BINOMIAL THEOREM

* Binomial Theorem for integral index:

If n is a positive integer then $(x + a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots. + {}^nC_r\, x^{n-r}\, a^r$

$+ \ldots. + {}^nC_n\, a^n$.

* The expansion of $(x + a)^n$ contains $(n + 1)$ terms.
* In the expansion, the sum of the powers of x and a in each term is equal to n.
* In the expansion, the coefficients ${}^nC_0, {}^nC_1, {}^nC_2 \ldots {}^nC_n$ are called binomial coefficients and these are simply denoted by $C_0, C_1, C_2 \ldots. C_n$.

  ${}^nC_0 = 1, {}^nC_n = 1, {}^nC_1 = n, {}^nC_r = {}^nC_{n-r}$

* In the expansion, $(r+1)^{th}$ term is called the general term. It is denoted by

  T_{r+1}. Thus $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$.

* $(x + a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$.

* $(x - a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, (-a)^r = \sum_{r=0}^{n} (-1)^n\, {}^nC_r\, x^{n-r} a^r = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2} a^2 - \ldots.+ (-1)^n\, {}^nC_n\, a^n$

* $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r = {}^nC_0 + {}^nC_1 x + \ldots + {}^nC_n x^n = C_0 + C_1 x + C_2 x^2 + \ldots C_n x^n$

* Middle term(s) in the expansion of $(x + a)^n$.

i) If n is even, then $\left(\frac{n}{2}+1\right)$th term is the middle term

ii) If n is odd, then $\frac{n+1}{2}$ th and $\frac{n+3}{2}$ th terms are the middle terms.

* Numerically greatest term in the expansion of $(1 + x)^n$:

i) If $\frac{(n+1)|x|}{|x|+1}$ = p, a integer then p^{th} and $(p + 1)^{th}$ terms are the numerically greatest terms in the expansion of $(1 + x)^n$.

ii) If $\frac{(n+1)|x|}{|x|+1}$ = p + F where p is a positive integer and 0 < F < 1 then (p+1) th term is the numerically greatest term in the expansion of $(1 + x)^n$.

* Binomial Theorem for rational index: If n is a rational number and

$|x| < 1$, then $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots = (1 + x)^n$

* If $|x| < 1$ then

i) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$

ii) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

iii) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r (r + 1)x^r + \ldots.$

iv) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r + 1)x^r + \ldots.$

v) $(1 - x)^{-n} = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$

vi) $(1 - x)^{-n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$

* If $|x| < 1$ and n is a positive integer, then

i) $(1 - x)^{-n} = 1 + {}^{n}C_1 x + {}^{(n+1)}C_2 x^2 + {}^{(n+2)}C_3 x^3 + \ldots.$

ii) $(1 + x)^{-n} = 1 - {}^{n}C_1 x + {}^{(n+1)}C_2 x^2 - {}^{(n+2)}C_3 x^3 + \ldots.$

* When $|x| < 1$,

$$(1 - x)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots\ldots\ \infty$$

* When $|x| < 1$,

$$(1 + x)^{-p/q} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots\ldots\ \infty$$

Binomial Theorem:

Let *n* be *a* positive integer and *x*, *a* be real numbers,

then $(x+a)^n = {}^{n}C_0.x^n a^0 + {}^{n}C_1.x^{n-1}a^1 + {}^{n}C_2.x^{n-2}a^2 + \ldots + {}^{n}C_r.x^{n-r}a^r + \ldots\ldots + {}^{n}C_n.x^0 a^n$

Proof:

We prove this theorem by using the principle of mathematical induction (on *n*).

When $n = 1, (x+a)^n = (x+a)^1 = x + a = {}^{1}C_0 x^1 a^0 + {}^{1}C_1 x^0 a^1$

Thus the theorem is true for $n = 1$

Assume that the theorem is true for $n = k \geq 1$ (where *k* is a positive integer). That is

$$(x+a)^k = {}^{k}C_0.x^k.a^0 + {}^{k}C_1.x^{k-1}.a^1 + {}^{k}C_2.x^{k-2}.a^2 + \ldots + {}^{k}C_r.x^{k-r}.a^r + \ldots + {}^{k}C_k.x^0.a^k$$

Now we prove that the theorem is true when $n = k + 1$ also

$(x+a)^{k+1} = (x+a)(x+a)^k$

$= (x+a)({}^kC_0.x^k.a^0 + {}^kC_1.x^{k-1}.a^1 + {}^kC_2.x^{k-2}.a^2 + ... + {}^kC_r.x^{k-r}.a^r + ... + {}^kC_k.x^0.a^k)$

$= {}^kC_0.x^{k+1}.a^0 + {}^kC_1.x^k.a^1 + {}^kC_2.x^{k-1}.a^2 + ... + {}^kC_r.x^{k-r+1}.a^r + ... + {}^kC_k.x^1.a^k + {}^kC_0.x^k.a^1 + {}^kC_1.x^{k-1}.a^2 + ...$
$+ {}^kC_{r-1}.x^{k-r+1}.a^r + ... + {}^kC_{k-1}.x^1.a^k + {}^kC_k.x^0.a^{k+1}$

$= {}^kC_0.x^{k+1}.a^0 + ({}^kC_1 + {}^kC_0).x^k.a^1 + ({}^kC_2 + {}^kC_1).x^{k-1}.a^2 + ... + ({}^kC_r + {}^kC_{r-1}).x^{k-r+1}.a^r + ... + ({}^kC_k + {}^kC_{k-1}).x^1.a^k$
$+ {}^kC_k.x^0.a^{k+1}$

Since ${}^kC_0 = 1 = {}^{k+1}C_0, {}^kC_r + {}^kC_{r-1} = {}^{(k+1)}C_r$ for $1 \le r \le k$, ${}^kC_k = 1 = {}^{(k+1)}C_{(k+1)}$

$(x + a)^{k+1}$
$= {}^{(k+1)}C_0.x^{k+1}.a^0 + {}^{(k+1)}C_1.x^k.a^1 + {}^{(k+1)}C_2.x^{k-1}.a^2 + ... + {}^{(k+1)}C_r.x^{k-r+1}.a^r + ... +$
${}^{(k+1)}C_k.x^1.a^k + {}^{k+1}C_{k+1}.x^0.a^{k+1}$

Therefore the theorem is true for $n = k + 1$

Hence, by mathematical induction, it follows that the theorem is true of all positive integer n

Very Short Answer Questions

1. Expand the following using binomial theorem.

(i) $(4x + 5y)^7$ **(ii)** $\left(\frac{2}{3}x + \frac{7}{4}y\right)^5$

(iii) $\left(\frac{2p}{5} - \frac{3p}{7}\right)^6$ **(iv)** $(3 + x - x^2)^4$

i) $(4x + 5y)^7$

Sol. $(4x+5y)^7 =$

$${}^7C_0(4x)^7(5y)^0 + {}^7C_1(4x)^6(5y)^1 + {}^7C_2(4x)^5(5y)^2 + {}^7C_3(4x)^4(5y)^3 + {}^7C_4(4x)^3(5y)^4 + {}^7C_5(4x)^2(5y)^5 + {}^7C_6(4x)^1(5y)^6 + {}^7C_7(4x)^0 + (5y)^7$$

$$= \sum_{r=0}^{7} {}^7C_r(4x)^{7-r}(5y)^r$$

ii) $\left(\frac{2}{3}x + \frac{7}{4}y\right)^5$

Sol. $\left(\frac{2}{3}x + \frac{7}{4}y\right)^5 =$

$${}^5C_0\left(\frac{2}{3}x\right)^5 + {}^5C_1\left(\frac{2}{3}x\right)^4\left(\frac{7}{4}y\right) + {}^5C_2\left(\frac{2}{3}x\right)^3\left(\frac{7}{4}y\right)^2 + {}^5C_3\left(\frac{2}{3}x\right)^2\left(\frac{7}{4}y\right)^3 + {}^5C_4\left(\frac{2}{3}x\right)^1\left(\frac{7}{4}y\right)^4 + {}^5C_5\left(\frac{7}{4}y\right)^5$$

$$= \sum_{r=0}^{5} {}^5C_r\left(\frac{2}{3}x\right)^{5-r}\left(\frac{7}{4}y\right)^r$$

iii) $\left(\frac{2p}{5}-\frac{3p}{7}\right)^6$

$$=\sum_{r=0}^{6}(-1)^r\ {}^6C_r\left(\frac{2p}{5}\right)^{6-r}\left(\frac{3q}{7}\right)^r$$

iv) $(3 + x - x^2)^4$

$$81+108x-54x^2-96x^3+19x^4+32x^5-6x^6-4x^7+x^8$$

2. Write down and simplify

i) 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$

ii) 7^{th} term in $(3x - 4y)^{10}$

iii) 10^{th} term in $\left(\frac{3p}{4}-5q\right)^{14}$

iv) r^{th} term in $\left(\frac{3a}{5}+\frac{5b}{7}\right)^8\ (1\le r\le 9)$

i) 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$

Sol. 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$

The general term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$ is

$$T_{r+1}={}^9C_r\left(\frac{2x}{3}\right)^{9-r}\left(\frac{3y}{2}\right)^r$$

Put r = 5

$$T_6 = {}^9C_5\left(\frac{2x}{3}\right)^4\left(\frac{3y}{2}\right)^5 = {}^9C_5\left(\frac{2}{3}\right)^4\left(\frac{3}{x}\right)^5 x^4y^5$$

$$= \frac{9\times8\times7\times6}{1\times2\times3\times4}\frac{(2^4)}{3^4}\frac{3^5}{2^5}x^4y^5 = 189x^4y^5$$

ii) **Ans.** $280(12)^5x^4y^6$

iii) **Ans.** $\frac{-(2002)3^5\cdot5^9}{4^5}p^5q^9$

iv) **Ans.** ${}^8C_{(r-1)}\left(\frac{3a}{5}\right)^{9-r}\left(\frac{5b}{7}\right)^{r-1}; 1\le r\le 9$

3. Find the number of terms in the expansion of

(i) $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ **(ii)** $(3p+4q)^{14}$ **(iii)** $(2x+3y+z)^7$

i) $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$

Sol. Number of terms in $(x + a)^n$ is $(n + 1)$, where n is a positive integer.

Hence number of terms in $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ are:

9 + 1 = 10

iii) $(2x+3y+z)^7$

Sol. Number of terms in $(a + b + c)^n$ are $\frac{(n+1)(n+2)}{2}$, where n is a positive integer.

Hence number of terms in $(2x+3y+z)^7$ are: $\frac{(7+1)(7+2)}{2} = \frac{8\times9}{2} = 36$

4. Find the range of x for which the binomial expansions of the following are valid.

(i) $(2 + 3x)^{-2/3}$ (ii) $(5 + x)^{3/2}$ (iii) $(7 + 3x)^{-5}$ (iv) $\left(4-\frac{x}{3}\right)^{-1/2}$

Sol. (i) $(2 + 3x)^{-2/3}$ =

$$\left[2\left(1+\frac{3}{2}x\right)\right]^{-2/3} = 2^{-2/3}\left(1+\frac{3}{2}x\right)^{-2/3}$$

$\therefore$ The binomial expansion of $(2 + 3x)^{-2/3}$ is valid when $\left|\frac{3}{2}x\right|<1$.

i.e. $|x| < \frac{2}{3}$ i.e. $x\in\left(-\frac{2}{3},\frac{2}{3}\right)$

ii) $(5+x)^{3/2} = \left[5\left(1+\frac{x}{5}\right)\right]^{3/2} = 5^{3/2}\left(1+\frac{x}{5}\right)^{3/2}$

$\therefore$ The binomial expansion of $(5 + x)^{3/2}$ is valid when $\left|\frac{x}{5}\right|<1$.

i.e. $|x| < 5$

i.e. $x \in (-5, 5)$

iii) $(7+3x)^{-5} = 7\left[\left(1+\frac{3}{7}x\right)\right]^{-5} = 7^{-5}\left(1+\frac{3}{7}x\right)^{-5}$

$(7 + 3x)^{-5}$ is valid when $\left|\frac{3x}{7}\right|<1$

$\Rightarrow |x| < \frac{7}{3} \Rightarrow x\in\left(\frac{-7}{3},\frac{7}{3}\right)$

iv) $\left(4-\frac{x}{3}\right)^{-1/2} = \left[4\left(1-\frac{x}{12}\right)\right]^{-1/2}$

$\left(4-\frac{x}{3}\right)^{-1/2}$ is valid when $\left|\frac{-x}{12}\right|<1$

$\Rightarrow |x| < 12 \Rightarrow x \in (-12, 12)$

5. Find the (i) 6th term of $\left(1+\frac{x}{2}\right)^{-5}$.

Sol. T_{r+1} in $(1 + x)^{-n} = (-1)^r \frac{(n)(n+1)(n+2)...(n+r-1)}{1\cdot 2\cdot 3\cdot ...r}\cdot x^r$

Put r = 5, n = 5, x by x/2

$$T_6 = (-1)^5 \frac{(5)(5+1)(5+2)(5+3)(5+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\left(\frac{x}{2}\right)^5$$

$$= \frac{-5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5\cdot x^5 = \frac{-63}{16}\cdot x^5$$

ii) 7th term of $\left(1-\frac{x^2}{3}\right)^{-4}$

Sol. T_{r+1} in $(1 - x)^{-n}$ =

$$= \frac{(n)(n+1)(n+2)...(n+r-1)}{1\cdot 2\cdot 3\cdot ...r}\cdot x^r$$

Put r = 6, n = 4, x by $\frac{x^2}{3}$

Then 7th term in $\left(1-\frac{x^2}{3}\right)^{-4}$ is

$$= \frac{(4)(4+1)(4+2)(4+3)(4+4)(4+5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{-x^2}{3}\right)^6$$

$$= \frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{x^{12}}{3^6} = \frac{28}{243}\cdot x^{12}$$

iii) 10th term of $(3 - 4x)^{-2/3}$.

