac{1}{n}}+(p-q)^{\frac{1}{n}}=\{r(\cos\theta+i\sin\theta)\}^{\frac{1}{n}}+\{r(\cos\theta-i\sin\theta)\}^{\frac{1}{n}}$

$=r^{\frac{1}{n}}\left\{\cos\dfrac{\theta}{n}+i\sin\dfrac{\theta}{n}+\cos\dfrac{\theta}{n}-i\sin\dfrac{\theta}{n}\right\}$

$=\left(\sqrt{p^2+q}^{\,2}\right)^{\frac{1}{n}}\left\{2\cos\dfrac{1}{n}\theta\right\}$

$=2\left(p^2+q^2\right)^{\frac{1}{2n}}\cos\left(\dfrac{1}{n}\tan^{-1}\dfrac{q}{p}\right)$

7. Show that $\left\{\dfrac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right\}^{8/3}=-1$

Solution : -

$$\text{LHS }=\left\{\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right\}^{8/3}$$

$$\left\{\frac{1+\cos\left(\frac{\pi}{2}-\pi/8\right)+i\sin\left(\frac{\pi}{2}-\pi/8\right)}{1+\cos\left(\frac{\pi}{2}-\pi/8\right)-1\sin\left(\frac{\pi}{2}-\pi/3\right)}\right\}^{8/3}$$

$$\left\{\frac{1+\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}}{1+\cos\frac{3\pi}{8}-i\sin\frac{3\pi}{8}}\right\}^{8/3}=\left\{\frac{2\cos^2\frac{3\pi}{16}+2i\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}{2\cos^2\frac{3\pi}{16}-2i\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}\right\}^{8/3}$$

$$\left[\frac{2\cos\frac{3\pi}{16}\left\{\cos\frac{3\pi}{16}+1\sin\frac{3\pi}{16}\right\}}{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)}\right]^{\frac{8}{3}}$$

$$\left[\frac{\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)}{\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)\left(\cos\left(\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\right)}\right]^{8/3}$$

$$\left[\frac{\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)^2}{\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}}\right]^{8/3}$$

$$\left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)^{8/3}$$

$$\cos\pi+i\sin\pi=-1$$

8. **Find the common roots of $x^{12} - 1 = 0$ and $x^4 + x^2 + 1 = 0$.**

Sol: $x^{12}-1=0$

$$x=(1)^{1/12}$$

$$x=[\cos(2n\pi)+i\sin(2n\pi)]^{1/12}$$

$$n=0,1,2,\ldots 11\ \ldots(1)$$

$$x^4+x^2+1=0$$

$$(x^2-1)(x^4+x^2+1)=0$$

$$x^6-1=0$$

$$x=(1)^{1/6}$$

$$x=[\cos(2n\pi)+i\sin(2n\pi)]^{1/6}$$

$$=\cos\frac{2n\pi}{6}+i\sin\frac{2n\pi}{6},\ n=0,1,2,3,4,5\ldots(2)$$

Common roots to (1) and (2)

$\text{cis}\frac{\pi}{3}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{4\pi}{3}, \text{cis}\frac{5\pi}{3}$.

9. Find the number of 15th roots of unity, which are also 25th roots of unity.

Sol: $x = (1)^{1/15}$

$x = [\cos 2n\pi + i\sin 2n\pi]^{1/15}$

$x = \cos\frac{2n\pi}{15} + i\sin\frac{2n\pi}{15}$

$n = 0, 1, 2, 3, \ldots 14$

$n = 3, m = 5$

$x = \cos\frac{2\pi}{25} + i\sin\frac{2\pi}{25}$

$n = 9, m = 15$

$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$x = (1)^{1/25}$

$x = [\cos 2m\pi + i\sin 2m\pi]^{1/25}$

$x = \cos\frac{2m\pi}{25} + i\sin\frac{2m\pi}{25}$

$m = 0, 1, 2, 3, \ldots 24$

$n = 6, m = 10$

$x = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$n = 12, m = 20$

$\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

$n = 0$, $m = 0$

5 roots common.

