COMPLEX NUMBERS

1. Any number of the form x+iy where x, y $\in$ R and $i^2 = -1$ is called a **Complex Number**.

2. In the complex number x+iy, x is called the real part and y is called the imaginary part of the complex number.

3. A complex number is said to be purely imaginary if its real part is zero and is said to be purely real if its imaginary part is zero.

4. (a) Two complex numbers are said to be equal if their real parts are equal and their imaginary parts are equal.

(b) In the set of complex numbers, there is no meaning to the phrases one complex is greater than or less than another i.e. If two complex numbers are not equal, we say they are unequal.

(c) a+ ib > c + id is meaningful only when b = 0, d = 0.

5. Two complex numbers are conjugate if their sum and product are both real. They are of the form a+ib, a-ib.

6. $\text{cis}\theta_1 \ \text{cis}\theta_2 = \text{cis}(\theta_1 + \theta_2)$, $\frac{\text{cis}\,\theta_1}{\text{cis}\,\theta_2} = \text{cis}(\theta_1 - \theta_2)$, $\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$.

7. $\frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^{\,2} + b_2^{\,2}}$

8. $\frac{1+i}{1-i} = i, \frac{1-i}{1+i} = -i$

9. $\sqrt{x^2 + y^2}$ is called the modulus of the complex number x + iy and is denoted by r or |x + iy|

10. Any value of θ obtained from the equations $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ is called an amplitude of the complex number.

11.i) The amplitude lying between -π and π is called the principal amplitude of the complex number. Rule for choosing the principal amplitude.

ii) If θ is the principal amplitude, then $-\pi < \theta \le \pi$

$\pi-\theta$	θ
$-(\pi-\theta)$	$-\theta$

12. If α is the principle amplitude of a complex number, general amplitude = $2n\pi + \alpha$ where $n \in Z$.

13.i) $\text{Amp}(Z_1 Z_2) = \text{Amp } Z_1 + \text{Amp} Z_2$

ii) $\text{Amp}\left(\frac{Z_1}{Z_2}\right) = \text{Amp}Z_1 - \text{Amp}Z_2$

iii) $\text{Amp } z + \text{Amp } \bar{z} = 2\pi$ (when z is a negative real number) = 0 (otherwise)

14. (a) $r(\cos\theta + i \sin\theta)$ is the modulus amplitude form of $x+iy$.

15. If the amplitude of a complex number is $\frac{\pi}{2}$, its real part is zero.

16. If the amplitude of a complex number is $\frac{\pi}{4}$, its real part is equal to its imaginary part.

Very Short Answer Questions

1. **If** $z_1 = (2,-1)\ z_2 = (6,3)$ **then** $z_1 - z_2$

Solution: - **Given** $z_1 = z - i \qquad z_2 = 6 + 3i$

$z_1 - z_2 = -4 - 4i = (-4, -4)$

2. **If** $z_1 = (3,5)$ *and* $z_2 = (2,6)$ **find** z, z_2

Solution: -

Given $z_1 = 3 + 5i \qquad z_2 = 2 + 6i$

$z_1 - z_2 = (3+5i)(2+6i) = 6 + 28^0 C + 30i^2 = 6 - 30 + 28i$

$z_1 . z_2 = -24 + 28i\ = (-24, 28)$

3. **Write the additive inverse of the following**

(i) Additive inverse of $(\sqrt{3}, 5)$

Additive inverse is $(-\sqrt{3}, -5)$

(ii) Additive Inverse of $(-6,5) + (10,-4)$

Given complex no is (4,1)

Additive inverse is $(-4,-1)$

(iii) Additive inverse of $(2,1)(-4,6)$

Let $z_1 = 2 + i \quad z_2 = -4 + 6i$

$z_1 z_2 = (2+l)(-4+6i) = -8 + 8i - 6i^2$

$$= -2 + 8i$$

Additive inverse is (-2, -8)

4. **If** $z_1 = (6,-3)$ *and* $z_2 = (2,-1)$ **then find** z_1 / z_2

Solution: -

Given $z_1 = 6 + 3i \quad z_2 = 2 - i$

$$\frac{z_1}{z_2} = \frac{6+3i}{z-i} = \frac{(6+3i)(2+i)}{4-i^2} = \frac{8+12i+3i^2}{5}$$

$$1 + \frac{12}{5}i = \left(1, \frac{12}{5}\right)$$

5. **If** $z = \cos\theta + i\sin\theta$ **then find** $z - \frac{1}{z}$

Solution: -

$$z = \cos\theta + i\sin\theta$$
$$\Rightarrow \frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$
$$= \cos\theta - i\sin\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

6. Write the multiplicative inverse of the following complex numbers

(i) $(3,4)$ **(ii)** $(\sin\theta, \cos\theta)$ **(iii)** $(7,24)$ **(iv** $(-2,1)$

Solution: -

(i) Let $z = 3+4i$

Multiplicative inverse of z is $\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-3i)}$

$= \frac{3-4i}{25} = \left(\frac{3}{25}, \frac{-4}{25}\right)$

(ii) Let $z = \sin\theta + i\cos\theta$

Multiplicative inverse of $z = \frac{1}{\sin\theta + 1\cos\theta} = \frac{\sin\theta - i\cos\theta}{(\sin\theta - i\cos\theta)}$

$$= \frac{\sin\theta - i\cos\theta}{\sin^2\theta + \cos^2\theta} = \sin\theta - i\cos\theta$$