Sol. $(3-4x)^{-2/3} = \left[3\left(1-\frac{4}{3}x\right)\right]^{-2/3} \quad = (3)^{-2/3}\left(1-\frac{4}{3}x\right)^{-2/3} \quad ...(1)$

First find 10th term of $\left(1-\frac{4}{3}x\right)^{-2/3}$

The general term of $(1 - x)^{-p/q}$ is $T_{r+1} = \frac{(p)(p+q)(p+2q)+...+[p+(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$

Here p = 2, q = 3, r = 9

$$\frac{x}{q}=\left(\frac{(4/3)x}{3}\right)\left(\frac{4}{9}x\right)$$

$$T_{10}=\frac{(2)(2+3)(2+6)...[2+(9-1)3]}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}\left(\frac{4}{9}x\right)^9$$

$$=\frac{2\cdot5\cdot8\cdot...(26)}{9!}\left(\frac{4x}{9}\right)^9$$

10^{th} term in $(3-4x)^{-2/3}=3^{-2/3}\left[\frac{2\cdot5\cdot8\cdot...(26)}{9!}\left(\frac{4x}{9}\right)^9\right]$

iv) 5^{th} term of $\left(7+\frac{8y}{3}\right)^{7/4}$

Sol. $\left(7+\frac{8y}{3}\right)^{7/4}=\left[7\left(1+\frac{8y}{21}\right)\right]^{7/4}$

General term of $(1+x)^{p/q}$

$$T_{r+1}=\frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 7, q = 4, r = 4, $\frac{x}{q}=\frac{(8y/21)}{4}=\frac{2y}{21}$

$\therefore T_5$ of $\left(1+\frac{8y}{21}\right)^{7/4}$ is

$$=\frac{(7)(7-4)(7-2\times4)(7-3\times4)}{1\times2\times3\times4}\left(\frac{2y}{21}\right)^4$$

$$=\frac{7(3)(-1)(-5)}{1\times2\times3\times4}\cdot\frac{2^4y^4}{(21)^4}=70\left(\frac{y}{21}\right)^4$$

$\therefore 5^{th}$ term of $\left(7+\frac{8y}{3}\right)^{7/4}$ is $7^{7/4}(70)\left(\frac{y}{21}\right)^4$

$\therefore T_5$ in $\left(7+\frac{8y}{3}\right)^{7/4}=7^{7/4}(70)\left(\frac{y}{21}\right)^4$

6. Write down the first 3 terms in the expansion of

(i)$(3 + 5x)^{-7/3}$, **(ii) $(1 + 4x)^{-4}$,**

(iii) $(8 - 5x)^{2/3}$, **(iv) $(2 - 7x)^{-3/4}$.**

Sol.i)$(3 + 5x)^{-7/3}$ $= \left[3\left(1+\frac{5}{3}x\right)\right]^{-7/3} = (3)^{-7/3}\left(1+\frac{5}{3}x\right)^{-7/3}$

Now we have

$$(1 + x)^{-p/q} = 1-\frac{p}{11}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2+...$$

Here p = 7, q = 3, $\frac{x}{q}=\frac{(5/3)x}{3}=\frac{5}{9}x$

$\therefore (3 + 5x)^{-7/3} =$

$$(3)^{-7/3}\left[1-\frac{7}{1!}\left(\frac{5}{9}x\right)+\frac{(7)(10)}{1\cdot 2}\left(\frac{5}{9}x\right)^2+...\right]$$

$$=3^{-7/3}\left[1-\frac{35}{9}x+\frac{875}{81}x^2-......\right]$$

$\therefore$ The first 3 terms of $(3 + 5x)^{-7/3}$ are

$$3^{-7/3},\frac{-3^{7/3}\cdot 35x}{9},3^{-7/3}\frac{875}{81}x^2$$

ii) $(1 + 4x)^{-4}$ Try your self

iii) $(8 - 5x)^{2/3}$

Sol. $\left[8\left(1-\frac{5}{8}x\right)^{2/3}\right]=(2^3)^{2/3}\left[1-\frac{5}{8}x\right]^{2/3}$

$$=4\left[\left(1-\frac{5x}{8}\right)^{2/3}\right]$$

We know that

$$(1-X)^{p/q}=1-p\left(\frac{X}{q}\right)+\frac{(p)(p-q)}{1\cdot 2}\left(\frac{X}{q}\right)^2-\ldots$$

Here $X=\frac{5x}{8}, p=2, q=3, \frac{X}{q}=\frac{(5x/8)}{3}=\frac{5x}{24}$

$\therefore (8-5x)^{2/3}=$

$$4\left[1-2\left(\frac{5x}{24}\right)+\frac{(2)(2-3)}{1\cdot 2}\left(\frac{5x}{24}\right)^2-\ldots\right]$$

$$=4\left[1-\frac{5x}{12}-\left(\frac{5x}{24}\right)^2+\ldots\right]$$

$\therefore$ The first 3 terms of $(8-5x)^{2/3}$ are

$$4, \frac{-5x}{3}, \frac{-25}{144}x^2$$

iv) $(2-7x)^{-3/4}$ Try your self

7. Find the general term (r + 1)th term in the expansion of

(i) $(4+5x)^{-3/2}$ (ii) $\left(1-\frac{5x}{3}\right)^{-3}$ (iii) $\left(1+\frac{4x}{5}\right)^{5/2}$ (iv) $\left(3-\frac{5x}{4}\right)^{-1/2}$

i) $(4+5x)^{-3/2}$

Sol. Write $(4+5x)^{-3/2}=\left[4\left(1+\frac{5}{4}x\right)\right]^{-3/2}$

$$=(2^2)^{-3/2}\left[\left(1+\frac{5}{4}x\right)^{-3/2}\right]=\frac{1}{8}\left[\left(1+\frac{5}{4}x\right)^{-3/2}\right]$$

General term of $(1+x)^{-p/q}$ is

$T_{r+1}=(-1)^r$

$$\frac{(p)(p+q)(p+2q)+...+[p+(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 3, q = 2, $\frac{X}{p}=\frac{\left(\frac{5x}{4}\right)}{2}=\frac{5x}{8}$

$\therefore$ T_{r+1} in $(4 + 5x)^{-3/2}$ is

$$(-1)^r\frac{1}{8}\left[\frac{(3)(3+2)(3+2\times2)...[3+(r-1)2]}{r!}\right]\left(\frac{5x}{8}\right)^r$$

$$=(-1)^r\frac{3\cdot5\cdot7......(2r+1)}{r!}\frac{(5x)^r}{(8)^{r+1}}$$

ii) $\left(1-\frac{5x}{3}\right)^{-3}$

Sol. General term of $(1 - x)^{-n}$ is

$$T_{r+1}=\frac{(n)(n+1)(n+2)...(n+r-1)}{1\cdot2\cdot3\cdot...r}\cdot X^r$$

iii) $\left(1+\frac{4x}{5}\right)^{5/2}$

Sol. General term of $(1 + X)^{p/q}$ is $T_{r+1}=\frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$

iv) $\left(3-\frac{5x}{4}\right)^{-1/2}$

Sol. Write $\left(3-\frac{5x}{4}\right)^{-1/2}=\left[3\left(1-\frac{5x}{12}\right)^{-1/2}\right]$

$$=3^{-1/2}\left[\left(1-\frac{5x}{12}\right)^{-1/2}\right]$$

General term of $(1 - X)^{-p/q}$ is $T_{r+1} = \frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$

8. Find the largest binomial coefficients in the expansion of

(i) $(1 + x)^{19}$ (ii) $(1 + x)^{24}$

Sol. i) Here n = 19 is an odd integer. Hence the largest binomial coefficients are

$^nC_{\left(\frac{n-1}{2}\right)}$ and $^nC_{\left(\frac{n+1}{2}\right)}$

i.e. $^{19}C_9$ and $^{19}C_{10}$ $\left(^{19}C_9 = {}^{19}C_{10}\right)$

ii) Here n = 24 is an even integer. Hence the largest binomial coefficient is

$^nC_{\left(\frac{n}{2}\right)}$ i.e. $^{24}C_{12}$

9. If $^{22}C_r$ is the largest binomial coefficient in the expansion of $(1 + x)^{22}$, find the value of $^{13}C_r$.

Sol. Here n = 22 is an even integer. There is only one largest binomial coefficient and it is

$^nC_{(n/2)} = {}^{22}C_{11} = {}^{22}C_r \Rightarrow r = 11$

$\therefore {}^{13}C_r = {}^{13}C_{11} = {}^{13}C_2 = \frac{13 \times 12}{1 \times 2} = 78$

10. Find the 7^{th} term in the expansion of $\left(\frac{4}{x^3} + \frac{x^2}{2}\right)^{14}$.

Sol. The general term in the expansion of $(X + a)^n$ is

$T_{r+1} = {}^nC_r (X)^{n-r} a^r$

Put $X = \frac{4}{x^3}, a = \frac{x^2}{2}, n = 14, r = 6$

$$T_7 \text{ in } \left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14} \text{ is} = {}^{14}C_6\left(\frac{4}{x^3}\right)^{14-6}\left(\frac{x^2}{2}\right)^6$$

$$= {}^{14}C_6\,\frac{4^8}{2^6}\cdot\frac{x^{12}}{x^{24}} = {}^{14}C_6\cdot 4^5\cdot\frac{1}{x^{12}}$$

11. Find the 3rd term from the end in the expansion of $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$.

Sol. Comparing with $(X + a)^n$, we get

$$X = x^{-2/3},\ a = \frac{-3}{x^2}, n = 8$$

In the given expansion $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$, we have n + 1 = 8 + 1 = 9 terms.

Hence the 3rd term from the end is 7th term from the beginning.

$$\therefore\ T_7 = {}^nC_6\,(X)^{n-6}\,(a^6)$$

$$= {}^8C_6(x^{-2/3})^{8-6}\left(\frac{-3}{x^2}\right)^6 = {}^8C_6 x^{-4/3}\cdot\frac{3^6}{x^{12}}$$

$$= \frac{8\times7}{1\times2}\cdot 3^6\cdot x^{-4/3-12} = 28\cdot 3^6\cdot x^{-40/3}$$

12. Find the coefficient of x^9 and x^{10} in the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$.

Sol. If we write $X = 2x^2$ and $a = -\frac{1}{x}$, then the general term in the expansion of

$$\left(2x^2-\frac{1}{x}\right)^{20} = (X+a)^{20} \text{ is}$$

$$T_{r+1} = {}^nC_r X^{n-r} a^r = {}^{20}C_r(2x^2)^{20-r}\left(-\frac{1}{x}\right)^r$$

$$= (-1)^r\ {}^{20}C_r\ 2^{20-r}\ x^{40-3r}$$

Now x^9 coefficient is x^{40-3r}

$$\Rightarrow x^9 = 40 - 3r = 9 \Rightarrow 3r = 31 \Rightarrow r = \frac{31}{3}$$

Since r = 31/3 which is impossible since r must be a positive integer. Thus there is no term containing x^9 in the expansion of the given expression. In other words the coefficient of x^9 is 0.

Now to find the coefficient of x^{10}.

Put 40 – 3r = 10 ⇒ r = 10

$$T_{10+1} = (-1)^{10}\ {}^{20}C_{10}\, 2^{20-10}\, x^{40-30} = {}^{20}C_{10}\, 2^{10}\, x^{10}$$

∴ The coefficient of x^{10} is ${}^{20}C_{10}\, 2^{10}$.

13. Find the term independent of x (that is the constant term) in the expansion of

$$\left(\frac{\sqrt{x}}{3} + \frac{3}{2x^2}\right)^{10}.$$

Sol. $T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r = \frac{{}^{10}C_r \cdot 3^{\frac{3r-10}{2}}}{2^r} \cdot x^{\frac{10-5r}{2}}$

To find the term independent of x, put

$$\frac{10-5r}{2} = 10 \Rightarrow r = 2$$

$$\therefore T_3 = \frac{{}^{10}C_2\, 3^{\frac{6-10}{2}}}{2^2} \cdot x^{\frac{10-10}{2}} = \frac{{}^{10}C_2\, 3^{-2} x^0}{2^2} = \frac{5}{4}$$

14. Find the set E of x for which the binomial expansions for the following are valid

(i) $(3 - 4x)^{3/4}$ **(ii) $(2 + 5x)^{-1/2}$**

(iii) $(7 - 4x)^{-5}$ **(iv) $(4 + 9x)^{-2/3}$**

(v) $(a + bx)^{r}$

Sol.**i)** $(3-4x)^{3/4} = 3^{3/4}\left(1-\frac{4x}{3}\right)^{3/4}$

The binomial expansion of $(3 - 4x)^{3/4}$ is valid, when $\left|\frac{4x}{3}\right| < 1$.

i.e. $|x| < \frac{3}{4}$

i.e. $E = \left(\frac{-3}{4}, \frac{3}{4}\right)$

ii) $(2+5x)^{-1/2} = 2^{-1/2}\left(1+\frac{5x}{2}\right)^{-1/2}$

The binomial expansion of $(2 + 5x)^{-1/2}$ is valid when $\left|\frac{5x}{3}\right| < 1 \Rightarrow |x| < \frac{2}{5}$

i.e. $E = \left(\frac{-2}{5}, \frac{2}{5}\right)$

iii) $(7-4x)^{-5} = 7^{-5}\left(1-\frac{4x}{7}\right)^{-5}$

The binomial expansion of $(7 - 4x)^{-5}$ is valid when $\left|\frac{4x}{7}\right| < 1 \Rightarrow |x| < \frac{7}{4}$

i.e. $E = \left(\frac{-7}{4}, \frac{7}{4}\right)$

iv) $(4+9x)^{-2/3} = 4^{-2/3}\left(1+\frac{9x}{4}\right)^{-2/3}$

The binomial expansion of $(4 + 9x)^{-2/3}$ is valid when $\left|\frac{9x}{4}\right| < 1 \Rightarrow |x| < \frac{4}{9}$

$\Rightarrow x \in \left(\frac{-4}{9}, \frac{4}{9}\right)$

i.e. $E = \left(\frac{-4}{9}, \frac{4}{9}\right)$

v) For any non zero reals a and b, the set of x for which the binomial expansion of $(a + bx)^r$ is valid when $r \notin Z^+ \cup \{0\}$, is $\left(-\frac{|a|}{|b|}, \frac{|a|}{|b|}\right)$.

15. Find the

i) 9^th^ term of $\left(2+\frac{x}{3}\right)^{-5}$

ii) 10^th^ term of $\left(1-\frac{3x}{4}\right)^{4/5}$

iii)8^th^ term of $\left(1-\frac{5x}{2}\right)^{-3/5}$

iv)6^th^ term of $\left(3+\frac{2x}{3}\right)^{3/2}$

i) 9^th^ term of $\left(2+\frac{x}{3}\right)^{-5}$

Sol. $\left(2+\frac{x}{3}\right)^{-5} = \left[2\left(1+\frac{x}{6}\right)\right]^{-5} = 2^{-5}\left(1+\frac{x}{6}\right)^{-5}$ ...(1)

Compare $\left(1+\frac{x}{6}\right)^{-5}$ with $(1 + x)^{-n}$, we get X = x/6, n = 5

The general term in the binomial expansion of $(1 + x)^{-n}$ is

$$T_{r+1} = (-1)^n\ {}^{(n+r-1)}C_r \cdot x^r$$

Put r = 8

$$T_9 = (-1)^8\ {}^{(5+8-1)}C_8 \cdot x^8 = {}^{12}C_8\left(\frac{x}{6}\right)^8$$

From (1), the 9th term of $\left(2+\frac{x}{3}\right)^{-5}$ is

$$= 2^{-5}\ {}^{13}C_8\left(\frac{x}{6}\right) = \frac{495}{32}\cdot\left(\frac{x}{6}\right)^8$$

ii) 10th term of $\left(1-\frac{3x}{4}\right)^{4/5}$

Sol. Compare $\left(1-\frac{3x}{4}\right)^{4/5}$ with $(1 - x)^{p/q}$, we get $x = \frac{3x}{4}, p = 4, q = 5, \frac{x}{q} = \frac{3x}{20}$.