10. If the cube roots of unity are 1, ω, ω^2 then find the roots of the equation (x–1)^3 + 8 = 0.

Sol: $(x-1)^3 = -8$

$$(x-1) = (-8)^3$$
$$x-1=-2 \Rightarrow x=-1$$
$$x-1=-2\omega \Rightarrow x=-2\omega+1$$
$$x-1=-2\omega^2 \Rightarrow x=-2\omega^2+1$$

11. Find the product of all values of (1 + i)$^{4/5}$.

Sol: $(1+i)^{4/5}$

$$=(\sqrt{2})^{4/5}\left[\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]^{4/5}$$

$$=(2)^{2/5}\left[\cos\left(2n\pi+\frac{\pi}{4}\right)+i\sin\left(2n\pi+\frac{\pi}{4}\right)\right]^{4/5}$$

$n=0,1,2,3,4$

$$=(2)^{2/5}\left[\cos\frac{\pi}{4}\cdot\frac{4}{5}+i\sin\frac{\pi}{4}\cdot\frac{4}{5}\right] \qquad ...(1)$$

$$=(2)^{2/5}\left[\cos\frac{9\pi}{4}\cdot\frac{4}{5}+i\sin\frac{9\pi}{4}\cdot\frac{4}{5}\right] \qquad ...(2)$$

$$\text{Product} = (2^{2/5})^5 e^{i\left[\frac{\pi}{5}(1+9+17+25+33)\right]}$$

$$=2^2 e^{i\frac{\pi}{5}(85)}$$
$$=2^2[\cos 17\pi + i\sin 17\pi]$$
$$=-4$$

12. If z^2 + z + 1 = 0, where z is a complex number, prove that

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\left(z^4+\frac{1}{z^4}\right)^2+\left(z^5+\frac{1}{z^5}\right)^2+\left(z^6+\frac{1}{z^6}\right)^2=12$$

Sol: **let** z **=** ω then

L.H.S. =

$$\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega^2+\frac{1}{\omega^2}\right)^2+\left(\omega^3+\frac{1}{\omega^3}\right)^2+\left(\omega^4+\frac{1}{\omega^4}\right)^2+\left(\omega^5+\frac{1}{\omega^5}\right)^2+\left(\omega^6+\frac{1}{\omega^6}\right)^2$$

$$=(-1)^2+(-1)^2+(2)^2+(-1)^2+(-1)^2+(2)^2$$

$$=12$$

Long Answer Questions

1. $1,\alpha,\alpha^2,\alpha^3,\ldots\alpha^{n-1}$ **be the nth roots of unity then prove that**

$$1^p+\alpha^p+(\alpha^2)^p+(\alpha^3)^p+\ldots+(\alpha^{n-1})^p=\begin{cases}0 \text{ if } p\neq kn\\ n \text{ if } p=kn\end{cases}$$ **Where p, k ∈ N.**

Sol: Now $x^n-1=0\Rightarrow x=(1)^{1/n}$

$$x=[\cos 2n\pi+i\sin 2n\pi]^{1/n}$$

$$x=\cos\frac{2m\pi}{n}+i\sin\frac{2m\pi}{n}$$

$$\alpha=\cos\frac{2m\pi}{n}+i\sin\frac{2m\pi}{n}$$

$$\alpha^p=\cos\frac{2mp\pi}{n}+i\sin\frac{2mp\pi}{n}$$

Now p = kn

1 + 1 + 1 + … n terms = n

If p ≠kn value

$$1^p+\alpha^p+(\alpha^2)^p+(\alpha^3)^p+\ldots+(\alpha^{n-1})^p=0.$$

2. Prove the sum of 99th powers of the roots of the equation x^7 – 1 = 0 is zero and hence deduce the roots of $x^6+x^5+x^4+x^3+x^2+x+1=0$.

Sol: $x^7-1=0\Rightarrow x=(1)^{1/7}$

$$x=(\cos 2k\pi+i\sin 2k\pi)^{1/7}$$

$$x=\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}$$

$$k=0,1,2,3,4,5,6$$

$x_1 = 1, x_2 = e^{\frac{2\pi}{7}i}, x_3 = e^{\frac{4\pi}{7}i}, \ldots, x_6 = e^{\frac{12\pi}{7}i}$

$x_1^{99} + x_2^{99} + x_3^{99} + \ldots$

$1^{99} + e^{\frac{2\pi}{7}\cdot 99i} + e^{\frac{4\pi}{7}\cdot 99i} + \ldots e^{\frac{12\pi}{7}\cdot 99i} = 0$

Roots of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ be $\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$; k = 1, 2, 3, 4, 5, 6.