(iii) Let $z = 7+24i$

Multiplicative inverse of $z = \frac{1}{7+24i} = \frac{7-24i}{(7+24i)(7-24i)}$

$$= \frac{7-24i}{625}$$

7. Write the following complex numbers in the form A + iB.

i) $(2 - 3i)(3 + 4i)$

ii) $(1 + 2i)^3$

iii) $\frac{a - ib}{a + ib}$

iv) $\frac{4 + 3i}{(2 + 3i)(4 - 3i)}$

v) $(-5i)\left(\frac{i}{8}\right)$

vi) $\frac{2 + 5i}{3 - 2i} + \frac{2 - 5i}{3 + 2i}$

Sol: **i)** $z = (2 - 3i)(3 + 4i)$

$z = 6 + 8i - 9i + 12$

$z = 18 - i$

$= 18 + (-1)i$

ii) $z = (1 + 2i)^3$

$= 1^3 + 3(1)^2(2i) + 3(1)(2i)^2 + (2i)^3$

$= 1 + 6i + 6i^2 + 8i^3$

$= 1 + 6i - 6 - 8i$

$= -5 - 2i$

iii) $z = \frac{a - ib}{a + ib}$

$z = \frac{a - ib}{a + ib} \times \frac{a - ib}{a - ib}$

$= \frac{(a - ib)^2}{a^2 - (ib)^2}$

$= \frac{a^2 + (ib)^2 - 2(a)(ib)}{a^2 + b^2}$

$= \frac{a^2 - b^2}{a^2 + b^2} - i\frac{2ab}{a^2 + b^2}$

iv)z = $\frac{4+3i}{(2+3i)(4-3i)}$

$$=\frac{4+3i}{8-6i+12i+9}$$

$$=\frac{4+3i}{17+6i}$$

$$=\frac{4+3i}{17+6i}\times\frac{17-6i}{17-6i}$$

$$=\frac{(68+18)+i(-24+51)}{(17)^2-(6i)^2}$$

$$=\frac{86+27i}{289+36}$$

$$=\frac{86+27i}{325}$$

$$=\frac{86}{325}+i\frac{27}{325}$$

v) z = $(-5i)\left(\frac{i}{8}\right)$

$$z=\frac{-5}{8}i^2$$

$$z=\frac{5}{8}+0i$$

vi)z = $\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}$

$$z=\frac{(2+5i)(3+2i)}{(3-2i)(3+2i)}+\frac{(2-5i)(3-2i)}{(3-2i)(3-2i)}$$

$$z=\frac{6+4i+15i+10i^2}{9+4}+\frac{6-4i-15i+10i^2}{9+4}$$

$$z=\frac{-4+19i-4-19i}{13}$$

$$z=\frac{-8}{13}+0i$$

8. Write the conjugate of the following complex numbers.

i) (3 + 4i)

ii) (2 + 5i)(–4 + 6i)

Sol: i)z = 3 + 4i

$\overline{z} = 3 - 4i$

ii)z = (2 + 5i)(–4 + 6i)

$z = -8 + 12i - 20i + 30i^2$

$z = -38 - 8i$

$\overline{z} = -38 + 8i$

9. Simplify

i) $i^2 + i^4 + i^6 + ...(2n+1)$ **Terms**

ii) $i^{18} - 3.i^7 + i^2\left(1 + i^4\right)\cdot(-i)^{26}$

Sol: i) $i^2 + i^4 + i^6 + ...(2n+1)$ terms

$= i^2 + i^{2.2} + i^{2.3} + ...i^{2.(2n+1)}$

$= -1 + 1 + (-1) + ...2n \text{ terms} + i^2$

$= 0 - 1 = -1$

ii) $i^{18} - 3.i^7 + i^2\left(1 + i^4\right)\cdot(-i)^{26}$

$= (i^2)^9 - 3i^6 i + i^2(1 + i^4)(i)^{26}$

$= -1 + 3i + (-1)(1+1)(-1)^{13}$

$= 3i - 1 + 2$

$= 3i + 1$

10. Find a square root for the following complex numbers.

i) 7 + 24i

ii) $-47+i8\sqrt{3}$

Sol: i) z = 7 + 24i

Let square root of z be a + ib

$$a+ib=\sqrt{7+24i}$$

$$(a+ib)^2=7+24i$$

$$a^2-b^2+2abi=7+24i$$

$$a^2-b^2=8, 2ab=24 \quad ...(1)$$

$$|a+ib|=|\sqrt{7+24i}|$$

Squaring on both sides,

$$|a+ib|^2=|7+24i|$$

$$a^2+b^2=\sqrt{49+576}$$

$$a^2+b^2=\sqrt{625}=25 \quad ...(2)$$

$$a^2-b^2=7$$

Adding

$$\frac{a^2+b^2=25}{2a^2=7+25}$$

$$a^2=16$$

$$a=\pm 4$$

$2b^2 = 25 - 7$

$b^2 = 9$

$b = \pm 3$

a + ib = ±(4 + 3i).

ii) $-47+i8\sqrt{3}$

Let the square root of z be a + ib,

$$(a+ib)^2=-47+i8\sqrt{3}$$

$$a^2-b^2=-47, 2ab=8\sqrt{3}$$

$$|a+ib|=|\sqrt{-47+i8\sqrt{3}}|$$

Squaring on both sides,

$$a^2+b^2=\sqrt{(-47)^2+(8\sqrt{3})^2}$$

$$=\sqrt{2209+192}=\sqrt{2401}$$

$$a^2+b^2=49$$

$$a^2-b^2=-47$$

$$2a^2=2$$

$$a^2=1$$

$$a=\pm 1$$

$$2b^2=96$$

$$b^2=48$$

$$b=\pm 4\sqrt{3}$$

$$a+ib=\pm(1+4\sqrt{3}i)$$

11. Find the multiplicative inverse of the following complex numbers.

i) $\sqrt{5}+3i$ **ii) –i** **iii)** i^{-35}

Sol: **i) z =** $\sqrt{5}+3i$

Let a + ib be multiplicative inverse then

(a + ib)z = 1

$$z=\frac{1}{a+ib} \text{ or } a+ib=\frac{1}{z}$$

$$a+ib=\frac{\bar{z}}{(z\bar{z})}$$

$$a+ib=\frac{\bar{z}}{|z|^2}=\frac{\sqrt{5}-3i}{5+9}=\frac{1}{14}(\sqrt{5}-3i)$$

ii) z = –i

Let a + ib be multiplicative inverse then

(a + ib)z = 1

$$a+ib=\frac{1}{z}$$

$$= \frac{1}{-i}$$

$$= \frac{i}{-i \cdot i}$$

$$a + ib = i$$

iii) z = i^{-35}

Let a + ib be multiplicative inverse then

(a + ib)z = 1

$$a + ib = \frac{1}{z} = \frac{1}{i^{-35}} = i^{35}$$

$$(a + ib) = i^{35} = (i^2)^{17}.i = -i$$

12. Express the following complex numbers in modules – amplitude form.

i) 1 – i **ii)** $1 + i\sqrt{3}$

iii) $-\sqrt{3} + i$ **iv)** $-1 - i\sqrt{3}$

Sol: **i) 1-i**

Let 1-i = $r\cos\theta + ri\sin\theta$

r cos θ=1 and rsin θ=-1

r^2=1+1=2 $\Rightarrow r = \sqrt{2}$

$\sin\theta = -\frac{1}{r} = -\frac{1}{\sqrt{2}}$ And $\cos\theta = \frac{1}{r} = \frac{1}{\sqrt{2}}$

Therefore, $\theta = -\frac{\pi}{3}$

1-i= $\sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

ii) $1+i\sqrt{3}=r\cos\theta+ri\sin\theta$

$r\cos\theta=1$

$r\sin\theta=\sqrt{3}$

$r^2(\cos^2\theta+\sin^2\theta)=1+3$

$r^2=4$

$4=\pm2$

$\tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$

$2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

iii) $-\sqrt{3}+i=r\cos\theta+ri\sin\theta$

$r\cos\theta=-\sqrt{3}$

$r\sin\theta=1$

$r^2(\cos^2\theta+\sin^2\theta)=3+1$

$r=\pm2$

$\tan\theta=\frac{-1}{\sqrt{3}}$

$2\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right)$

$2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

iv) $-1-\sqrt{3}i=r\cos\theta+ri\sin\theta$

$r\cos\theta=-1$

$r\sin\theta=-\sqrt{3}$

$r^2(\cos^2\theta+\sin^2\theta)=4$

$r=\pm2$

$\tan\theta=\sqrt{3}$

$\theta=\frac{2\pi}{3}$

Hence $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

13. Simplify –2i(3 + i)(2 + 4i)(1 + i) and obtain the modulus of that complex number.

Sol: z = –2i(3 + i)(2 + 4i)(1 + i)