The general term in $(1 - x)^{p/q}$ is

$$T_{r+1} = \frac{(-1)^4\,[p(p-q)(p-2q)...p-(r-1)q]}{r!}\left(\frac{x}{q}\right)^r$$

Put r = 9

$$T_{10} = \frac{(-1)^9\,[4(4-5)(4-10)...(4-40)]}{9!}\left(\frac{3x}{20}\right)^9$$

$$= -4\,\frac{(-10(-6)(-11)(-16)(-21)\,(-26)(-31)(-36)}{9!}\left(\frac{3x}{20}\right)^9$$

$$= \frac{-4\times1\times6\times11\times16\times21\times26\times31\times36}{9!}\left(\frac{3x}{20}\right)^9$$

iii) 8^{th} term of $\left(1-\frac{5x}{2}\right)^{-3/5}$

Sol. Compare $\left(1-\frac{5x}{2}\right)^{-3/5}$ with $(1 - x)^{-p/q}$, we get $X=\frac{5x}{2}, p=3, q=5, \frac{x}{q}=\frac{\frac{5x}{2}}{5}=\frac{x}{2}$.

The general term in $(1 - x)^{-p/q}$ is

$$T_{r+1}=\frac{[p(p+q)(p+2q)...p+(r-1)q]}{r!}\left(\frac{x}{q}\right)^r$$

Put r = 7

$$T_8=\frac{(3)(3+5)(3+2\times5)....[3+(7-1)5]}{7!}\left(\frac{x}{2}\right)^7$$

$$=\frac{(3\cdot8\cdot13\cdot18\cdot23\cdot28\cdot33)}{7!}\left(\frac{x}{2}\right)^7$$

iv) 6^{th} term of $\left(3+\frac{2x}{3}\right)^{3/2}$

Sol. $\left(3+\frac{2x}{3}\right)^{3/2}=\left[3\left(1+\frac{2x}{9}\right)\right]^{3/2}$

$$=3^{3/2}\left(1+\frac{2x}{9}\right)^{3/2} ...(1)$$

Compare $\left(1+\frac{2x}{9}\right)^{3/2}$ with $(1 + x)^{p/q}$, we get

$$X=\frac{2x}{9}, p=3, q=2 \Rightarrow \frac{x}{q}=\frac{(2x/9)}{2}=\frac{x}{9}$$

The general term of $(1 + x)^{p/q}$ is

$$T_{r+1}=\frac{[p(p-q)(p-2q)...p-(r-1)q]}{r!}\left(\frac{x}{q}\right)^r$$

Put r = 5, we get

$$T_6 = \frac{(3)(3-2)(3-2\times2)(3-3\times2)(3-4\times2)}{5!}\left(\frac{x}{9}\right)^5$$

$$= \frac{(3)(1)(-1)(-3)(-5)}{5!}\left(\frac{x}{9}\right)^5 = -\frac{3}{8}\left(\frac{x}{9}\right)^5$$

From (1), the 6th term of $\left(3+\frac{2x}{3}\right)^{3/2}$ is $= 3^{3/2}\left(-\frac{3}{8}\right)\left(\frac{x}{9}\right)^5 = -\frac{9\sqrt{3}}{8}\left(\frac{x}{9}\right)^5$

16. Write the first 3 terms in the expansion of

(i) $\left(1+\frac{x}{2}\right)^{-5}$**,** **(ii) $(3 + 4x)^{-2/3}$,** **(iii) $(4 - 5x)^{-1/2}$**

i) $\left(1+\frac{x}{2}\right)^{-5}$

Sol. We have

$$(1+X)^{-n} = 1 - nX + \frac{(n)(n+1)}{1\cdot 2}(X)^2 + \ldots$$

$$\therefore \left(1+\frac{x}{2}\right)^5 = 1 - \frac{5x}{2} + \frac{(5)(6)}{1\cdot 2}\left(\frac{x}{2}\right)^2 - \ldots$$

$$= 1 - \frac{5x}{2} + \frac{15}{4}x^2 - \ldots$$

$\therefore$ The first terms in the expansion of

$\left(1+\frac{x}{2}\right)^{-5}$ are $1, \frac{-5x}{2}, \frac{15}{4}x^2$

ii) $(3 + 4x)^{-2/3}$

Sol. $(3+4x)^{-2/3} = \left[3\left(1+\frac{4}{3}\right)\right]^{-2/3}$

$$= 3^{-2/3}\left(1+\frac{4}{3}x\right)^{-2/3} \quad \ldots(1)$$

We have

$$(1+X)^{-p/q} = 1 - \frac{p}{1}\frac{x}{q} + \frac{(p)(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 - \ldots$$

$$\therefore \left(1+\frac{4x}{3}\right)^{-2/3} = 1 - \frac{2}{1}\cdot\frac{4x}{9} + \frac{2\cdot 5}{1\cdot 2}\left(\frac{4x}{9}\right)^2 - \ldots$$

$\therefore$ From (1), the first 3 terms of $(3 + 4x)^{-2/3}$ is

$$3^{-2/3}\left\{1 - \frac{8x}{9} + \frac{80}{81}x^2 - \ldots..\right\}$$

i.e. $3^{-2/3}, -3^{-2/3-2}(8x), 3^{-2/3-4}(80x^2)$

$$\Rightarrow 3^{-2/3} - 3^{-8/3}(8x), 3^{-14/3}(80x^2)$$

iv) $(4 - 5x)^{-1/2}$

Sol. $(4-5x)^{-1/2} = \left[4\left(1-\frac{5}{4}x\right)\right]^{-1/2}$

$$= 4^{-1/2}\left(1-\frac{5}{4}x\right)^{-1/2} \quad \ldots(1)$$

We have

$$(1-X)^{-p/q} = 1 + \frac{p}{1}\frac{x}{q} + \frac{(p)(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \ldots$$

Here $p = 1$, $q = 2$, $X = \frac{5}{4}x \Rightarrow \frac{x}{q} = \frac{5}{8}x$

$$\therefore \left(1-\frac{5}{4}x\right)^{-1/2} = 1 + \frac{1}{1}\left(\frac{5x}{8}\right) + \frac{1\cdot 3}{1\cdot 2}\left(\frac{5x}{8}\right)^2 + \ldots$$

$$= 1 + \frac{5x}{8} + \frac{75}{128}x^2 + \ldots$$

From (1),

$$(4-5x)^{-1/2}=2^{-1/2}\left(1+\frac{5x}{8}+\frac{75}{128}x^2+...\right)$$

$\therefore$ The first 3 terms of $(4-5x)^{-1/2}$ are:

$$\frac{1}{2},\frac{5x}{16},\frac{75}{256}x^2$$

17. Write the general term of

(i) $\left(3+\frac{x}{2}\right)^{-2/3}$ **(ii)** $\left(2+\frac{3x}{4}\right)^{4/5}$

(iii) $(1-4x)^{-3}$ **(iv) $(2-3x)^{-1/3}$**

Sol.i) $\left(3+\frac{x}{2}\right)^{-2/3}$

$$\left(3+\frac{x}{2}\right)^{-2/3}=\left[3\left(1+\frac{x}{6}\right)\right]^{-2/3}$$

$$=3^{-2/3}\left(1+\frac{x}{6}\right)^{-2/3} \quad ...(1)$$

The general term of $(1+x)^{-p/q}$

$$T_{r+1}=(-1)^r$$

$$\left\{\frac{(p)(p+q)(p+2q)....(p)+(r-1)q}{(r)!}\left(\frac{x}{q}\right)^r\right\}$$

Here p = 2, q = 3, $X=\frac{x}{6}\Rightarrow\frac{x}{q}=\frac{\left(\frac{x}{6}\right)}{3}=\frac{x}{18}$

$\therefore$ T_{r+1} of $\left(3+\frac{x}{2}\right)^{-2/3}$ is

$$T_{r+1} = (3)^{-2/3}(-1)^r$$

$$\left[\frac{(2)(2+3)(2+2\cdot 3)+...[2+(r-1)3]}{r!}\left(\frac{x}{18}\right)^r\right]$$

$$\frac{1}{\sqrt{27}}\left\{\frac{(-1)^r(2)(5)(8)...(3r-1)}{r!}\left(\frac{x}{18}\right)^r\right\}$$

ii) $\left(2+\frac{3x}{4}\right)^{4/5}$

Sol. $\left(2+\frac{3x}{4}\right)^{4/5} = \left[2\left(1+\frac{3x}{8}\right)\right]^{4/5}$

$$= 2^{4/5}\left(1+\frac{3x}{8}\right)^{4/5} \qquad \ldots(1)$$

T_{r+1} of $(1 + X)^{p/q}$ is

$$T_{r+1} = \frac{[p(p-q)(p-2q)...(p-(r-1)q)]}{r!}\left(\frac{X}{q}\right)^r$$

Here p = 4, q = 5,

$$X = \frac{3x}{8}, \frac{x}{q} = \frac{\left(\frac{3x}{8}\right)}{5} = \frac{3x}{40}$$

$\therefore$ T_{r+1} of $\left(1+\frac{3x}{8}\right)^{4/5}$ is

$$T_{r+1} = \frac{(4)(4-5)(4-2\times 5)...(4-(r-1)5)}{r!}\left(\frac{3x}{40}\right)^r$$

$$= \frac{4(-1)(-6).....(-5r+9)}{r!}\left(\frac{3x}{40}\right)^r$$

$$= (-1)^{r-1}\frac{(4)(1)(6)...(5r-9)}{r!}\left(\frac{3x}{40}\right)^r$$

$\therefore$ The general term of $\left(2+\frac{3x}{4}\right)^{4/5}$ is

$$2^{4/5}\left[(-1)^{r-1}\frac{4\cdot 1\cdot 6...(5r-9)}{r!}\right]\left(\frac{3x}{40}\right)^{r}$$

iii) $(1-4x)^{-3}$

Sol. $(1-4x)^{-3}=(1-X)^{-n}$, here X = 4x, n = 3.

The general term of $(1-X)^{-n}$ is

$$T_{r+1}={}^{n+r-1}C_r\cdot X^r$$
$$={}^{(3+r-1)}C_r(4x)^r$$
$$={}^{(r+2)}C_r(4x)^r$$

$\therefore$ General term of $(1-4x)^{-3}$ is

$$T_{r+1}={}^{(r+2)}C_r(4x)^r$$

iv) $(2-3x)^{-1/3}$

Sol. $(2-3x)^{-1/3}=\left[2\left(1-\frac{3}{2}x\right)\right]^{-1/3}$

$$=2^{-1/3}\left(1-\frac{3}{2}x\right)^{-1/3}$$

General term of $(1-x)^{-p/q}$

$$T_{r+1}=\frac{(p)(p+q)(p+2q)....(p)+(r-1)q}{(r)!}\left(\frac{x}{q}\right)^{r}$$

Here p = 1, q = 3, $X=\frac{3}{2}x\Rightarrow\frac{x}{q}=\frac{\frac{3}{2}x}{3}=\frac{x}{2}$

$\therefore$ General term of $(2-3x)^{-1/3}$ is

$$T_{r+1} = 2^{-1/3}\left[\frac{(1)(1+3)(1+2\cdot 3)\ldots[1+(r-1)3]}{r!}\left(\frac{x}{2}\right)^r\right]$$

$$= \frac{1}{\sqrt[3]{2}}\left[\frac{(1)(4)(7)\ldots(3r-2)}{r!}\right]\left(\frac{x}{2}\right)^2$$

Short Answer Questions

1. Find the coefficient of

i) x^{-6} in $\left(3x-\frac{4}{x}\right)^{10}$

ii) x^{11} in $\left(2x^2+\frac{3}{x^3}\right)^{13}$

iii) x^2 in $\left(7x^3-\frac{2}{x^2}\right)^9$

iv) x^{-7} in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$

i) x^{-6} in $\left(3x-\frac{4}{x}\right)^{10}$

Sol. The general term in $\left(3x-\frac{4}{x}\right)^{10}$ is

$$T_{r+1} = (-1)^r\, {}^{10}C_r (3x)^{10-r}\left(\frac{4}{x}\right)^r$$

$$= (-1)^r\, {}^{10}C_r 3^{10-r}(4)^r x^{10-r-r}$$

$$= (-1)^r\, {}^{10}C_r 3^{10-r}(4)^r x^{10-2r} \quad \ldots(1)$$

For coefficient of x^{-6}, put 10 – 2r = –6

$\Rightarrow$ 2r = 10+6 = 16 $\Rightarrow$ r = 8

Put r = 8 in (1)

$$T_{8+1} = (-1)^8\,{}^{10}C_8\,3^{10-8}(4)^8 x^{10-16} = {}^{10}C_8\,3^2 4^8 x^{-6}$$

$\therefore$ Coefficient of x^{-6} in $\left(3x - \frac{4}{x}\right)^{10}$ is

$$\begin{aligned}{}^{10}C_8\,3^2 4^8 &= {}^{10}C_2\,3^2 4^8\\ &= \frac{10\times 9}{1\times 2}\times 9\times 4^8 = 405\times 4^8\end{aligned}$$

ii) x^{11} in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$

Sol. The general term in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$ is:

$$\begin{aligned}T_{r+1} &= {}^{13}C_r(2x^2)^{13-r}\left(\frac{3}{x^3}\right)^r\\ &= {}^{13}C_r(2)^{13-r}3^r x^{26-2r}x^{-3r}\\ &= {}^{13}C_r(2)^{13-r}(3)^r x^{26-5r} \qquad \ldots(1)\end{aligned}$$

For coefficient of x^{11}, put 26 – 5r = 11

$\Rightarrow$ 5r = 15 $\Rightarrow$ r = 3

Put r = 3 in (1)

$$T_{3+1} = {}^{13}C_3(2)^{10}(3)^3 x^{26-15}$$

$$T_4 = \frac{13\times 12\times 11}{1\times 2\times 3}\cdot 2^{10}\cdot 3^3\cdot x^{11}$$