$\because (x^7) - 1 =$

$(x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0$

x = 1 is one root

$\Rightarrow \text{cis}\frac{2k\pi}{7}$; k = 1, 2, 3, 4, 5, 6 are roots of

$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

3. Solve $(x - 1)^n = x^n$, n is a positive integer.

Sol: $\left(\frac{x-1}{x}\right)^n = 1$

$\frac{x-1}{x} = (1)^{1/n}$

$\frac{x-1}{x} = [\cos 2m\pi + i\sin 2m\pi]^{1/n}$

$\frac{x-1}{x} = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}$

$1 - \frac{1}{x} = e^{i\frac{2m}{n}\pi}$

$1 - \cos\frac{2m\pi}{n} - i\sin\frac{2m\pi}{n} = \frac{1}{x}$

$2\sin^2\frac{m\pi}{n} - 2i\sin\frac{m\pi}{n}\cos\frac{m\pi}{n} = \frac{1}{x}$

$2\sin\frac{m\pi}{n}\left[\sin\frac{m\pi}{n} - i\cos\frac{m\pi}{n}\right] = \frac{1}{x}$

$$x = \frac{1}{2\sin\frac{m\pi}{n}\left[\sin\frac{m\pi}{n} - i\cos\frac{m\pi}{n}\right]}$$

$$= \frac{1}{2\sin\frac{m\pi}{n}\left[\sin\frac{m\pi}{n} - i\cos\frac{m\pi}{n}\right]} X \frac{\left[\sin\frac{m\pi}{n} + i\cos\frac{m\pi}{n}\right]}{\left[\sin\frac{m\pi}{n} + i\cos\frac{m\pi}{n}\right]}$$

$$= \frac{\left[\sin\frac{m\pi}{n} + i\cos\frac{m\pi}{n}\right]}{2\sin\frac{m\pi}{n}\left[\sin^2\frac{m\pi}{n} + \cos^2\frac{m\pi}{n}\right]}$$

$$= \frac{\left[\sin\frac{m\pi}{n} + i\cos\frac{m\pi}{n}\right]}{2\sin\frac{m\pi}{n}}$$

$\frac{1}{2}\left[1 + i\cot\frac{m\pi}{n}\right]$; m = 1, 2, 3, …(n – 1)

4. If m, n are integers and x = cosα+ *i*sinα, y = cosβ + isinβ then prove that

$x^m y^n + \frac{1}{x^m y^n} = \cos(m\alpha + n\beta)$ **and** $x^m y^n - \frac{1}{x^m y^n} = 2i\sin(m\alpha + n\beta)$.

Sol. $x^m = (\cos\alpha + i\sin\alpha)^m$

$= \cos m\alpha + i\sin m\alpha$

$y^n = (\cos\beta + i\sin\beta)^n$

$= \cos n\beta + i\sin n\beta$

$\therefore x^m y^n = (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$

$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$ ... (1)

$\frac{1}{x^m y^n} = \frac{1}{\cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)}$

$= \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$ ...(2)

By adding (1) and (2), we get

$x^m y^n + \frac{1}{x^m y^n} = \cos(m\alpha + n\beta)$

By subtracting (2) from (1), we get

$$x^m y^n - \frac{1}{x^m y^n} = 2i\sin(m\alpha + n\beta)$$

5. If n is a positive integer, show that (1 + i)^n + (1 – i)^n = $2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$**.**

Sol. $(1+i) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$(1-i) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$(1+i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad ...(1)$$

$$(1-i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n$$

$$= 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \quad ...(2)$$

By adding (1) and (2), we get

$$(1+i)^n + (1-i)^n = 2^{n/2}\left(2\cos\frac{n\pi}{4}\right) = 2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$$