$=-2i(2+14i)(1+i)$

$=-2i(2+2i+14i-14)$

$=-2i(-12+16i)$

$=24i+32$

$=8(4+3i)$

$|z|^2=64\cdot 25$

$|z|=8\times 4=40$

14. i) If z ≠ 0 find $\operatorname{Arg} z+\operatorname{Arg}\bar{z}$.

ii) If z_1 = –1 and z_2 = –i then find Arg(z_1z_2)

iii) If z_1 = –1 and z_2 = i then find $\operatorname{Arg}\left(\frac{z_1}{z_2}\right)$.

Sol: i) let $z=x+iy\Rightarrow\bar{z}=x-iy$

$\operatorname{Arg} z=\tan^{-1}\frac{y}{x}$ and $\operatorname{Arg}\bar{z}=\tan^{-1}\left(\frac{-y}{x}\right)$

$\operatorname{Arg} z+\operatorname{Arg}\bar{z}=\tan^{-1}\frac{y}{x}+\tan^{-1}\left(\frac{-y}{x}\right)$

$=\tan^{-1}\frac{y}{x}-\tan^{-1}\left(\frac{y}{x}\right)$

$=0$

ii) $z_1 = -1$ and $z_2 = -i$

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$$
$$= \mathrm{Tan}^{-1}\frac{0}{-1} + \mathrm{Tan}^{-1}\frac{-1}{0}$$
$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

iii) $z_1 = -1$ and $z_2 = i$

$$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2$$
$$\mathrm{Tan}^{-1}\frac{0}{-1} - \mathrm{Tan}^{-1}\frac{1}{0}$$
$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

15. i) If $(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta) = \cos\theta + i\sin\theta$ then find the value of θ.

ii) If $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$ then find the value of θ in radian measure.

iii) If $x + iy = \mathrm{cis}\alpha \,.\, \mathrm{cis}\,\beta$ then find the value of $x^2 + y^2$.

iv) If $\frac{z_2}{z_1}; z_1 \neq 0$ is an imaginary number then find the value of $\left|\frac{2z_1 + z_2}{2z_1 - z_2}\right|$.

v) If $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$ then show that $a^2 + b^2 = 4$.

Sol: **i)** $(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta) = \cos\theta + i\sin\theta$

$$= \cos 2\alpha \cos 2\beta + i\sin 2\alpha \cos 2\beta + \cos 2\alpha i \sin 2\beta + i^2 \sin 2\alpha \sin 2\beta$$
$$= \cos 2\alpha \cdot \cos 2\beta - \sin 2\alpha \sin 2\beta + i(\sin 2\alpha \cdot \cos 2\beta + \cos 2\alpha \sin 2\beta)$$
$$= \cos(2(\alpha + \beta)) + i\sin(2(\alpha + \beta))$$
$$= \cos\theta + i\sin\theta$$
$$\therefore \theta = 2(\alpha + \beta)$$

ii) let $\sqrt{3}+i = r(\cos\theta + i\sin\theta)$ **then**

$$r\cos\theta = \sqrt{3}$$
$$r\sin\theta = 1$$
$$r^2(\sin^2\theta + \cos^2\theta) = 4$$
$$r = \pm 2$$
$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

iii) x + iy = $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta - \sin\alpha\sin\beta)$$
$$x + iy = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$
$$x = \cos(\alpha+\beta)$$
$$y = \sin(\alpha+\beta)$$
$$x^2 + y^2 = 1$$

iv) Given $\frac{z_2}{z_1}; z_1 \neq 0$ **is an imaginary number =ki say**

$$\left|\frac{2z_1+z_2}{2z_1-z_2}\right| = \left|\frac{2+\frac{z_2}{z_1}}{2-\frac{z_2}{z_1}}\right|$$

$$= \left|\frac{2+ki}{2-ki}\right| = \frac{\sqrt{4+k^2}}{\sqrt{4+k^2}} = 1$$

v) $(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$

$$|\sqrt{3}+i|^{100} = 2^{99}\,|a+ib|$$
$$(\sqrt{4})^{100} = 2^{99}\sqrt{a^2+b^2}$$
$$2^{100} = 2^{99}\sqrt{a^2+b^2}$$
$$4 = a^2 + b^2$$

16. i) If z = x + iy and |z| = 1, then find the locus of z.

Sol: i) z = x + iy

$|z| = \sqrt{x^2 + y^2}$ =1, given

$1 = x^2 + y^2$

Locus is circle.

ii) If the amplitude of (z – 1) is π/2 then find the locus of z.

$z - 1 = (x - 1) + iy$

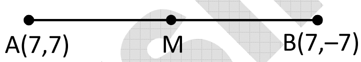
Given $\text{Tan}^{-1} \dfrac{y}{x-1} = \dfrac{\pi}{2}$

x – 1 = 0, y ≠ 0 also y > 0.

17. i) Find the equation of the perpendicular bisector of the line segment joining the points 7 + 7i, 7 – 7i.

Sol: i)z_1 = 7 + 7i, z_2 = 7 – 7i

Let A (7, 7),B(7, –7) be the points which represents above complex nos.



Midpoint of AB is $M\left(\dfrac{7+7}{2}, \dfrac{7-7}{2}\right) = M(7,0)$

Slope of AB = $\dfrac{7+7}{7-7} \to \infty$

Slope of the line perpendicular to AB is zero.

Therefore equation of the line passing through (7,0) and having slope 0 is y = 0 .

ii) Find the equation of the straight line joining the points –9 + 6i, 11 – 4i in the Argand plane.