$\therefore$ Coefficient of x^{11} in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$ is: $(286)(2^{10})(3^3)$

iii) x^2 in $\left(7x^3-\frac{2}{x^2}\right)^9$ **Ans.** Coefficient of x^2 in $\left(7x^3-\frac{2}{x^2}\right)^9$ is $-126\times 7^4\times 2^5$.

iv) x^{-7} in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$

Sol. The general term in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$ is

$$T_{r+1}=(-1)^r\cdot {}^7C_r\left(\frac{2x^2}{3}\right)^{7-r}\left(\frac{5}{4x^5}\right)^r$$

$$=(-1)^r\cdot {}^7C_r\left(\frac{2}{3}\right)^{7-r}\left(\frac{5}{4}\right)^r x^{14-2r}x^{-5r}$$

$$\therefore T_{r+1}=(-1)^r\,{}^7C_r\left(\frac{2}{3}\right)^{7-r}\left(\frac{5}{4}\right)^r x^{14-7r}\quad ...(1)$$

For coefficient of x^{-7}, put $14-7r=-7$

$\Rightarrow 7r=21\Rightarrow r=3$

Put $r = 3$ in equation (1)

$$T_{3+1}=(-1)^3\,{}^7C_3\left(\frac{2}{3}\right)^4\left(\frac{5}{4}\right)^3 x^{14-21}$$

$$=\frac{-7\times 6\times 5}{1\times 2\times 3}\left(\frac{2}{3}\right)^4\left(\frac{5}{4}\right)^3 x^{-7}$$

$\therefore$ Coefficient of x^{-7} in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$ is:

$$=-35\times\frac{1}{3^4}\cdot\frac{5^3}{2^2}=\frac{-4375}{324}$$

2. Find the term independent of x in the expansion of

(i) $\left(\frac{x^{1/2}}{3}-\frac{4}{x^2}\right)^{10}$ **(ii)** $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$

(iii) $\left(4x^3+\frac{7}{x^2}\right)^{14}$ **(iv)** $\left(\frac{2x^2}{5}+\frac{15}{4x}\right)^{9}$

i) $\left(\frac{x^{1/2}}{3}-\frac{4}{x^2}\right)^{10}$

Sol. The general term in $\left(\frac{x^{1/2}}{3}-\frac{4}{x^2}\right)^{10}$ is

$$T_{r+1}=(-1)^r\,{}^{10}C_r\left(\frac{x^{1/2}}{3}\right)^{10-r}\left(\frac{4}{x^2}\right)^r$$

$$=(-1)^r\,{}^{10}C_r\left(\frac{1}{3}\right)^{10-r}(4)^r\cdot x^{5-\frac{r}{2}}\cdot x^{-2r}$$

$$=(-1)^r\,{}^{10}C_r\left(\frac{1}{3}\right)^{10-r}(4)^r\cdot x^{5-\frac{r}{2}-2r}$$

$$=(-1)^r\,{}^{10}C_r\left(\frac{1}{3}\right)^{10-r}(4)^r\cdot x^{\frac{10-5r}{2}}\quad\ldots(1)$$

For the term independent of x,

Put $\frac{10-5r}{2}=0\Rightarrow 5r=10\Rightarrow r=2$

Put r = 2 in eq.(1)

$$T_{2+1}=(-1)^2\,{}^{10}C_2\left(\frac{1}{3}\right)^8 4^2\cdot x^0$$

$$T_3=\frac{80}{729}$$

ii) $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$

Sol. The general term in $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$ is

$$T_{r+1} = {}^{25}C_r\left(\frac{3}{\sqrt[3]{x}}\right)^{25-r}(5\sqrt{x})^r$$

$$= {}^{25}C_r(3)^{25-r}(5)^r \cdot x^{-1/3(25-r)}x^{r/2}$$

$$= {}^{25}C_r(3)^{25-r}(5)^r \cdot x^{-\frac{25}{3}+\frac{r}{3}+\frac{r}{2}}$$

$$= {}^{25}C_r(3)^{25-r}(5)^r \cdot x^{-\frac{50+2r+3r}{6}} \quad ...(1)$$

For term independent of x, put

$$\frac{-50+5r}{6}=0 \Rightarrow 5r=50 \Rightarrow r=10$$

Put r = 10 in equation (1),

$$T_{10+1} = {}^{25}C_{10}(3)^{15}(5)^{10}x^0$$

i.e. $T_{11} = {}^{25}C_{10}(3)^{15}(5)^{10}$

iii) $\left(4x^3+\frac{7}{x^2}\right)^{14}$

Sol. The general term in $\left(4x^3+\frac{7}{x^2}\right)^{14}$ is

$$T_{r+1} = {}^{14}C_r(4x^3)^{14-r}\left(\frac{7}{x^2}\right)^r$$

$$= {}^{14}C_r(4)^{14-r}(7)^r x^{42-3r}x^{-2r}$$

$$= {}^{14}C_r(4)^{14-r}(7)^r x^{42-5r} \quad ...(1)$$

For term independent of x,

Put 4x – 5r = 0 $\Rightarrow$ r = 42/5 which is not an integer.

Hence term independent of x in the given expansion does not exist.

iv) $\left(\frac{2x^2}{5}+\frac{15}{4x}\right)^9$

Ans.

$$T_{6+1} = {}^9C_6\left(\frac{2}{5}\right)^3\left(\frac{15}{4}\right)^6 x^0 = {}^9C_6\cdot\frac{2^3}{5^3}\cdot\frac{3^6\times 5^6}{4^6}$$

$$=\frac{9\times 8\times 7}{1\times 2\times 3}\cdot\frac{3^6\times 5^6}{4^6}=\frac{3^7\times 5^3\times 7}{2^7}$$

3. Find the middle term(s) in the expansion of

(i) $\left(\frac{3x}{7}-2y\right)^{10}$ **(ii)** $\left(4a+\frac{3}{2}b\right)^{11}$

(iii) $(4x^2+5x^3)^{17}$ **(iv)** $\left(\frac{3}{a^3}+5a^4\right)^{20}$

Sol. The middle term in $(x + a)^n$ when n is even is $T_{\left(\frac{n+1}{2}\right)}$, when n is odd, we have two middle terms,

i.e. $T_{\left(\frac{n+1}{2}\right)}$ and $T_{\left(\frac{n+3}{2}\right)}$.

i) $\left(\frac{3x}{7}-2y\right)^{10}$

Sol. n = 10 is even, we have only one middle term.

i.e. $\frac{10}{2}+1=6^{th}$ term

$\therefore$ T_6 in $\left(\frac{3x}{7}-2y\right)^{10}$ is :

$$={}^{10}C_5\left(\frac{3x}{7}\right)^5(-2y)^5=-({}^{10}C_5)\frac{3^5}{7^5}\cdot 2^5(xy)^5$$

$$=-{}^{10}C_5\left(\frac{6}{7}\right)^5x^5y^5$$

ii) $\left(4a+\frac{3}{2}b\right)^{11}$

Sol.Here n = 11 is an odd integer, we have two middle terms, i.e. $\frac{n+1}{2}$ and $\frac{n+3}{2}$ terms = 7^{th} and 7^{th} terms are middle terms.

T_6 in $\left(4a+\frac{3}{2}b\right)^{11}$ is:

$$={}^{11}C_5(4a)^6\left(\frac{3}{2}b\right)^5={}^{11}C_5(4)^6\frac{3^5}{2^5}a^6b^5$$

$$=\frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5}2^7\cdot3^5\cdot a^6b^5$$

$$=77\times2^8\times3^6\times a^6b^5$$

T_7 in $\left(4a+\frac{3}{2}b\right)^{11}$ is:

$$={}^{11}C_6(4a)^5\left(\frac{3}{2}b\right)^6={}^{11}C_5(4)^5\frac{3^6}{2^6}a^5b^6$$

$$=\frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5}2^4\cdot3^6\cdot a^5b^6$$

$$=77\times2^5\times3^7\times a^5b^6$$

iii) $(4x^2+5x^3)^{17}$ **Try yourself.**

iv) $\left(\frac{3}{a^3}+5a^4\right)^{20}$ **Try your self**

4. Fin the numerically greatest term (s) in the expansion of

i) $(4 + 3x)^{15}$ when $x = \frac{7}{2}$

ii) $(3x + 5y)^{12}$ when $x = \frac{1}{2}$ and $y = \frac{4}{3}$

iii) $(4a - 6b)^{13}$ when a = 3, b = 5

iv) $(3 + 7x)^n$ when $x = \frac{4}{5}$, n = 15

i) $(4 + 3x)^{15}$ when $x = \frac{7}{2}$

Sol. Write $(4 + 3x)^{15} = \left[4\left(1+\frac{3}{4}x\right)\right]^{15}$

$$= 4^{15}\left(1+\frac{3}{4}x\right)^{15} \quad \ldots(1)$$

First we find the numerically greatest term in the expansion of $\left(1+\frac{3}{4}x\right)^{15}$

Write $X = \frac{3}{4}x$ and calculate $\frac{(n+1)|x|}{1+|x|}$

Here $|X| = \left(\frac{3}{4}X\right) = \frac{3}{4}\times\frac{7}{2} = \frac{21}{8}$

Now $\frac{(n+1)|x|}{1+|x|} = \frac{15+1}{1+\frac{21}{8}}\cdot\frac{21}{8}$

$$= \frac{16\times 21}{29} = \frac{336}{29} = 11\frac{17}{29}$$

Its integral part $m = \left[11\frac{17}{29}\right] = 11$

T_{m+1} is the numerically greatest term in the expansion $\left(1+\frac{3}{4}x\right)^{15}$ and

$$T_{m+1} = T_{12} = {}^{15}C_{11}\left(\frac{3}{4}x\right)^{4} = {}^{15}C_{11}\left(\frac{3}{4}\cdot\frac{7}{2}\right)^{11}$$

$\therefore$ Numerically greatest term in $(4 + 3x)^{15}$

$$= 4^{15}\left[{}^{15}C_{11}\left(\frac{21}{8}\right)^{11}\right] = {}^{15}C_{4}\frac{(21)^{11}}{2^{3}}$$

ii) $(3x + 5y)^{12}$ when $x = \frac{1}{2}$ and $y = \frac{4}{3}$

Sol. Write $(3x + 5y)^{12} = \left[3x\left(1+\frac{5y}{3x}\right)\right]^{12}$

$$= 3^{12}x^{12}\left(1+\frac{5}{3}\frac{y}{x}\right)^{12}$$

On comparing $\left(1+\frac{5}{3}\frac{y}{x}\right)^{12}$ with $(1 + x)^n$, we get

$$n = 17,\ x = \frac{5}{3}\cdot\frac{y}{x} = \frac{5}{3}\frac{(4/3)}{(1/2)} = \frac{5}{3}\cdot\frac{8}{3} = \frac{40}{9}$$

$$\text{Now } \frac{(n+1)|x|}{1+|x|} = \frac{(12+1)\left(\frac{40}{9}\right)}{1+\frac{40}{9}}$$

$$= \frac{13\times 40}{49} = \frac{520}{49} = 10\frac{30}{49}$$

Which is not an integer.

$$\therefore\ m = \left[10\frac{30}{49}\right] = 10$$

N.G. term in $\left(1+\frac{5y}{3x}\right)^{12}$ is

$$T_{m+1} = T_{11} = {}^{12}C_{10}\left(\frac{5}{3}\frac{y}{x}\right)^{10} = {}^{12}C_{10}\left(\frac{5}{3}\times\frac{(4/3)}{(1/2)}\right)^{10}$$

$$= {}^{12}C_{10}\left(\frac{5}{3}\times\frac{8}{3}\right)^{10} = {}^{12}C_{10}\left(\frac{40}{9}\right)^{10}$$

$\therefore$ N.G. term in $(3x + 5y)^{12}$ is

$$= 3^{12}\left(\frac{1}{2}\right)^{12}\ {}^{12}C_{10}\left(\frac{40}{9}\right)^{10}$$

$$= {}^{12}C_{10}\frac{3^{12}}{2^{12}}\frac{(2^2)^{10}\times(10)^{10}}{(3^2)^{10}} = {}^{12}C_{10}\left(\frac{3}{2}\right)^{2}\left(\frac{20}{3}\right)^{10}$$

iii) $(4a - 6b)^{13}$ when a = 3, b = 5

Sol. Write $(4a - 6b)^{13} = \left[4a\left(1-\frac{6b}{4a}\right)\right]^{13}$

$$= (4a)^{13}\left(1-\frac{3}{2}\frac{b}{a}\right)^{13}$$

On comparing $\left(1-\frac{3}{2}\frac{b}{a}\right)^{13}$ with $(1 + x)^n$

We get n = 13, $x = \frac{-3}{2}\left(\frac{b}{a}\right)$

$$x = \frac{-3}{2}\times\frac{5}{3} = \frac{-5}{2}$$

Now $\frac{(n+1)|x|}{1+|x|} = \frac{(13+1)\left|\frac{-5}{2}\right|}{1+\left|\frac{-5}{2}\right|} = \frac{14\times\frac{5}{2}}{1+\frac{5}{2}}$

$= \frac{70}{7} = 10$ which is an integer.

Hence we have two numerically greatest terms namely T_{10} and T_{11}.