6. If n is an integer then show that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$

Sol.L.H.S. =

$$(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n =$$

$$= \left(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n + \left(2\cos^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$= 2^n \cos^n \frac{\theta}{2}\left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n + \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)^n\right]$$

$$= 2^n \cos^n \frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right)$$

$$= 2^n \cos^n \frac{\theta}{2}\left(2\cos\frac{n\theta}{2}\right)$$

$$= 2^{n+1} \cos^n \frac{\theta}{2} \cos\frac{n\theta}{2} = \text{R.H.S.}$$

7. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$ prove that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Sol. $(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0$

$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$ ...(1)

Let $x = \text{cis}\alpha, y = \text{cis}\beta, z = \text{cis}\gamma$ then

$x + y + z = 0$ by (1), then

$x^2 + y^2 + z^2 = -2(xy + yz + zx)$

$$= -2xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

$= -2xyz[\cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma]$

$= -2xyz[(\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)]$

$= -2xyz(0 - i0) = 0$

$\therefore x^2 + y^2 + z^2 = 0$

$\Rightarrow (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

$\Rightarrow \cos 2\alpha + i\sin 2\alpha + \cos 2\beta + i\sin 2\beta + \cos 2\gamma + i\sin 2\gamma = 0$

$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$

$\therefore \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$

$2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$

$$1-\sin^2\alpha + 1-\sin^2\beta + 1-\sin^2\gamma = \frac{3}{2}$$

$$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}.$$

8. Find all the value of $(\sqrt{3}+i)^{1/4}$.

Sol. The modulus amplitude form of

$$\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = 2(\cos 30° + i\sin 30°)$$

Hence $(\sqrt{3}+i)^{1/4} = \left(2\text{cis}\frac{\pi}{6}\right)^{1/4}$

$$= 2^{1/4}\left(\text{cis}\frac{2k\pi+\frac{\pi}{6}}{4}\right), \text{ k = 0, 1, 2, 3}$$

$$= 2^{1/4}\text{cis}\left(\frac{12k\pi+\pi}{24}\right), \text{ k = 0, 1, 2, 3}$$

$$= 2^{1/4}\text{cis}(12k+1)\frac{\pi}{24}, \text{ k = 0, 1, 2, 3}$$

$\therefore$ All the values of are $(\sqrt{3}+i)^{1/4}$ are

$$2^{1/4}\text{cis}\frac{\pi}{24}, 2^{1/4}\text{cis}\frac{13\pi}{24},$$

$$2^{1/4}\text{cis}\frac{25\pi}{24}, 2^{1/4}\text{cis}\frac{37\pi}{24}$$

9. If 1, ω, ω² are the cube roots of unity, prove that

i) $(1-\omega+\omega^2)^6+(1-\omega^2+\omega)^6=128$

$=(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7$

Sol: **We use** $1+\omega+\omega^2$ **=** $1+\frac{(-1+i\sqrt{3})}{2}+\frac{(-1-i\sqrt{3})}{2}=0$ **and** $\omega^3=\left(\text{cis}\frac{2\pi}{3}\right)^3=\text{cis}2\pi=1$

i) $(1-\omega+\omega^2)^6+(1-\omega^2+\omega)^6$

$=(-\omega-\omega)^6+(-\omega^2-\omega^2)^6$

$=2^6(\omega^6+\omega^{12})=2^6(2)=128$

$(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7$

$=(-\omega-\omega)^7+(-\omega^2-\omega^2)^7$

$=(-2)^7(\omega^7+\omega^{14})$

$=(-2)^7(\theta+\theta^2)$

$=(-128)(-1)=128.$

ii) $(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)=a^3+b^3$**.**

$(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$

$=(a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3)$

$=(a+b)\left(a^2+ab(\omega+\omega^2)+b^2\right)$

$=(a+b)(a^2-ab+b^2)$

$=a^3+b^3$

iii) $x^2+4x+7=0$ **where** $x=\omega-\omega^2-2$**.**

$x=\omega-\omega^2-2$

$(x+2)=\omega-\omega^2$

$\Rightarrow(x+2)^2=\omega^2+\omega^4-2\omega^3$

$\Rightarrow x^2+4x+4=\omega^2+\omega-2=-1-2=-3$

$\Rightarrow x^2+4x+7=0.$