Sol. Given complex nos. are A(–9, 6) B(11, –4)

Slope of AB = $\frac{6+4}{-9-11}=\frac{10}{-20}=\frac{-1}{2}$

Equation of line AB:

$y-6=\frac{-1}{2}(x+9)$

$2y-12=-x-9$

$x+2y=3$

18. If z = x + iy and if the point P in the Argand plane represents z, then describe geometrically the locus of z satisfying the equations

i) |z – 2– 3i| = 5

ii) 2|z – 2| = |z – 1|

iii)Img z^2 = 4

iv) $\text{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$

Sol: i)|z – 2 – 3i| = 5

|x+iy – 2– 3i| = 5

$|(x-2)+(y-3)|=5$

$(x-2)^2+(y-3)^2=25$

$x^2+y^2-4x-6y+4+9-25=0$

$x^2+y^2-4x-6y-12=0$

ii) 2|z – 2| = |z – 1|

Squaring on both sides 4|z – 2|^2 = |z – 1|^2

$$4(z-2)(\overline{z}-2)=(z-1)(\overline{z}-1)$$
$$4z\overline{z}-8z-8\overline{z}+16=z\overline{z}-z-\overline{z}+1$$
$$3z\overline{z}-7z-7\overline{z}+15=0$$
$$3(x^2+y^2)-7(2x)+15=0$$

iii)Img z^2 = 4

z = x + iy

$$z^2=(x+iy)^2$$
$$z^2=x^2-y^2+2xyi$$
$$\text{Im}(z^2)=2xy$$
$$2xy=4$$

xy = 2 rectangular hyperbola.

iv) $\text{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$

$$z_1=\frac{z-1}{z+1}=\frac{(x-1)+iy}{(x+1)+iy}$$
$$=\frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$$
$$=\frac{(x-1)(x+1)+y^2+iy(x+1-x+1)}{(x+1)^2+y^2}$$
$$\text{Arg}z_1=\text{Tan}^{-1}\frac{2y}{x^2+y^2-1}=\frac{\pi}{4}$$
$$\frac{2y}{x^2+y^2-1}=1$$
$$2y=x^2+y^2-1$$
$$x^2+y^2-2y-1=0$$

19. Show that the points in the Argand diagram represented by the complex numbers 2 + 2i, –2 – 2i, $-2\sqrt{3}+2\sqrt{3}\,i$ are the vertices of an equilateral triangle.

Sol: **Let** A(2, 2), B(–2, –2), C($-2\sqrt{3}, 2\sqrt{3}$) be points represents given complex numbers in the argand plane .

$$AB = \sqrt{(2+2)^2 + (2+2)^2} = 4\sqrt{2}$$

$$BC = \sqrt{(-2+2\sqrt{3})^2 + (-2-2\sqrt{3})^2}$$

$$BC = \sqrt{4+12-8\sqrt{3}+4+12+8\sqrt{3}} = 4\sqrt{2}$$

$$AC = \sqrt{(2+2\sqrt{3})^2 + (2-2\sqrt{3})^2} = 4\sqrt{2}$$

AB = AC = BC

ΔABC is equilateral.

20. Find the eccentricity of the ellipse whose equation is $|z-4| + \left|z - \frac{12}{5}\right| = 10$

Sol. Given equation is of the form $SP + S^1P = 2a$

Where $S(4,0)$ $S'\left(\frac{12}{5}, 0\right)$ and 2a = 10 $\Rightarrow$ a=5

SS'= 2ae

$$\Rightarrow 4 - \frac{12}{5} = 2X\,5e \Rightarrow \frac{8}{5} = 10e \Rightarrow e = \frac{4}{5}$$

21. Find the real and imaginary parts of the complex number $\frac{a+ib}{a-ib}$.

Sol: $$\frac{a+ib}{a-ib} = \frac{(a+ib)\,(a+ib)}{(a-ib)\,(a+ib)}$$

$$= \frac{(a)^2 + (ib)^2 + 2a(ib)}{(a)^2 - (ib)^2}$$

$$= \frac{a^2 - b^2 + 2iab}{a^2 + b^2}$$

$$= \frac{a^2 - b^2}{a^2 + b^2} + i\frac{2ab}{a^2 + b^2}$$

$\therefore$ Real part = $\dfrac{a^2 - b^2}{a^2 + b^2}$

Imaginary part = $\dfrac{2ab}{a^2 + b^2}$.

22. If 4x + i(3x – y) = 3 – 6i where x and y are real numbers, then find the values of x and y.

Sol: We have 4x + i(3x – y) = 3 + i(–6). Equating the real and imaginary parts in the above equation, we get 4x = 3, 3x – y = –6. Upon solving the simultaneous equations, we get x = 3/4 and y = 33/4.

23. If z = 2 – 3i, then show that z^2 – 4z + 13= 0.

Sol: $z = 2 - 3i \Rightarrow z - 2 = -3i \Rightarrow (z - 2)^2 = (-3i)^2$

$\Rightarrow z^2 + 4 - 4z = -9$

$\Rightarrow z^2 - 4z + 13 = 0.$

Short Answer Questions

1.

i) If $(a+ib)^2 = x+iy$**, find $x^2 + y^2$.**

Sol: **i)** $(a+ib)^2 = x+iy$

$$a^2 - b^2 + 2abi = x + iy$$

$$a^2 - b^2 = x$$

$$2ab = y$$

Now $x^2 = (a^2 - b^2)^2$

$$y^2 = 4a^2 b^2$$

$$x^2 + y^2 = (a^2 - b^2)^2 + 4a^2b^2 = (a^2 + b^2)^2$$

ii) If $x + iy = \dfrac{3}{2 + \cos\theta + i\sin\theta}$ **then show that $x^2 + y^2 = 4x - 3$.**

$x + iy = \dfrac{3}{2+\cos\theta + i\sin\theta}$ **rationalizing the dr.**

$$= \frac{3(2+\cos\theta - i\sin\theta)}{(2+\cos\theta)^2 - i^2\sin^2\theta}$$

$$= \frac{3(2+\cos\theta - i\sin\theta)}{4+\cos^2\theta + 4\cos\theta + \sin^2\theta}$$

$$= \frac{6+3\cos\theta - 3i\sin\theta}{5+4\cos\theta}$$

$$= \frac{6+3\cos\theta}{5+4\cos\theta} + \frac{-3i\sin\theta}{5+4\cos\theta}$$

$$x = \frac{6+3\cos\theta}{5+4\cos\theta},\ y = \frac{-3\sin\theta}{5+4\cos\theta}$$

L.H.S. =

$$x^2 + y^2 = \left(\frac{6+3\cos\theta}{5+4\cos\theta}\right)^2 + \left(\frac{-3\sin\theta}{5+4\cos\theta}\right)^2$$

$$= \frac{36 + 9\cos^2\theta + 36\cos\theta + 9\sin^2\theta}{(5+4\cos\theta)^2}$$

$$= \frac{45+36\cos\theta}{(5+4\cos\theta)^2}$$

$$= \frac{9(5+4\cos\theta)}{(5+4\cos\theta)^2}$$

$$x^2 + y^2 = \frac{9}{5+4\cos\theta}$$

R.H.S. =

$$4x-3 = \frac{4\,(6+3\cos\theta)}{5+4\cos\theta} - 3$$

$$= \frac{24+12\cos\theta - 15 - 12\cos\theta}{5+4\cos\theta}$$

$$= \frac{9}{5+4\cos\theta}$$

$\therefore\ x^2 + y^2 = 4x - 3.$

iii)If $x+iy = \frac{1}{1+\cos\theta+i\sin\theta}$ then show that $4x^2 - 1 = 0$.