T_{10} in $\left(1-\frac{3}{2}\frac{b}{a}\right)^{13} = {}^{13}C_9\left|-\frac{3}{2}\cdot\frac{b}{a}\right|^9$

$$= {}^{13}C_9\left(\frac{3}{2}\cdot\frac{5}{3}\right)^9 = {}^{13}C_9\left(\frac{5}{2}\right)^9$$

T_{10} in $(4a - 6b)^{13}$ is

$$= (4a)^{13}\cdot{}^{13}C_9\left(\frac{5}{2}\right)^9 = (4\times 3)^{13}\cdot{}^{13}C_9\left(\frac{5}{2}\right)^9$$

$$= {}^{13}C_9(12)^4(12)^9\left(\frac{5}{2}\right)^9 = {}^{13}C_9(12)^4(30)^9$$

T_{11} in $\left(1-\frac{3}{2}\frac{b}{a}\right)^{13}$ is $= {}^{13}C_{10}\left(\frac{-3}{2}\cdot\frac{b}{a}\right)^{10}$

$$= {}^{13}C_{10}\left(\frac{3}{2}\times\frac{5}{3}\right)^{10} = {}^{13}C_{10}\left(\frac{5}{2}\right)^{10}$$

$\therefore$ N.G. term in $(4a - 6b)^{13}$ is

$$= (4a)^{13}\cdot{}^{13}C_{10}\left(\frac{5}{2}\right)^{10} = (4\times 3)^{13}\cdot{}^{13}C_{10}\left(\frac{5}{2}\right)^{10}$$

$$= (12)^{13}\cdot{}^{13}C_{10}\frac{5^{10}}{2^{10}} = {}^{13}C_{10}(12)^3\cdot(12)^{10}\cdot\frac{5^{10}}{2^{10}}$$

$$= {}^{13}C_{10}(12)^3(30)^{10}$$

iv) $(3 + 7x)^n$ when $x = \frac{4}{5}$, n = 15 Try your self

5. Prove the following

i) $2\cdot C_0 + 5\cdot C_1 + 8\cdot C_2 + ... + (3n+2)\cdot C_n$

$$= (3n+4)\cdot 2^{n-1}$$

Sol. Let S = $2\cdot C_0 + 5\cdot C_1 + 8\cdot C_2 + ... \quad ... + (3n-1)\cdot C_{n-1} + (3n+2)C_n$

$\because C_n = C_0, C_{n-1} = C_1 ...$

$S = (3n+2)C_0 + (3n-1)C_1 + (3n-4)C_2 + ... + 5C_{n-1} + 2\cdot C_n$

$2S = (3n+4)C_0 + (3n+4)C_1 + (3n+4)C_2 + ... + (3n+4)C_n$

Adding $= (3n+4)(C_0 + C_1 + C_2 + ... + C_n) = (3n+4)2^n$

$\therefore S = (3n+4)\cdot 2^{n-1}$

ii) $C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ... = 0$

Sol. 1, 4, 7, 10 … are in A.P.

$T_{n+1} = a + nd = 1 + n(3) = 3n + 1$

$\therefore\ C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ...(n+1)\text{terms}$

$= C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ... + (-1)^n(3n+1)C_n$

$= \sum_{r=0}^{n}(-1)^r(3r+1)C_r = \sum_{r=0}^{n}\left\{(-1)^r(3r)C_r + (-1)^r C_r\right\}$

$= 3\cdot\sum_{r=0}^{n}(-1)^r r\cdot C_r + \sum_{r=0}^{n}(-1)^r\cdot C_r = 3(0) + 0 = 0$

$\therefore C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ... = 0$

iii) $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\frac{C_7}{8}+...=\frac{2^n-1}{n+1}$

Sol. $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\frac{C_7}{8}+.........$

$$=\frac{{}^nC_1}{2}+\frac{{}^nC_3}{4}+\frac{{}^nC_5}{6}+\frac{{}^nC_7}{8}+...$$

$$=\frac{n}{2}+\frac{n(n-1)(n-2)}{4\times 3!}+\frac{n(n-1)(n-2)(n-3)(n-4)}{6\times 5!}+...$$

$$=\frac{1}{n+1}\left[\frac{(n+1)n}{2!}+\frac{(n+1)n(n-1)(n-2)}{4!}+....\frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{6!}+...\right]$$

$$=\frac{1}{n+1}\left[{}^{(n+1)}C_2+{}^{(n+1)}C_4+{}^{(n+1)}C_6+...\right]$$

$$=\frac{1}{n+1}\left[{}^{(n+1)}C_0+{}^{(n+1)}C_2+{}^{(n+1)}C_4+...-{}^{(n+1)}C_0\right]=\frac{1}{n+1}\left[2^n-1\right]=\frac{2^n-1}{n+1}$$

$$\therefore\ \frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\frac{C_7}{8}+...=\frac{2^n-1}{n+1}$$

iv) $C_0+\frac{3}{2}C_1+\frac{9}{3}C_2+\frac{27}{4}C_3+...+\frac{3^n}{n+1}C_n$

$$=\frac{4^{n+1}-1}{3(n+1)}$$

Sol. Let S =

$$C_0+\frac{3}{2}C_1+\frac{3^2}{3}C_2+\frac{3^3}{4}C_3+...+C_n\frac{3^n}{n+1}\ ...(1)$$

$$\Rightarrow 3\,S=\ C_0\cdot 3+\frac{3^2}{2}C_1+\frac{3^3}{3}C_2+\frac{3^4}{4}C_3+...+C_n\frac{3^{n+1}}{n+1}\ ...(2$$

$$\Rightarrow (n+1)3\cdot S$$

$$=(n+1)C_0\cdot 3+(n+1)C_1\cdot\frac{3^2}{2}+(n+1)C_2\cdot\frac{3^3}{3}+(n+1)C_3\cdot\frac{3^4}{3}+...+(n+1)C_n\cdot\frac{3^{n+1}}{n+1}$$

$\Rightarrow (n+1)3\cdot S$

$= {}^{(n+1)}C_1\cdot 3 + {}^{(n+1)}C_2\cdot 3^2 + {}^{(n+1)}C_3\cdot 3^3 + ... + {}^{(n+1)}C_{n+1}\cdot 3^{n+1}$

$= (1+3)^{n+1} - {}^{(n+1)}C_0 = 4^{n+1} - 1$

$\therefore S = \dfrac{4^{n+1}-1}{3(n+1)}$

v) $C_0 + 2\cdot C_1 + 4\cdot C_2 + 8\cdot C_3 + ... + 2^n\cdot C_n = 3^n$

Sol. L.H.S.= $C_0 + 2\cdot C_1 + 4\cdot C_2 + 8\cdot C_3 + ... + 2^n\cdot C_n$

= $C_0 + C_1(2) + C_2(2^2) + C_3(2^3) + ... + C_n(2^n)$

= $(1 + 2)^n = 3^n$

[$(1+x)^n = C_0 + C_1\cdot x + C_2x^2 + ... + C_nx^n$]

6. Using binomial theorem, prove that 50^n – 49n – 1 is divisible by 49^2 for all positive integers n.

Sol. $50^n - 49n - 1 = (49 + 1)^n - 49n - 1$

$= [{}^nC_0(49)^n + {}^nC_1(49)^{n-1} + {}^nC_2(49)^{n-2} + ... + {}^nC_{n-2}(49)^2 + {}^nC_{n-1}(49) + {}^nC_n(1)] - 49n - 1$

$= (49)^n + {}^nC_1(49)^{n-1} + {}^nC_2(49)^{n-2} + ... + {}^nC_{n-2}(49)^2 + (n)(49) + 1 - 49n - 1$

$= 49^2[(49)^{n-2} + {}^nC_1(49)^{n-3} + {}^nC_2(49)^{n-4} + ... + ... + ... + {}^nC_{n-2}]$

$= 49^2$ [a positive integer]

Hence 50^n – 49n – 1 is divisible by 49^2 for all positive integers of n.

7. Using binomial theorem, prove that $5^{4n} + 52n - 1$ is divisible by 676 for all positive integers n.

Sol. $5^{4n} + 52n - 1 = (5^2)^{2n} + 52n - 1$

$= (25)^{2n} + 52n - 1 = (26-1)^{2n} + 52n - 1$

$= [\,{}^{2n}C_0(26)^{2n} - {}^{2n}C_1(26)^{2n-1} + {}^{2n}C_2(26)^{2n-2} - ..... + {}^{2n}C_{2n-2}(26)^2 - {}^{2n}C_{2n-1}(26) + {}^{2n}C_{2n}(1)] + 52n - 1$

$= {}^{2n}C_0(26)^{2n} - {}^{2n}C_1(26)^{2n-1} + {}^{2n}C_2(26)^{2n-2} - ..... + {}^{2n}C_{2n-2} - 2n(26) + 1 + 52n - 1$

$= (26)^2[\,{}^{2n}C_0(26)^{2n-2} - {}^{2n}C_1(26)^{2n-3} + {}^{2n}C_2(26)^{2n-4} + ... + {}^{2n}C_{2n-2}]$

is divisible by $(26)^2 = 676$

$\therefore 5^{4n} + 52n - 1$ is divisible by 676, for all positive integers n.

8. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then prove that

i) $a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$

ii) $a_0 + a_2 + a_4 + ... + a_{2n} = \dfrac{3^n + 1}{2}$

iii) $a_1 + a_3 + a_5 + ... + a_{2n-1} = \dfrac{3^n - 1}{2}$

iv) $a_0 + a_3 + a_6 + a_9 + ... = 3^{n-1}$

Sol. $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Put x = 1,

$\therefore a_0 + a_1 + a_2 + \dots + a_{2n} = (1+1+1)^n = 3^n \dots(1)$

Put x = –1,

$a_0 - a_1 + a_2 - \dots + a_{2n} = (1-1+1)^n = 1 \dots(2)$

i) $a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$

ii)(1) + (2) $\Rightarrow 2(a_0 + a_2 + a_4 + ... + a_{2n}) = 3^n + 1$

$\therefore\ a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n + 1}{2}$

iii) $(1) - (2) \Rightarrow\ 2(a_1 + a_3 + a_5 + ... + a_{2n-1}) = 3^n - 1$

$\therefore\ a_1 + a_3 + a_5 + ... + a_{2n-1} = \frac{3^n - 1}{2}$

iv) Put x = 1

$a_0 + a_1 + a_2 + ... + a_{2n} = 3^n \qquad ...(a)$

Hint: $1 + \omega + \omega^2 = 0\ ;\ \omega^3 = 1$

Put $x = \omega$

$a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + ... + a_{2n}\omega^{2n} = 0 \quad ...(b)$

Put $x = \omega^2$

$a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + ... + a_{2n}\omega^{4n} = 0 \quad ...(c)$

Adding (a), (b), (c)

$3a_0 + a_1(1+\omega+\omega^2) + a_2(1+\omega^2+\omega^4) + a_3(1+\omega^3+\omega^6) + ... + a_{2n}(1+\omega^{2n}+\omega^{4n}) = 3^n$

$\Rightarrow 3a_0 + a_1(0) + a_2(0) + 3a_3 + ... + ... = 3^n$

$\therefore a_0 + a_3 + a_6 + a_9 + ... = \frac{3^n}{3} = 3^{n-1}$

9. If the coefficients of $(2r + 4)^{th}$ term and $(3r + 4)^{th}$ term in the expansion of $(1 + x)^{21}$ are equal, find r.

Sol. T_{2r+4} in $(1 + x)^{21}$ is $= {}^{21}C_{2r+3}(x)^{2r+3}$ …(1)

T_{3r+4} in $(1 + x)^{21}$ is $= {}^{21}C_{3r+3}(x)^{3r+3}$ ...(2)

$\Rightarrow$ Coefficients are equal

$\Rightarrow {}^{21}C_{2r+3} = {}^{21}C_{3r+3}$

$\Rightarrow 21 = (2r + 3) + (3r + 3)$ (or) $2r + 3 = 3r + 3$

$\Rightarrow 5r = 15 \Rightarrow r = 3$ (or) $r = 0$

Hence r = 0, 3.

10. If the coefficients of x^{10} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-10} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$; find the relation between a and b where a and b are real numbers.

Sol. The general term in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r (ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$$

$$= {}^{11}C_r a^{11-b}\left(\frac{1}{b}\right)^r x^{22-2r-r}$$

To find the coefficient of x^{10}, put

$22 - 3r = 10 \Rightarrow 3r = 12 \Rightarrow r = 4$

Hence the coefficient of x^{10} in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is $= {}^{11}C_7 \cdot a^7\left(\frac{1}{b}\right)^4 = {}^{11}C_7 \frac{a^7}{b^4}$ ...(1)

The general term in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r (ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r$$

$$= (-1)^r\, {}^{11}C_r a^{11-r}\left(\frac{1}{b}\right)^r x^{11-r-2r}$$

For the coefficient of x^{-10} put

$11 - 3r = -10 \Rightarrow 3r = 21 \Rightarrow r = 7$

$\therefore$ The coefficient of x^{-10} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is

$$= (-1)^7 \cdot {}^{11}C_7 (a)^4 \left(\frac{1}{b}\right)^7 = (-1)\, {}^{11}C_7 \frac{(a^4)}{b^7} \ldots(2)$$

Given that the coefficients are equal.

Hence from (1) and (2), we get

$${}^{11}C_7 \cdot \frac{a^7}{a^4} = -\, {}^{11}C_7 \cdot \frac{a^4}{b^7}$$

$$\Rightarrow a^3 = \frac{-1}{b^3} \Rightarrow a^3 b^3 = -1 \Rightarrow ab = -1$$

11. If the k^{th} term is the middle term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$, find T_k and T_{k+3}.

Sol. The general term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$ is

$$T_{r+1} = {}^{20}C_r (x^2)^{20-r}\left(\frac{-1}{2x}\right)^r \qquad \ldots (1)$$

$\because$ The given expansion has (20 + 1) = 21 times, $\left(\frac{n}{2}+1\right)^{th}$ term, i.e. $\left(\frac{20}{2}+1\right) = 11^{th}$ term is the only middle term.