$$x+iy = \frac{1}{1+\cos\theta+i\sin\theta}$$

$$x+iy = \frac{1+\cos\theta - i\sin\theta}{(1+\cos\theta)^2 - i^2\sin^2\theta}$$

$$x = \frac{1+\cos\theta}{2+2\cos\theta} = \frac{1+\cos\theta}{2(1+\cos\theta)}$$

$$x = \frac{1}{2} \Rightarrow 2x = 1$$

$$\Rightarrow 4x^2 = 1$$

$$\Rightarrow 4x^2 - 1 = 0$$

iv) If $u+iv=\frac{2+i}{z+3}$ **and z = x + iy find u, v.**

$$u+iv=\frac{2+i}{z+3}$$

$$=\frac{2+i}{(x+3)+iy}$$

$$=\frac{(2+i)(x+3-iy)}{(x+3)^2+y^2}$$

$$=\frac{(x+3)2+y}{(x+3)^2+y^2}+\frac{i(x+3-2y)}{(x+3)^2+y^2}$$

$$u=\frac{2(x+3)+y}{(x+3)^2+y^2}, v=\frac{x+3-2y}{(x+3)^2+y^2}$$

2. i) If z = 3 + 5i then show that $z^3 - 10z^2 + 58z - 136 = 0$.

Sol: **i)** z = 3 + 5i

$$(z-3)^2 = (5i)^2$$

$$z^2-6z+9=25i^2$$

$$z^2-6z+9=-25$$

$$z^2-6z+34=0$$

$$z^3-6z^2+34z=0$$

$$(z^3-10z^2+58z-136)+4z^2-24z+136=0$$

$$(z^3-10z^2+58z-136)+4(z^2-6z+34)=0$$

$$\therefore z^3-10z^2+58z-136=0$$

ii) If z = $2-i\sqrt{7}$ **then show that**

$3z^3 - 4z^2 + z + 88 = 0$.

$$z=2-i\sqrt{7}$$

$$(z-2)^2 = (-i\sqrt{7})^2$$

$$z^2 - 4z + 4 = -7$$

$$z^2 - 4z + 11 = 0$$

$$z^3 - 4z^2 + 11z = 0$$

$$3z^3 - 12z^2 + 33z = 0$$

$$(3z^3 - 4z^2 + z + 88) + (-8z^2 + 32z - 88) = 0$$

$$(3z^3 - 4z^2 + z + 88) - 8(z^2 - 4z + 11) = 0$$

$$3z^3 - 4z^2 + z + 88 = 0$$

iii) Show that $\frac{2-i}{(1-2i)^2}$ and $\frac{-2-11i}{25}$ **are conjugate to each other.**

$z = \frac{2-i}{(1-2i)^2}$ **(if z = a + ib,** $\bar{z} = a - ib$ **)**

$$= \frac{2-i}{(1-4)-4i}$$

$$= \frac{2-i}{-3-4i}$$

$$= \frac{(2-i)(-3+4i)}{(-3-4i)(-3+4i)}$$

$$= \frac{-6+8i+3i+4}{9+16}$$

$$= \frac{-2+11i}{25}$$

$\bar{z} = \frac{-2+11i}{25}$ [Conjugate to z is $\bar{z}$]

3. i) If $(x-iy)^{1/3} = a - ib$**, then show that** $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$**.**

Sol: i) $(x-iy)^{1/3} = a - ib$

$$x - iy = (a-ib)^3$$

$$x - iy = a^3 + ib^3 + 3a(-ib)(a-ib)$$

$$\Rightarrow x = a^3 - 3ab^2 \ \& \ -iy = ib^3 - 3a^2bi$$

$$\Rightarrow x = a^3 - 3ab^2 \ \& \ y = b^3 - 3a^2b$$

$y = b^3 - 3a^2 b$

$\frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = b^2 - 3a^2$

$\frac{x}{a} - \frac{y}{b} = 4\left(a^2 - b^2\right)$

ii) Write $\left(\frac{a+ib}{a-ib}\right)^2 - \left(\frac{a-ib}{a+ib}\right)^2$ in the form x + iy.

$\left(\frac{a+ib}{a-ib}\right)^2 - \left(\frac{a-ib}{a+ib}\right)^2$

$z = \left(\frac{(a+ib)(a+ib)}{(a-ib)(a+ib)}\right)^2 - \left(\frac{(a-ib)(a-ib)}{(a+ib)(a-ib)}\right)^2$

$= \left(\frac{a^2 - b^2 + 2abi}{a^2 + b^2}\right)^2 - \left(\frac{a^2 - b^2 - 2abi}{a^2 + b^2}\right)^2$

$= \frac{1}{(a^2+b^2)^2}\left(8ab(a^2 - b^2)i\right)$

iii)If x and y are real numbers such that $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$ then determine the values of x and y.

$\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

$\frac{[(1+i)x-2i](3-i)}{(3+i)(3-i)} + \frac{[(2-3i)y+i](3+i)}{(3-i)(3+i)} = i$

$\frac{x(3-i+3i+1)+y(6+2i-9i+3)-6i-2+3i+1^2}{9+1} = i$

$\frac{i(2x+2-7y-3)}{10} + \frac{4x+9y-3}{10} = i$

$4x + 9y - 3 = 0$

$2x - 7y - 3 = 10$

Solving we get, x = 3, y = –1.

4. i)Find the least positive integer n, satisfying $\left(\frac{1+i}{1-i}\right)^n = 1$.

Sol: **i)** $\left(\frac{1+i}{1-i}\right)^n = 1$

$\left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^n = 1$

$\left(\frac{2i}{2}\right)^n = 1$

$i^n = 1$

$(\because i^n = 1 = -1 \times -1 = i^2 \times i^2 = i^4)$

$n = 4$

$i^4 - 1$

Least value of n = 4.

ii) If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$ **find x and y**

$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$

$\left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^3 - \left(\frac{(1-i)(1-i)}{(1+i)(1-i)}\right)^3 = x + iy$

$\left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3$

$-i - 1 = x + iy$

$x = 0$

$y = -2$

iii)Find the real values of θ in order that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is a

(a) Real Number (b) Purely Imaginary Number.

z = $\frac{3+2i\sin\theta}{1-2i\sin\theta}$

$$z=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$z=\frac{3-4\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$$

Z is purely real $\Rightarrow$ imaginary part = 0

$$\frac{8\sin\theta}{1+4\sin^2\theta}=0$$

$$\sin\theta=0$$

$$\theta=n\pi, n\in I$$

z is purely imaginary $\Rightarrow$ Real part zero

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta}=0$$

$$\sin^2\theta=\frac{3}{4}$$

$$\sin\theta=\pm\frac{\sqrt{3}}{2}$$

$$\theta=2n\pi\pm\frac{\pi}{3}, n\in I$$

iv)Find the real value of x and y if $\frac{x-1}{3+i}+\frac{y-1}{3-i}=i$**.**

$$\frac{x-1}{3+i}+\frac{y-1}{3-i}=i\Rightarrow\frac{(x-1)(3-i)}{(3+i)(3-i)}+\frac{(y-1)(3+i)}{(3-i)(3+i)}=i$$

$$\frac{(x-1)(3-i)}{9+1}+\frac{(y-1)(3+i)}{9+1}=i$$

$$\frac{3(x-1)+3(y-1)}{10}+\frac{i(1-x+y-1)}{10}=i$$

$$\frac{3x+3y-6}{10}+\frac{-x+y}{10}=i$$

$$3x+3y-6=0$$

$$y-x=10$$

Solving we get, x = –4, y = 6.

5. Simplify the following complex numbers and find their modulus.

$$\frac{(2+4i)(-1+2i)}{(-1-i)(3-i)}$$

Sol: $z = \dfrac{(2+4i)(-1+2i)}{(-1-i)(3-i)}$

$$=\frac{-2+4i-4i+8i^2}{-3+1-3i+i^2}$$

$$=\frac{-10}{-4-2i}=\frac{5}{2+i}$$

$$=\frac{5(2-i)}{(2+i)(2-i)}$$

$$=\frac{5(2-i)}{4+1}=2-i$$

$$|z|=\sqrt{4+1}=\sqrt{5}$$

6. i) If $(1-i)(2-i)(3-i)...(1-ni)=x-iy$ **then prove that** $2\cdot 5\cdot 10...(1+n^2)=x^2+y^2$**.**

Sol: **i)** $(1-i)(2-i)(3-i)...(1-ni)=x-iy$

Taking modulus both sides.

$$|(1-1)||(2-i)|...|1-ni|=|x-iy|$$

$$\sqrt{2}\cdot\sqrt{5}......\sqrt{1+n^2}=\sqrt{x^2+y^2}$$

$$2\cdot 5\cdot ...\cdot(1+n^2)=x^2+y^2$$

ii) If the real part of $\frac{z+1}{z+i}$ is 1, then find the locus of z.

$\frac{z+1}{z+i}$ = $\mathbf{k_1 + k_2 i}$

$$\frac{(x+1)+iy}{(x)+(y+1)i} = k_1 + k_2 i$$

$$\frac{[(x+1)+iy][x-(y+1)i]}{[x+(y+1)i][x-(y+1)i]} = k_1 + k_2 i$$

$$\frac{x(x+1)+y(y+1)+i(xy-(x+1)(y+1))}{x^2+(y+1)^2}$$

Given real part = 1

$$x^2+y^2+x+y = x^2+(y+1)^2$$

$$x^2+y^2+x+y = x^2+y^2+2y+1$$

$$x-y=1$$

iii)If |z – 3 + i| = 4 determine the locus of z.

|z – 3 + i| = 4

$$|(x-3)+i(y+1)|=4$$

$$(x-3)^2+(y+1)^2=16$$

$$x^2+y^2-6x+2y+10=16$$

$$x^2+y^2-6x+2y-6=0$$

iv)If $|z+ai|=|z-ai|$ **then find the locus of z.**

|z + ai| = |z – ai|

$$|(x)+(y+a)i|=|x+(y-a)i|$$

$$x^2+(y+a)^2 = x^2+(y-a)^2$$

$$y=0$$

7. If z = x + iy and if the point P in the Argand plane represents z, then describe geometrically the locus of P satisfying the equations

i) |2z – 3| = 7

Sol: i)|2z – 3| = 7

$$|2(x)-3+2yi|=7$$

$$\sqrt{(2x-3)^2+4y^2}=7$$

$$4x^2-12x+9+4y^2=49$$

$$4x^2+4y^2-12x-40=0$$

$$x^2+y^2-3x-10=0$$

Centre $\left(\frac{3}{2},0\right)$, radius = $\frac{7}{2}$

ii) $|z|^2$ = 4 Re (z + 2)

$$|z|^2=4\,\text{Re}(z+2)$$

$x^2 + y^2 = 4(x + 2)$

$x^2 + y^2 - 4x - 8 = 0$

Circle centre (2, 0).

Radius = $\sqrt{12}=2\sqrt{3}$.

iii)$|z + i|^2 - |z - i|^2 = 2$

$$x^2+(y+1)^2-x^2-(y-1)^2=2$$

$$4y=2$$

$$2y=1\Rightarrow 2y-1=0$$

Line parallel to x-axis.

iv) |z + 4i| + |z – 4i| = 10

Sol: Given |z + 4i| + |z – 4i| = 10.

$$|(x+(y+4)i)|+|(x+(y-4)i)|=10$$

$$\sqrt{x^2+(y+4)^2}+\sqrt{x^2+(y-4)^2}=10$$

$$x^2+(y+4)^2=\left(10-\sqrt{x^2(y-4)^2}\right)^2$$

$$x^2+(y+4)^2=$$

$$100+x^2+(y-4)^2-20\sqrt{x^2+(y-4)^2}$$

Solving we get

$25x^2 + 9y^2 = 225$ is ellipse.

Centre (0, 0)

Eccentricity = e = $\sqrt{\frac{a^2-b^2}{a^2}}$

$$=\sqrt{\frac{25-9}{25}}$$

$$e=\frac{4}{5}.$$

8. If z_1, z_2 are two non-zero complex numbers satisfying

i) $|z_1 + z_2| = |z_1| + |z_2|$ then show that Arg z_1 – Arg z_2 = 0.

Sol: i)$|z_1 + z_2| = |z_1| + |z_2|$

Squaring both sides

$$|z_1+z_2|^2=(|z_1|+|z_2|)^2$$

$$(z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$=|z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+z_2\bar{z}_1$$

$$=|z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$z_1\bar{z}_2+z_2\bar{z}_1=2|z_1||z_2|$$

$$(x_1+iy_1)(x_2-iy_2)+(x_2+iy_2)(x_1-iy_1)$$

$$=2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$

Squaring on both sides we get

$(x_1x_2 + y_1y_2)^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$

$(x_1y_2 - y_1x_2)^2 = 0$

$\frac{y_2}{x_2} = \frac{y_1}{x_1}$

$\therefore \text{Arg}\, z_1 - \text{Arg}\, z_2 = 0$

ii) If z = x + iy and the point P represents z in the Argand plane and $\left|\frac{z-a}{z+a}\right| = 1$ Re(a) ≠ 0 then find the locus of P.