$\therefore$ k = 11

Put r = 10 in eq.(1)

$$T_{13+1} = {}^{20}C_{13}(x^2)^7\left(\frac{-1}{2x}\right)^{13} = (-1)\ {}^{20}C_{13}\frac{1}{2^{13}}x$$

12. If the coefficients of (2r + 4)th and (r – 2)nd terms in the expansion of (1 + x)^{18} are equal, find r.

Sol. T_{2r+4} term of (1 + x)^{18} is

$$T_{2r+4} = {}^{18}C_{2r+3}(x)^{2r+3}$$

T_{r-2} term of $(1+x)^{18}$

$$T_{r-2} = {}^{18}C_{r-3}(x)^{r-3}$$

Given that the coefficients of (2r + 4)th term = The coefficient of (r – 2)nd term.

$$\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3}$$
$$\Rightarrow 2r+3 = r-3 \text{ (or) } (2r+3)+(r-3) = 18$$
$$\Rightarrow r = -6 \text{ (or) } 3r = 18 \Rightarrow r = 6$$

13. Find the coefficient of x^{10} in the expansion of $\frac{1+2x}{(1-2x)^2}$.

Sol. $\frac{1+2x}{(1-2x)^2} = (1+2x)(1-2x)^{-2}$

$$= (1+2x)[1+2(2x)+3(2x)^2+4(2x)^3+5(2x)^4+6(2x)^5+7(2x)^6+8(2x)^7+9(2x)^8+10(2x)^9$$
$$+11(2x)^{10}+\ldots+(r+1)(2x)^r+\ldots]$$

$\therefore$ The coefficient of x^{10} in $\frac{1+2x}{(1-2x)^2}$ is

$$= (11)(2)^{10}+10(2)(2^9) = 2^{10}(11+10) = 2\times1^{10}$$

14. Find the coefficient of x^4 in the expansion of $(1 - 4x)^{-3/5}$.

Sol. General term in $(1 - x)^{-p/q}$ is

$$T_{r+1} = \frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 3, q = 5, $\frac{X}{q} = \left(\frac{4x}{5}\right)$

Put r = 4

$$T_{4+1} = \frac{(3)(3+5)(3+2\times5)(3+3\times5)}{1\times2\times3\times4}\left(\frac{4x}{5}\right)^4$$

$\therefore$ Coefficient of x^4 in $(1 - 4x)^{-3/5}$ is

$$\frac{(3)(8)(13)(18)}{1\times2\times3\times4}\left(\frac{4}{5}\right)^4 = \frac{234\times256}{625} = \frac{59904}{625}$$

15. Find the sum of the infinite series

i) $1+\frac{1}{3}+\frac{1\cdot3}{3\cdot6}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9}+...$

Sol. The given series can be written as

$$S = 1+\frac{1}{1}\cdot\frac{1}{3}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+...$$

The series of the right is of the form

$$1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{1\cdot2\cdot3}\left(\frac{x}{q}\right)^3+...$$

Here p = 1, q = 2, $\frac{x}{q} = \frac{1}{3} \Rightarrow x = \frac{2}{3}$

The sum of the given series $S = (1 - x)^{-p/q} = \left(1-\frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2} = \sqrt{3}$

ii) $\frac{3}{4}+\frac{3\cdot 5}{4\cdot 8}+\frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+......$

Sol. Let S = $\frac{3}{4}+\frac{3\cdot 5}{4\cdot 8}+\frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+......$

$$=\frac{3}{1}\cdot\frac{1}{4}+\frac{3\cdot 5}{1\cdot 2}\left(\frac{1}{4}\right)^2+\frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{4}\right)^3+......$$

$$\Rightarrow 1+S=1+\frac{3}{1}\cdot\frac{1}{4}+\frac{3\cdot 5}{1\cdot 2}\left(\frac{1}{4}\right)^2+......$$

Comparing (1 + S) with

$$(1-x)^{-p/q}=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2+...$$

Here p = 3, q = 2, $\frac{x}{p}=\frac{1}{4}\Rightarrow x=\frac{1}{2}$

$$\therefore\ 1+S=(1-x)^{-p/q}=\left(1-\frac{1}{2}\right)^{-3/2}$$

$$=\left(\frac{1}{2}\right)^{-3/2}=2^{3/2}=\sqrt{8}$$

$$\therefore S=2\sqrt{2}-1$$

iii) $1-\frac{4}{5}+\frac{4\cdot 7}{5\cdot 10}-\frac{4\cdot 7\cdot 10}{5\cdot 10\cdot 15}+......$

Sol. Let S = $1-\frac{4}{5}+\frac{4\cdot 7}{5\cdot 10}-\frac{4\cdot 7\cdot 10}{5\cdot 10\cdot 15}+......$

$$=1+\frac{4}{1}\left(-\frac{1}{5}\right)+\frac{4\cdot 7}{1\cdot 2}\left(-\frac{1}{5}\right)^2+\frac{4\cdot 7\cdot 10}{1\cdot 2\cdot 3}\left(-\frac{1}{5}\right)^3+...$$

Comparing S with $(1-x)^{-p/q}=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2+...$

Here p = 4, q = 3, $\frac{x}{q} = -\frac{1}{5} \Rightarrow x = \frac{-3}{5}$

$$\therefore S = (1-x)^{-p/q} = \left(1+\frac{3}{5}\right)^{-4/3} = \left(\frac{8}{5}\right)^{-4/3}$$

$$= \left(\frac{5}{8}\right)^{4/3} = \frac{5^{4/3}}{8^{4/3}} = \frac{\sqrt[3]{5^4}}{2^4} = \frac{\sqrt[3]{625}}{16}$$

$$\therefore 1 - \frac{4}{5} + \frac{4\cdot 7}{5\cdot 10} - \frac{4\cdot 7\cdot 10}{5\cdot 10\cdot 15} + ... = \frac{\sqrt[3]{625}}{16} = \frac{5^{4/3}}{16}$$

iv) $\frac{3}{4\cdot 8} - \frac{3\cdot 5}{4\cdot 8\cdot 12} + \frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16} - ....$

Sol.Let S = $\frac{3}{4\cdot 8} - \frac{3\cdot 5}{4\cdot 8\cdot 12} + \frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16} - ....$

$$= \frac{1\cdot 3}{4\cdot 8} - \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12} + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16} - ....$$

Add $1-\frac{1}{4}$ on both sides,

$$1 - \frac{1}{4} + S = 1 - \frac{1}{4} + \frac{1\cdot 3}{4\cdot 8} - \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12} + ...$$

$$\Rightarrow \frac{3}{4} + S = 1 - \frac{1}{1}\cdot\frac{1}{4} + \frac{1\cdot 3}{1\cdot 2}\left(\frac{1}{4}\right)^2 - \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{1}{4}\right)^3 + ...$$

Here p = 1, q = 2, $\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{1}{2}$

$$= 1 - \frac{p}{1}\frac{x}{q} + \frac{(p)(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 - \frac{(p)(p+q)(p+2q)}{1\cdot 2\cdot 3}\left(\frac{x}{q}\right)^3 + ...$$

$$= (1+x)^{-p/q} = \left(1+\frac{1}{2}\right)^{-1/2} = \left(\frac{3}{2}\right)^{-1/2} = \sqrt{\frac{2}{3}}$$

$$\therefore\ S = \sqrt{\frac{2}{3}} - \frac{3}{4}$$

16. Find an approximate value of the following corrected to 4 decimal places.

(i) $\sqrt[5]{242}$ **(ii)** $\sqrt[7]{127}$ **(iii)** $\sqrt[5]{32.16}$ **(iv)** $\sqrt[7]{199}$

(v) $\sqrt[3]{1002}-\sqrt[3]{998}$

Sol. i) $\sqrt[5]{242} = (243-1)^{1/5} = (243)^{1/5}\cdot\left(1-\frac{1}{243}\right)^{1/5}$

$$=(3^5)^{1/5}\left[1-\frac{1}{5}\cdot\frac{1}{243}+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{1\cdot 2}\left(\frac{1}{243}\right)^2-\ldots\right]$$

$$=3\left\{1-\frac{1}{5}(0.00243)-\frac{2}{25}(0.00243)^2-\ldots\right\}$$

$$\because \frac{1}{243}=\left(\frac{1}{3}\right)^5=(0.3)^5=0.00243$$

$$\simeq 3-\frac{3}{5}(0.00243)-\frac{6}{25}(0.00243)^2-\ldots$$

$$\simeq 3-0.001458-0.000001417176$$

$$\simeq 2.998541$$

ii) $\sqrt[7]{127}$ **Try yourself** **iii)** $\sqrt[5]{32.16}$ **Try yourself** **iv)** $\sqrt{199}$ **Try yourself**

v) $\sqrt[3]{1002}-\sqrt[3]{998}$ **Try yourself**

17. If |x| is so small that x^2 and higher powers of x may be neglected then find the approximate values of the following.

i) $\dfrac{(4+3x)^{1/2}}{(3-2x)^2}$

Sol. $$\frac{(4+3x)^{1/2}}{(3-2x)^2}=\frac{\left[4\left(1+\frac{3}{4}x\right)\right]^{1/2}}{\left[3\left(1-\frac{2}{3}x\right)\right]^2}$$

$$=\frac{2}{9}\left(1+\frac{3}{4}x\right)^{1/2}\left(1-\frac{2}{3}\right)^{-2}$$

$$=\frac{2}{9}\left(1+\frac{1}{2}\cdot\frac{3}{4}x\right)\left(1-(-2)\frac{2}{3}x\right)$$

(After neglecting x^2 and higher powers of x)

$$=\frac{2}{9}\left(1+\frac{3}{8}x\right)\left(1+\frac{4}{3}x\right)=\frac{2}{9}\left(1+\frac{3}{8}x+\frac{4}{3}x\right)$$

(Again by neglecting x^2 term)

$$=\frac{2}{9}\left(1+\frac{41}{24}x\right)=\frac{2}{9}+\frac{41}{108}x$$

$$\therefore\ \frac{(4+3x)^{1/2}}{(3-2x)^2}=\frac{2}{9}+\frac{82}{108}x=\frac{2}{9}+\frac{41}{108}x$$

ii) $\dfrac{\left(1-\dfrac{2x}{3}\right)^{3/2}(32+5x)^{1/5}}{(3-x)^3}$

Sol. $\dfrac{\left(1-\dfrac{2x}{3}\right)^{3/2}(32+5x)^{1/5}}{(3-x)^3}$

$$=\frac{\left(1-\frac{2}{3}x\right)^{3/2}(32)^{1/5}\left(1+\frac{5}{32}x\right)^{1/5}}{3^3\left(1-\frac{x}{3}\right)^3}$$

$$=\frac{2}{27}\left(1-\frac{2x}{3}\right)^{3/2}\left(1+\frac{5}{32}x\right)^{1/5}\left(1-\frac{x}{3}\right)^{-3}$$

$$=\frac{2}{27}\left(1-\frac{3}{2}\cdot\frac{2x}{3}\right)\left(1+\frac{1}{5}\frac{5}{32}x\right)\left(1+3\frac{x}{3}\right)$$

(By neglecting x^2 and higher powers of x)

$$= \frac{2}{27}(1-x)\left(1+\frac{x}{32}\right)(1+x)$$

$$= \frac{2}{27}\left(1-x^2\right)\left(1+\frac{x}{32}\right) = \frac{2}{27}\left(1+\frac{x}{32}\right)$$

iii) $\sqrt{4-x}\left(3-\frac{x}{2}\right)^{-1}$ **Try yourself**

iv) $\frac{\sqrt{4+x}+\sqrt[3]{8+x}}{(1+2x)+(1-2x)^{-1/3}}$ **Try yourself**

v) $\frac{(8+3x)^{2/3}}{(2+3x)\sqrt{4-5x}}$ **Try yourself**

18. Suppose s and t are positive and t is very small when compared to s, then find an approximate value of $\left(\frac{s}{s+t}\right)^{1/3}-\left(\frac{s}{s-t}\right)^{1/3}$**.**

Sol. Since t is very small when compared with s, t/s is very small.

$$\left(\frac{s}{s+t}\right)^{1/3}-\left(\frac{s}{s-t}\right)^{1/3} = \left[\frac{1}{1+\frac{t}{s}}\right]^{1/3}-\left[\frac{1}{1-\frac{t}{s}}\right]^{1/3}$$

$$= \left(1+\frac{t}{s}\right)^{-1/3}-\left(1-\frac{t}{s}\right)^{-1/3}$$

$$= \left(1+\left(-\frac{1}{3}\right)\left(\frac{t}{s}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{1\cdot 2}\left(\frac{t}{s}\right)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}\left(\frac{t}{s}\right)^3+\ldots\right)$$

$$= \left(1-\left(-\frac{1}{3}\right)\left(\frac{t}{s}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{1\cdot 2}\left(\frac{t}{s}\right)^2-\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}\left(\frac{t}{s}\right)^3+\ldots\right)$$

$$= 2\left[-\frac{1}{3}\left(\frac{t}{s}\right)-\frac{1\cdot 4\cdot 7}{27\times 6}\frac{t^3}{s^3}\right] = \frac{-2}{3}\frac{t}{s}-\frac{28}{81}\frac{t^3}{s^3}$$

19. Suppose p, q are positive and p is very small when compared to q. Then find an approximate value of $\left(\frac{q}{q+p}\right)^{1/2}+\left(\frac{q}{q-p}\right)^{1/2}$

Sol. Do it yourself. Same as above.

20. By neglecting x^4 and higher powers of x, find an approximate value of $\sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}$.

Sol. $\sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}$

$$=(64+x^2)^{1/3}-(27+x^2)^{1/3}$$

$$=(64)^{1/3}\left(1+\frac{x^2}{64}\right)^{1/3}-(27)^{1/3}\left(1+\frac{x^2}{27}\right)^{1/3}$$

$$=4\left(1+\frac{x^2}{192}\right)-3\left(1+\frac{x^2}{81}\right)$$

(By neglecting x^4 and higher powers of x)

$$=4+\frac{x^2}{48}-3-\frac{x^2}{27}=1+\frac{(27-48)}{48\times 27}x^2$$

$$=1+\left(\frac{-21}{48\times 27}\right)x^2=1-\frac{7x^2}{432}=1-\frac{7}{432}x^2$$

$$\therefore \sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}=1-\frac{7}{432}x^2$$

21. Expand $3\sqrt{3}$ in increasing powers of 2/3.

Sol. $3\sqrt{3}=3^{3/2}=\left(\frac{1}{3}\right)^{-3/2}=\left(1-\frac{2}{3}\right)^{-3/2}$

$$=1+\frac{\frac{3}{2}}{1}\cdot\left(\frac{2}{3}\right)+\frac{\frac{3}{2}\left(\frac{3}{2}+1\right)}{1\cdot 2}\left(\frac{2}{3}\right)^2+\ldots\ldots+\frac{\frac{3}{2}\left(\frac{3}{2}+1\right)\ldots\ldots\left(\frac{3}{2}+r-1\right)}{(1\cdot 2\cdot 3\ldots.r)2^r}\left(\frac{2}{3}\right)^r+\ldots\ldots$$

$$=1+\frac{3}{1\cdot 2}\left(\frac{2}{3}\right)+\frac{3\cdot 5}{(1\cdot 2)2^2}\left(\frac{2}{3}\right)^2+\ldots\ldots+\frac{3\cdot 5\ldots(2r+1)}{(1\cdot 2\cdot\ldots r)2^r}\left(\frac{2}{3}\right)^r+\ldots\ldots$$

$$=1+3\left(\frac{1}{3}\right)+\frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+......+\frac{3\cdot 5\cdot 7...(2r+1)}{r!}\left(\frac{1}{3}\right)^r+...$$

22. Prove that $2\cdot C_0+7\cdot C_1+12\cdot C_2+...+(5n+2)C_n=(5n+4)2^{n-1}$

Sol. First method:

The coefficients of C_0, C_1, C_2, …… C_n are in A.P. with first term a = 2, C.d. (d) = 5

$$\because a\cdot C_0+(a+d)C_1+(a+2d)C_2+...+(a+(n-1)d)C_{n-1}+(a+nd)C_n$$
$$=(2a+nd)2^{n-1}$$
$$=(2\times 2+n\cdot 5)\cdot 2^{n-1}=(4+5n)2^{n-1}$$