Sol. $\left|\frac{z-a}{z+a}\right| = 1$

$|z - a| = |z + a|$

Squaring on both sides

$(x - a)^2 + y^2 = (x + a)^2 + y^2$

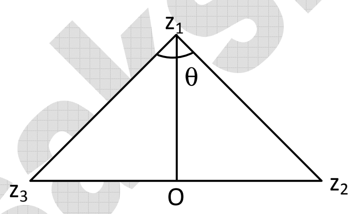
$4xa = 0$

$x = 0$

Parallel to y-axis.

9. If $\frac{z_3 - z_1}{z_2 - z_1}$ is a real number, show that the points represented by the complex numbers z_1, z_2, z_3 are collinear.

Sol:



$\text{Arg}\left(\frac{z_1 - z_3}{z_1 - z_2}\right) = 0$ then $\frac{z_1 - z_3}{z_1 - z_2}$ is real.

$\theta = 0$

$\therefore z_1, z_2, z_3$ are collinear.

10. Show that the four points in the Argand plane represented by the complex numbers 2 + i, 4 + 3i, 2 + 5i, 3i are the vertices of a square.

Sol: Let the four points represented by the given complex numbers be A, B, C, D then

A = (2, 1), B (4, 3), C (2, 5), D (0, 3)

$$AB = \sqrt{(4-2)^2+(3-1)^2} = \sqrt{4+4} = 2\sqrt{2}$$

$$BC = \sqrt{(4-2)^2+(3-5)^2} = \sqrt{4+4} = 2\sqrt{2}$$

$$AC = \sqrt{(2-2)^2+(5-1)^2} = \sqrt{0+16} = 4$$

$$BD = \sqrt{(4-0)^2+(3-3)^2} = \sqrt{16+0} = 4$$

AB = BC and AC = BD

Given Complex number are the vertices of a square.

11. Show that the points in the Argand diagram represented by the complex numbers $-2+7i$, $\frac{-3}{2}+\frac{1}{2}i$, $4-3i$, $\frac{7}{2}(1+i)$ are the vertices of a rhombus.

Sol: $A(-2, 7),\ B\left(\frac{-3}{2},\frac{1}{2}\right), C(4,-3), D\left(\frac{7}{2},\frac{7}{2}\right)$

$$AB = \sqrt{\left(-2+\frac{3}{2}\right)^2+\left(7-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{169}{4}} = \frac{\sqrt{170}}{2}$$

$$BC = \sqrt{\left(4+\frac{3}{2}\right)^2+\left(-3-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{121}{4}+\frac{49}{4}} = \frac{\sqrt{170}}{2}$$

$$CD = \sqrt{\left(4-\frac{7}{2}\right)^2+\left(-3-\frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{169}{4}} = \frac{\sqrt{170}}{2}$$

$$AD = \sqrt{\left(-2-\frac{7}{2}\right)^2 + \left(7-\frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{121}{4}+\frac{49}{4}} = \frac{\sqrt{170}}{2}$$

Slope of AC = $\frac{7+3}{-2-4} = \frac{10}{-6} = \frac{-5}{3}$

Slope of BD = $\frac{\frac{7}{2}-\frac{1}{2}}{\frac{7}{3}+\frac{3}{2}} = \frac{3}{5}$

AC ⊥ BD

∴ ABCD is rhombus.

12. Show that the points in the Argand diagram represented by the complex numbers z_1, z_2, z_3 are collinear if and only if there exists real numbers p, q, r not all zero, satisfying $pz_1 + qz_2 + rz_3 = 0$ and $p + q + r = 0$

Sol: $pz_1 + qz_2 + rz_3 = 0$

$pz_1 + qz_2 = -rz_3$

$$\left(\frac{pz_1+qz_2}{p+q}\right)(p+q) = -rz_3$$

Now $p + q = -r$

$$\frac{pz_1+qz_2}{p+q} = z_3$$

⇒ z_3 divides z_1 and z_2 is q : p ratio. ∴ z_1, z_2, z_3 are collinear.

13. The points P, Q denote the complex numbers z_1, z_2 in the Argand diagram. O is origin. If $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$ then show that ∠POQ = 90°.

Sol: $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$

$$\frac{z_1\bar{z}_2 + \bar{z}_1 z_2}{z_2\bar{z}_2} = 0 \Rightarrow \left(\frac{z_1}{z_2} + \overline{\left(\frac{z_1}{z_2}\right)}\right) = 0$$

$$\Rightarrow \text{Real of } \frac{z_1}{z_2} = 0$$

Or $\frac{z_1}{z_2}$ is purely imaginary

Let Imaginary part of $\left(\frac{z_1}{z_2}\right)$ is k.

$$\frac{z_1}{z_2} = ki.$$

$$\Rightarrow \text{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}.$$

14. Determine the locus of z, z ≠ 2i, such that $\left(\frac{z-4}{z-2i}\right) = 0$.

Sol: Let z = x + iy, then

$$\frac{z-4}{z-2i} = \frac{x+iy-4}{x+iy-2i} = \frac{(x-4)+iy}{x+i(y-2)}$$

$$= \frac{(x-4)+iy}{x+i(y-2)} \times \frac{x-i(y-2)}{x-i(y-2)}$$

$$= \frac{(x^2-4x+y^2-2y)+i(2x+4y-8)}{x^2+(y-2)^2}$$

Hence, real part of $\left(\frac{z-4}{z-2i}\right) = \frac{x^2+y^2-4x-2y}{x^2+(y-2)^2}$

The ratio on the RHS is zero.

i.e. $x^2 - 4x + y^2 - 2y = 0$ if and only if

$(x-2)^2 + (y-1)^2 = 5$.

Therefore $z \neq 2i$ and $\text{Re}\left(\frac{z-4}{z-2i}\right)=0$

$\Leftrightarrow$ $(x, y) \neq (0, 2)$ and $(x - 2)^2 + (y - 1)^2 = 5$.

Hence the locus of the given point representing the complex number is the circle with (2, 1) as centre and $\sqrt{5}$ units as radius except the point (0, 2).

15. Write z = $-7+i\sqrt{21}$ in the polar form.

Sol: If $z = -7+i\sqrt{21} = x + iy$

Then $x = -\sqrt{7}$, $y = \sqrt{21}$, $r=\sqrt{x^2+y^2}$

$$=\sqrt{7+21}=\sqrt{28}=2\sqrt{7}$$

$$\tan\theta=\frac{y}{x}=\frac{\sqrt{21}}{-\sqrt{7}}=-\sqrt{3}$$

Since the given point lies in the second quadrant, we look for a solution of $\tan\theta = -\sqrt{3}$ which lies in $\left(\frac{\pi}{2},\pi\right)$. We find that $\theta=\frac{2\pi}{3}$ is such a solution.

$$\therefore -\sqrt{7}+i\sqrt{21}=2\sqrt{7}\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$$

16. If the amplitude of $\left(\frac{z-2}{z-6i}\right)=\frac{\pi}{2}$, find its locus.

Sol: Let $z = x + iy$

Then $\frac{z-2}{z-6i}=\frac{x-2+iy}{x+i(y-6)}$

$$=\frac{[(x-2)+iy][x-(i(y-6)]}{[x+i(y-6)][x-i(y-6)]}$$

$$=\frac{x(x-2)+y(y-6)}{x^2+(y-6)^2}+i\frac{xy-(x-2)(y-6)}{x^2+(y-6)^2}$$

$= a + ib$ (say)

Then $a = \frac{x(x-2)+y(y-6)}{x^2+(y-6)^2}$

$b = \frac{xy-(x-2)(y-6)}{x^2+(y-6)^2}$

But by the hypothesis, amplitude of $a+ib = \frac{\pi}{2}$.

Hence a = 0 and b ≥ 0

$\therefore$ x(x – 2) + y(y – 6) = 0 or

$x^2 + y^2 - 2x - 6y = 0$ …(1)

and 3x +y – 6 ≥ 0 …(2)

The points satisfying (1) and (2) constitute the arc of the circle $x^2 + y^2 - 2x - 6y = 0$ intercepted by the diameter 3x + y – 6 = 0 not containing the origin and excluding the points (0, 6) and (2, 0). Hence this arc is the required locus.

17. Show that the equation of any circle in the complex plane is of the form

$z\bar{z} + b\bar{z} + \bar{b}z + c = 0, (b \in C; c \in R)$.

Sol: Assume the general form of the equation of a circle in Cartesian coordinates as

$x^2 + y^2 + 2gx + 2fy + c = 0, (g, f \in R)$ ... (1)

To write this equation in the complex variable form,

Let (x, y) = z. Then

$\frac{z+\bar{z}}{2} = x, \frac{z-\bar{z}}{2i} = y = -\frac{i(z-\bar{z})}{2}$

$x^2 + y^2 = |z|^2 = z\bar{z}$

Substituting these results in equation (1), we obtain

$z\bar{z} + g(z+\bar{z}) + f(z-\bar{z})(-i) + c = 0$

i.e. $z\bar{z} + (g - if)z + (g + if)\bar{z} + c = 0$ ... (2)

If g + if = b, then equation (2) can be written as: $z\bar{z} + \bar{b}z + b\bar{z} + c = 0$.

.

18. Show that the complex numbers z satisfying $z^2 + \bar{z}^2 = 2$ constitute a hyperbola.

Sol. Substituting z = x + iy in the given equation $z^2 + \bar{z}^2 = 2$, we obtain the Cartesian form of the given equation.

$\therefore (x+iy)^2 + (x-iy)^2 = 2$

i.e., $x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2$

Or $2x^2 - 2y^2 = 2$ i.e., $x^2 - y^2 = 1$.

Since this equation denotes a hyperbola all the complex numbers satisfying $z^2 + \bar{z}^2 = 2$ constitute the hyperbola $x^2 - y^2 = 1$.

19. Show that the points in the Argand diagram represented by the complex numbers 1+ 3i, 4 – 3i, 5 – 5i are collinear.

Sol: Let P = (1 + 3i) represented by (1, 3)

Q = 4 – 3i represented by (4, –3)

R = 5 – 5i represented by (5, –5)

If slope of PQ = slope QR then P, Q, R are collinear.

Slope of PQ = $\frac{3+3}{1-4} = \frac{6}{-3} = -2$

Slope of QR = $\frac{-3+5}{4-5} = -2$

$\therefore$ P, Q, R are collinear.

20. Find the equation of straight line joining the points represented by (–4+3i), (2 – 3i) in the Argand plane.

Sol: Take the given points as

A = – 4 + 3i = (– 4, 3)

B = 2 – 3i = (2, –3)

Equation of the straight line $\overline{AB}$ is

$$y - 3 = \frac{3+3}{-4-2}(x+4)$$

i.e., x + y + 1 = 0.

21. The point P resent a complex number z in the Argand plane. If the amplitude of z is π/4, determine the locus of P.

Sol: Let z = x + iy

By hypothesis, amplitude of $z = \frac{\pi}{4}$.

Hence $x = |z|\cos\frac{\pi}{4} = \frac{|z|}{\sqrt{2}}$ and $y = |z|\sin\frac{\pi}{4} = \frac{|z|}{\sqrt{2}}$.

Hence $x \geq 0, y \geq 0$ and x = y.

Clearly for any $x \in [0,\infty)$, the point x + ix has amplitude π/4.

∴ The locus of P is the ray

$\left\{(x,y) \in R^2 \mid x \geq 0,\ y \geq 0, x = y\right\}$.

22. If the point P denotes the complex number z = x + iy in the Argand plane and if $\frac{z-i}{z-1}$ is a purely imaginary number, find the locus of P.

Sol: We note that $\frac{z-i}{z-1}$ is not defined if z = 1.

Since, z = x + iy,

$$\frac{z-i}{z-1} = \frac{x+iy-i}{x+iy-1} = \frac{x+i(y-1)}{x-1+iy}$$

$$= \frac{[x+i(y-1)][(x-1)-iy]}{[(x-1)+iy]\,[(x-1)-iy]}$$

$$= \frac{x^2+y^2-x-y}{(x-1)^2+y^2} + i\left(\frac{1-x-y}{(x-1)^2+y^2}\right)$$

$\frac{z-i}{z-1}$ Will be purely imaginary, if and z ≠ 1 and $\frac{x^2+y^2-x-y}{(x-1)^2+y^2} = 0$.

i.e., $x^2+y^2-x-y=0$ and $(x,y) \neq (1,0)$.

∴ The locus of P is the circle $x^2+y^2-x-y=0$ excluding the point (1, 0).

23. The complex no. z has argument θ **,** $0<\theta<\frac{\pi}{2}$ **and satisfy the equation** $|z-3i|=3.$ **Then prove that** $\left(\cot\theta-\frac{6}{z}\right)=i$ **.**

Sol. Let z = x+iy3

$$|z-3i|=3 \Rightarrow |x+iy-3i|=3$$
$$\Rightarrow |x-i(y-3)|=3$$
$$\Rightarrow \sqrt{x^2+(y-3)^2}=3$$
$$\Rightarrow x^2+y^2-6y+9=9$$
$$\Rightarrow x^2+y^2-6y=0$$

$$\Rightarrow x^2+y^2=6y$$
$$\Rightarrow \frac{6y}{x^2+y^2}=1 \Rightarrow \frac{6y}{|z|^2}=1$$
$$\Rightarrow \frac{6y}{z\bar{z}}=1 \Rightarrow \frac{6}{z}=\frac{\bar{z}}{y}$$

$$\Rightarrow \frac{6}{z}=\frac{x-iy}{y}$$
$$\Rightarrow \frac{6}{z}=\frac{x}{y}-i \Rightarrow \frac{6}{z}=\cot\theta-i$$
$$\therefore \cot\theta-\frac{6}{z}=i$$