Second method:

General term in L.H.S.

i.e. $T_{r+1}=(5r+2)C_n$

23. Prove that

i) $C_0+3C_1+3^2C_2+...+3^nC_n=4^n$

ii) $\frac{C_1}{C_0}+2\cdot\frac{C_2}{C_1}+3\cdot\frac{C_3}{C_2}+...+n\frac{C_n}{C_{n-1}}=\frac{n(n+1)}{2}$

Sol. (i) We have

$$(1+x)^n=C_0+C_1x+C_2x^2+...+C_nx^n$$

Put x = 3, we get

$$(1+3)^n=C_0+C_1\cdot 3+C_2 3^2+...+C_n 3^n$$
$$\therefore C_0+3C_1+3^2C_2+...+3^nC_n=4^n$$

(ii) $\frac{C_1}{C_0}+2\cdot\frac{C_2}{C_1}+3\cdot\frac{C_3}{C_2}+\ldots+n\frac{C_n}{C_{n-1}}$

$$=\frac{{}^nC_1}{{}^nC_0}+2\left(\frac{{}^nC_2}{{}^nC_1}\right)+3\left(\frac{{}^nC_3}{{}^nC_2}\right)+\ldots+n\left(\frac{{}^nC_n}{{}^nC_1}\right)$$

$$=\frac{n}{1}+2\frac{n-1}{2}+3\frac{n-2}{3}+\ldots+n\frac{1}{n}$$

$$=n+(n-1)+(n-2)+\ldots+3+2+1$$

$$=1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

24. For n = 0, 1, 2, 3, … n, prove that $C_0\cdot C_r+\ C_1\cdot C_{r+1}+C_2\cdot C_{r+2}+\ldots+C_{n-r}\cdot C_n={}^{2n}C_{n+r}$ **and hence deduce that**

i) $C_0^2+C_1^2+C_2^2+\ldots+C_n^2={}^{2n}C_n$

ii) $C_0\cdot C_1+C_1\cdot C_2+C_2\cdot C_3+\ldots+C_{n-1}C_n={}^{2n}C_{n+1}$

Sol. We know that

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n\ \ldots(1)$$

On replacing x by 1/x in the above equation,

$$\left(1+\frac{1}{x}\right)^n=C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\ldots+\frac{C_n}{x^n}\ldots(2)$$

From (1) and (2)

$$\left(1+\frac{1}{x}\right)^n(1+x)^n=\left(C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\ldots+\frac{C_n}{x^n}\right)$$
$$(C_0+C_1x+C_2x^2+\ldots+C_nx^n)\ldots(3)$$

The coefficient of x^r in R.H.S. of (3)

$$=C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots+C_{n-r}C_n$$

The coefficient of x^r in L.H.S. of (3)

= the coefficient of x^r in $\frac{(1+x)^{2n}}{x^n}$

= the coefficient of x^{n+r} is $(1 + x)^{2n}$

$= {}^{2n}C_{n+r}$

From (3) and (4), we get

$$C_0 \cdot C_1 + C_1 \cdot C_2 + C_2 \cdot C_3 + ... + C_{n-1}C_n = {}^{2n}C_{n+1}$$

i) On putting r = 0 in (i), we get

$$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$$

ii) On substituting r = 1 in (i) we get

$$C_0 \cdot C_1 + C_1 \cdot C_2 + C_2 \cdot C_3 + ... + C_{n-1}C_n = {}^{2n}C_{n+1}$$

$$3 \cdot C_0^2 + 7 \cdot C_1^2 + 11 \cdot C_2^2 + ... + (4n+3)C_n^2$$
$$= (2n+3)\ {}^{2n}C_n$$

Sol. Let S = $3 \cdot C_0^2 + 7 \cdot C_1^2 + 11 \cdot C_2^2 + ... + (4n-1)C_{n-1}^2 + (4n+3)C_n^2$ ...(1)

$\because$ $C_0 = C_n$, $C_1 = C_{n-1}$ etc., on writing the terms of R.H.S. of (1) in the reverse order, we get

$$S = (4n+3)C_0^2 + (4n-1)C_1^2 + ... + 7C_{n-1}^2 + 3C_n^2 \quad ......(2)$$

Add (1) and (2)

$$2S = (4n+6)C_0^2 + (4n+6)C_1^2 + ... + (4n+6)C_n^2$$

$$\Rightarrow 2S = (4n+6)(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2)$$
$$= 2(2n+3)\ {}^{2n}C_n$$

$$\therefore S = (2n+3)\ {}^{2n}C_n$$

25. Find the numerically greatest term(s) in the expansion of

i) $(2 + 3x)^{10}$ when $x = \frac{11}{8}$

Sol. Write $(2+3x)^{10} = \left[2\left(1+\frac{3}{2}x\right)^{10}\right] = 2^{10}\left(1+\frac{3x}{2}\right)^{10}$

First find N.G. term in $\left(1+\frac{3x}{2}\right)^{10}$

Let $X = \frac{3x}{2} = \frac{3\times\frac{11}{8}}{2} = \frac{33}{16}$

Now consider

$$\frac{(n+1)|x|}{1+|x|} = \frac{(10+1)\left(\frac{33}{16}\right)}{\frac{33}{16}+1} = \frac{11\times 33}{48} = \frac{363}{48}$$

Its integral part $m = \left[\frac{363}{48}\right] = 7$

$\therefore$ T_{m+1} is the numerically greatest term in

$$\left(1+\frac{3x}{2}\right)^{10}$$

i.e. $T_{7+1} = T_8 = {}^{10}C_7\left(\frac{3x}{2}\right)^7$

$= {}^{10}C_7\left(\frac{3}{2}\times\frac{11}{8}\right)^7 = {}^{10}C_7\left(\frac{33}{16}\right)^7$

$\therefore$ N.G. term in the expansion of $(2 + 3x)^{10}$ is $= 2^{10}\cdot{}^{10}C_7\left(\frac{33}{16}\right)^7$.

ii) $(3x - 4y)^{14}$ when x= 8, y = 3.

Sol. $(3x-4y)^{14} = \left(3x\left(1-\frac{4y}{3x}\right)\right)^{14}$

$$= (3x)^{14}\left(1-\frac{4y}{3x}\right)^{14}$$

Write $X = \frac{-4y}{3x} = -\left(\frac{4\times 3}{3\times 8}\right) = -\frac{1}{2}$

$|X| = \frac{1}{2}$

Now $\frac{(n+1)|X|}{1+|X|} = \frac{(14+1)\frac{1}{2}}{1+\frac{1}{2}} = 5$, an integer.

Here $|T_5| = |T_6|$ are N.G. terms.

T_5 in the expansion of $\left(1-\frac{4y}{3x}\right)^{14}$ is

$$T_5 = {}^{14}C_4\left(\frac{-4y}{3x}\right)^4 = {}^{14}C_4\left(\frac{1}{2}\right)^4$$

and $T_6 = {}^{14}C_5\left(\frac{-4y}{3x}\right)^5 = -{}^{14}C_5\left(\frac{1}{2}\right)^5$

Here N.G. terms are T_5 and T_6. They are

$$T_5 = {}^{14}C_4\left(\frac{1}{2}\right)^4 (24)^{14}$$

$$T_6 = -{}^{14}C_5\left(\frac{1}{2}\right)^5 (24)^{14}$$

But $|T_5| = |T_6|$

26. Prove that $6^{2n} - 35n - 1$ is divisible by 1225 for all natural numbers of n.

Sol. $6^{2n} - 35n - 1 = (36)^n - 35n - 1$

$$= (35+1)^n - 35n - 1$$

$$= \left[(35)^n + {}^nC_1(35)^{n-1} + {}^nC_2(35)^{n-2} + ...+ {}^nC_{n-2}(35)^2 + {}^nC_{n-1}(35)^1 + {}^nC_n\right] - 35n - 1$$

$$= (35)^n + {}^nC_1(35)^{n-1} + {}^nC_2(35)^{n-2} + ...+ {}^nC_{n-2}(35)^2$$

$$= (35)^2\left[\begin{matrix}(35)^{n-2} + {}^nC_1(35)^{n-3} + {}^nC_2(35)^{n-4} \\ +...+ {}^nC_{n-2}\end{matrix}\right]$$

= 1225 (k), for same integer k.

Hence $6^{2n} - 35n - 1$ is divisible by 1225 for all integral values of n.

27. Find the number of terms with non-zero coefficients in $(4x - 7y)^{49} + (4x + 7y)^{49}$.

Sol: We know that

$$(4x - 7y)^{49} = {}^{40}C_0(4x)^{49} - {}^{49}C_1(4x)^{48}(7y) + {}^{49}C_2(4x)^{47}(7y)^2 - {}^{49}C_3(4x)^{46}(7y)^3 + ... - {}^{409}C_{49}(7y)^{49} ...(1)$$

$$(4x + 7y)^{49} = {}^{40}C_0(4x)^{49} + {}^{49}C_1(4x)^{48}(7y) + {}^{49}C_2(4x)^{47}(7y)^2 + {}^{49}C_3(4x)^{46}(7y)^3 + ... + {}^{409}C_{49}(7y)^{49} ...(2)$$

(1) + (2) $\Rightarrow$

$(4x - 7y)^{49} + (4x + 7y)^{49} = 2[{}^{49}C_0(4x)^{49} + {}^{49}C_2(4x)^{47}(7y)^2 + {}^{49}C_4(4x)^{45}(7y)^4 + ... + {}^{49}C_{48}(7y)^{48}]$ which contains 25 non-zero coefficients.

28. Find the sum of last 20 coefficients in the expansion of $(1 + x)^{39}$.

Sol: The last 20 coefficients in the expansion of $(1 - x)^{39}$ are ${}^{30}C_{20}, {}^{39}C_{21}, ..., {}^{39}C_{39}$.

We know that

$$\therefore {}^{39}C_0 + {}^{39}C_1 + {}^{39}C_2 + ... + {}^{39}C_{19} + {}^{39}C_{20} + ... + {}^{39}C_{39} = 2^{39}$$

$$\Rightarrow {}^{39}C_{39} + {}^{39}C_{38} + {}^{39}C_{37} + ... + {}^{39}C_{20} + {}^{39}C_{20} + {}^{39}C_{21} + ... + {}^{39}C_{39} = 2^{39}$$

$$(\because {}^nC_r = {}^nC_{n-r})$$

$$\Rightarrow 2[{}^{39}C_{20} + {}^{39}C_{21} + {}^{39}C_{22} + ... + {}^{39}C_{39}] = 2^{39} \Rightarrow [{}^{39}C_{20} + {}^{39}C_{21} + {}^{39}C_{22} + ... + {}^{39}C_{39}] = 2^{38}$$

$\therefore$ The sum of last 20 coefficients in expansion of $(1 + x)^{39}$ is 2^{38}.

29. **If A and B are coefficients of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then find the value of A/B.**

Sol: Coefficient of x^n in the expansion of $(1 - x)^{2n}$ is ${}^{2n}C_n$.

Coefficient of x^n in the expansion of

$(1+x)^{2n-1}$ is ${}^{2n-1}C_n$.

$\therefore A = {}^{2n}C_n$ and $B = {}^{2n-1}C_n$

$$\therefore \frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{2n!}{n!n!}}{\frac{(2n-1)!}{(n-1)!n!}}$$

$$= \frac{2n!}{(2n-1)!n!}(n-1)!$$

$$= \frac{2n}{n} = 2$$

$$\Rightarrow \frac{A}{B} = 2.$$

30. **Find the sum of the following:**

i) $\frac{{}^{15}C_1}{{}^{15}C_0} + 2\frac{{}^{15}C_2}{{}^{15}C_1} + 3.\frac{{}^{15}C_3}{{}^{15}C_2} + ... + 15.\frac{{}^{15}C_{15}}{{}^{15}C_{14}}$

ii) $C_0.C_3 + C_1.C_4 + C_2.C_5 + ... + C_{n-3}.C_n$

iii) $2^2 \cdot C_0 + 3^2 \cdot C_1 + 4^2 \cdot C_2 + ... + (n+2)^2 C_n$

iv) $3C_0 + 6C_1 + 12C_2 + ... + 3.2\ {}^nC_n$

Sol: **i)** We know that

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{(n-r)!r!} \times \frac{(r-1)!(n-r+1)!}{n!}$$

$$= \frac{n-r+1}{r}$$

$$\therefore \frac{^{15}C_1}{^{15}C_0}+2\frac{^{15}C_2}{^{15}C_1}+3.\frac{^{15}C_3}{^{15}C_2}+...+15.\frac{^{15}C_{15}}{^{15}C_{14}}$$

$$=\frac{15}{1}+2\left(\frac{14}{2}\right)+3\left(\frac{13}{3}\right)+...+10\times\frac{1}{10}$$

$$=15+14+13+...+1$$

$$=\frac{15\times16}{2}=120$$

ii) $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n \ ...(1)$

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n \ ...(2)$

$(1)\times(2) \Rightarrow (1+x)^{2n}$

$= (C_0 + C_1x + C_2x^2 + ... + C_nx^n)$

$\quad (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n)$

Comparing coefficients of x^{n-3} on both sides,

$^{2n}C_{n-3} = C_0\cdot C_3 + C_1\cdot C_4 + C_2\cdot C_5 + ... + C_{n-3}\cdot C_n$

i.e. $C_0\cdot C_3 + C_1\cdot C_4 + C_2\cdot C_5 + ... + C_{n-3}\cdot C_n$

$\quad = {}^{2n}C_{n-3} = {}^{2n}C_{n+3}\left[\because {}^nC_r = {}^nC_{n-r}\right]$

iii) $2^2\cdot C_0 + 3^2\cdot C_1 + 4^2\cdot C_2 + ... + (n+2)^2 C_n$

$$=\sum_{r=0}^{n}(r+2)^2 C_n$$

$$=\sum_{r=0}^{n}\left(r^2+4r+4\right)C_r$$

$$=\sum_{r=0}^{n} r^2C_r + 4\sum_{r=0}^{n} rC_r + 4\sum_{r=0}^{n} C_r$$

$$=\sum_{r=0}^{n} r(r-1)C_r + \sum_{r=0}^{n} rC_r + 4\sum_{r=0}^{n} {}^rC_r + 4\sum_{r=0}^{n} C_r$$

$$=\sum_{r=2}^{n} r(r-1)C_r + 5\sum_{r=1}^{n} rC_r + 4\sum_{r=0}^{n} C_r$$

$= n(n-1)2^{n-2} + 5n.2^{n-1} + 4.2^n$

$= (n^2 + 9n + 44)2^{n-2}$

$= (n^2 + 9n + 16)2^{n-2}.$

iv) $3C_0 + 6C_1 + 12C_2 + ... + 3.2\ {}^nC_n$

$= \sum_{r=0}^{n} 3 \cdot 2^r \cdot C_r$

$= 3\sum_{r=0}^{n} 2^r \cdot C_r$

$= 3[1 + C_1(2) + C_2(2^2) + C_3(2^3) + ... + C_n 2^n)$

$= 3[1+2]^n$

$= 3 \cdot 3^n$

$= 3^{n+1}.$

31. **If** $(1 + x + x^2 + x^3)^7 = b_0 + b_1x + b_2x^2 + \ldots + b_{21}x^{21}$**, then find the value of**

i) $b_0 + b_2 + b_4 + ... + b_{20}$

ii) $b_1 + b_3 + b_5 + ... + b_{21}$

Sol: Given

$(1 + x + x^2 + x^3)^7 = b_0 + b_1x + b_2x^2 + \ldots + b_{21}x^{21} \ldots(1)$

Substituting x = 1 in (1),

We get

$b_0 + b_1 + b_2 + ... + b_{20} + b_{21} = 4^7 \qquad \ldots(2)$

Substituting x = –1 in (1),

We get $b_0 - b_1 + b_2 + ... + b_{20} - b_{21} = 0 \qquad \ldots(3)$

i) (2) + (3)

$\Rightarrow 2b_0 + 2b_2 + 2b_4 + ... + 2b_{20} = 4^7$

ii) (2) – (3)

$$\Rightarrow 2b_1 + 2b_3 + 2b_5 + ... + 2b_{21} = 4^7$$

$$\Rightarrow b_1 + b_3 + b_5 + ... + b_{21} = 2^{13}.$$

32. **If the coefficients of x^{11} and x^{12} in the binomial expansion of $\left(2+\frac{8x}{3}\right)^n$ are equal, find n.**

Sol: We know that $\left(2+\frac{8x}{3}\right)^n = 2^n\left(1+\frac{4x}{3}\right)^n$

Coefficient of x^{11} in the expansion of

$\left(2+\frac{8x}{3}\right)^n$ is ${}^nC_{11}\cdot 2^n\left(\frac{4}{3}\right)^{11}$

Coefficient of x^{12} in the expansion of

$\left(2+\frac{8x}{3}\right)^n$ is ${}^nC_{12}\cdot 2^n\left(\frac{4}{3}\right)^{12}$

Given coefficients of x^{11} and x^{12} are same

$$\Rightarrow {}^nC_{11}\cdot 2^n\left(\frac{4}{3}\right)^{11} = {}^nC_{12}\cdot 2^n\left(\frac{4}{3}\right)^{12}$$

$$\Rightarrow \frac{n!}{(n-11)!11!} = \frac{n!}{(n-12)!12!}\left(\frac{4}{3}\right)$$

$$\Rightarrow 12 = (n-11)\frac{4}{3}$$

$$\Rightarrow 9 = n-11$$

$$\Rightarrow n = 20.$$

33. **Find the remainder when 2^{2013} is divided by 17.**

Sol: We have 2^{2013}

$$= 2(2^{2012})$$

$$= 2(2^4)^{503}$$

$$= 2(16)^{503}$$

$$= 2(17-1)^{503}$$

$$= 2[{}^{503}C_0 17^{503} - {}^{503}C_1 17^{502} + {}^{503}C_2 17^{501} - ...... + {}^{503}C_{502} 17 - {}^{503}C_{503}]$$

$= 2[{}^{503}C_0 17^{503} - {}^{503}C_1 17^{502} + {}^{503}C_2 17^{501} - ......+ {}^{503}C_{502} 17] - 2$

$= 17m - 2$ where m is some integer.

$\therefore 2^{2013} = 17m - 2 \text{ (or) } 17k + 15$

$\therefore$ The remainder is –2 or 15.

Long Answer Questions

1. If 36, 84, 126 are three successive binomial coefficients in the expansion of $(1 + x)^n$, find n.

Sol. Let ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are three successive binomial coefficients in the expansion of $(1 + x)^n$, find n.

Then ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$

Now $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{84}{36} \Rightarrow \frac{n-r+1}{r} = \frac{7}{3}$

$3n - 3r + 3 = 7r \Rightarrow 3n = 10r - 3$

$\Rightarrow \frac{3n+3}{10} = r \quad ...(1)$

$\Rightarrow \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{126}{84} \Rightarrow \frac{n-r}{r+1} = \frac{3}{2}$

$\Rightarrow 2n - 2r = 3r + 3 \Rightarrow 2n = 5r + 3 \quad ...(2)$

$\Rightarrow 2n = 5\left(\frac{3n+3}{10}\right) + 3$ from (1)

$\Rightarrow 2n = \frac{3n+3+6}{2} \Rightarrow 4n = 3n + 9 \Rightarrow n = 9$

2. If the 2^{nd}, 3^{rd} and 4^{th} terms in the expansion of $(a + x)^n$ are respectively 240, 720, 1080, find a, x, n.

Sol. $T_2 = 240 \Rightarrow {}^nC_1\, a^{n-1}\, x = 240 \quad \ldots(1)$

$T_3 = 720 \Rightarrow {}^nC_2\, a^{n-2}\, x^2 = 720 \quad \ldots(2)$

$T_4 = 1080 \Rightarrow {}^nC_3\, a^{n-3}\, x^3 = 1080 \ldots(3)$

$$\frac{(2)}{(1)} \Rightarrow \frac{{}^nC_2 a^{n-2} x^2}{{}^nC_1 a^{n-1} x} = \frac{720}{240}$$

$$\Rightarrow \frac{n-1}{2}\frac{x}{a} = 3 \Rightarrow (n-1)x = 6a \ \ldots(4)$$

$$\frac{(3)}{(2)} \Rightarrow \frac{{}^nC_3 a^{n-3} x^3}{{}^nC_2 a^{n-2} x^2} = \frac{1080}{720} \Rightarrow \frac{n-2}{3}\frac{x}{a} = \frac{3}{2} \Rightarrow 2(n-2)x = 9a \ldots(5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{(n-1)x}{2(n-2)x} = \frac{6a}{9a} \Rightarrow \frac{n-1}{2n-4} = \frac{2}{3}$$

$$\Rightarrow 3n-3 = 4n-8 \Rightarrow n = 5$$

From (4), $(5 - 1)x = 6a \Rightarrow 4x = 6a$

$$\Rightarrow x = \frac{3}{2}a$$

Substitute $x = \frac{3}{2}a$, $n = 5$ in (1)

$${}^5C_1 \cdot a^4 \cdot \frac{3}{2}a = 240 \Rightarrow 5 \times \frac{3}{2}a^5 = 240$$

$$a^5 = \frac{480}{15} = 32 = 2^5$$

$$\therefore a = 2, x = \frac{3}{2}a = \frac{3}{2}(2) = 3 \quad \therefore a = 2, x = 3, n = 5$$

3. If the coefficients of r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1 + x)^{th}$ are in A.P. then show that $n^2 - (4r + 1)n + 4r^2 - 2 = 0$.

Sol. Coefficient of $T_r = {}^nC_{r-1}$

Coefficient of $T_{r+1} = {}^nC_r$

Coefficient of $T_{r+2} = {}^nC_{r+1}$

Given ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in A.P.

$\Rightarrow 2\,{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

$$\Rightarrow 2\frac{n!}{(n-r)!r!} = \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r-1)!(r+1)!}$$

$$\Rightarrow \frac{2}{(n-r)r} = \frac{1}{(n-r+1)(n-r)} + \frac{1}{(r+1)r}$$

$$\Rightarrow \frac{1}{n-r}\left[\frac{2}{r} - \frac{1}{n-r+1}\right] = \frac{1}{(r+1)r}$$

$$\Rightarrow \frac{1}{n-r}\left[\frac{2n-2r+2-r}{r(n-r+1)}\right] = \frac{1}{r(r+1)}$$

$$\Rightarrow (2n-3r+2)(r+1) = (n-r)(n-r+1)$$

$$\Rightarrow 2nr + 2n - 3r^2 - 3r + 2r + 2 = n^2 - 2nr + r^2 + n - r$$

$$\Rightarrow n^2 - 4nr + 4r^2 - n - 2 = 0$$

$$\therefore n^2 - (4r+1)n + 4r^2 - 2 = 0$$

4. Find the sum of the coefficients of x^{32} and x^{-18} in the expansion of $\left(2x^3 - \frac{3}{x^2}\right)^{14}$.

Sol. The general term in $\left(2x^3 - \frac{3}{x^2}\right)^{14}$ is:

$$T_{r+1} = {}^{14}C_r (2x^3)^{14-r}\left(-\frac{3}{x^2}\right)^r$$

$$= (-1)^r\,{}^{14}C_r (2)^{14-r}\cdot(3)^r\cdot x^{42-r}\cdot x^{-2r}$$

$$= (-1)^r\cdot{}^{14}C_r\, 2^{14-r}(3)^r x^{42-5r} \qquad ...(1)$$

From coefficients of x^{32},

Put 42 – 5r = 32 ⇒ 5r = 10 ⇒ r = 2

Put r = 2 in equation (1)

$$T_3 = (-1)^2\, {}^{14}C_2 (2)^{12} (3)^2 \cdot x^{42-10}$$
$$= {}^{14}C_2 (2)^{12} (3)^2 \cdot x^{32}$$

Coefficient of x^{32} is ${}^{14}C_2 (2)^{12} (3)^2$ …(2)

For coefficient of x^{-18}

Put 42 – 5r = –18 ⇒ 5r = 60 ⇒ r = 12

Put r = 12 in equation (1)

$$T_{13} = (-1)^{12}\, {}^{14}C_{12} (2)^2 (3)^{12} \cdot x^{42-60}$$
$$= {}^{14}C_{12} (2)^2 (3)^{12} \cdot x^{-18}$$

∴ Coefficient of x^{-18} is ${}^{14}C_{12}(2)^2\, 3^{12}$

Hence sum of the coefficients of x^{32} and x^{-18} is ${}^{14}C_2 (2)^{12} (3)^2 + {}^{14}C_{12} (2)^2 (3)^{12}$.

5. If P and Q are the sum of odd terms and the sum of even terms respectively in the expansion of $(x + a)^n$ then prove that

(i) $P^2 - Q^2 = (x^2 - a^2)^n$

(ii) $4PQ = (x + a)^{2n} - (x - a)^{2n}$

Sol. $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \ldots + {}^nC_{n-1}xa^{n-1} + {}^nC_na^n$

$$= ({}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots) + ({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + \ldots)$$
$$= P + Q$$

$$(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + \ldots + {}^nC_n(-1)^n a^n$$
$$= ({}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots) - ({}^nC_2x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + \ldots)$$
$$= P - Q$$

i) $P^2 - Q^2 = (P+Q)(P-Q)$

$= (x + a)^n (x - a)^n$

$= [(x + a)(x - a)]^n = (x^2 - a^2)^n$

ii) 4PQ $= (P + Q)^2 - (P - Q)^2$

$= [(x + a)^n]^2 - [(x - a)^n]^2$

$= (x + a)^{2n} - (x - a)^{2n}$

6. If the coefficients of 4 consecutive terms in the expansion of $(1 + x)^n$ are a_1, a_2, a_3, a_4 respectively, then show that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

Sol. Given a_1, a_2, a_3, a_4 are the coefficients of 4 consecutive terms in $(1 + x)^n$ respectively.

Let $a_1 = {}^nC_{r-1}$, $a_2 = {}^nC_r$, $a_3 = {}^nC_{r+1}$, $a_4 = {}^nC_{r+2}$

L.H.S: $$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{a_1}{a_1\left(1+\frac{a_2}{a_1}\right)} + \frac{a_3}{a_3\left(1+\frac{a_4}{a_3}\right)}$$

$$= \frac{1}{1+\frac{{}^nC_r}{{}^nC_{r-1}}} + \frac{1}{1+\frac{{}^nC_{r+2}}{{}^nC_{r+1}}} = \frac{1}{1+\frac{n-r+1}{r}} + \frac{1}{1+\frac{n-r-1}{r+2}}$$

$$= \frac{r}{n+1} + \frac{r+2}{r+2+n-r-1} = \frac{r+r+2}{n+1} = \frac{2(r+1)}{n+1}$$

R.H.S: $$\frac{2a_2}{a_2+a_3} = \frac{2a_2}{a_2\left(1+\frac{a_3}{a_2}\right)}$$

$$\frac{2}{1+\frac{{}^nC_{r+1}}{{}^nC_r}} = \frac{2}{1+\frac{n-r}{r+1}} = \frac{2(r+1)}{n+1} = \text{L.H.S}$$

$$\therefore \frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$$

7. Prove that $({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-({}^{2n}C_3)^2+\ldots+({}^{2n}C_{2n})^2=(-1)^n\ {}^{2n}C_n$

Sol. $(x+1)^{2n}={}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+\ {}^{2n}C_2x^{2n-2}+\ldots+{}^{2n}C_{2n}$ ...(1)

$(x-1)^{2n}={}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2+\ldots+{}^{2n}C_{2n}x^{2n}$ ...(2)

Multiplying eq. (1) and (2), we get

$$({}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+{}^{2n}C_2x^{2n-2}+\ldots+{}^{2n}C_{2n})$$
$$({}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2+\ldots+{}^{2n}C_{2n}x^{2n})$$

$$=(x+1)^{2n}(1-x)^{2n}=[(1+x)(1-x)]^{2n}$$

$$=(1-x^2)^{2n}=\sum_{r=0}^{2n}{}^{2n}C_r(-x^2)^r$$

Equating the coefficients of x^{2n}

$$({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-({}^{2n}C_3)^2+\ldots+({}^{2n}C_{2n})^2=(-1)^n\ {}^{2n}C_n$$

8. Prove that $(C_0+C_1)(C_1+C_2)(C_2+C_3)\ldots(C_{n-1}+C_n)=\frac{(n+1)^n}{n!}\cdot C_0\cdot C_1\cdot C_2\cdot\ldots C_n$

Sol. $(C_0+C_1)(C_1+C_2)(C_2+C_3)\ldots(C_{n-1}+C_n)=$

$$=C_0\left(1+\frac{C_1}{C_0}\right)\cdot C_1\left(1+\frac{C_2}{C_1}\right)\ldots C_{n-1}\left(1+\frac{C_n}{C_{n-1}}\right)$$

$$=\left(1+\frac{{}^nC_1}{{}^nC_0}\right)\left(1+\frac{{}^nC_2}{{}^nC_1}\right)\ldots\ldots\left(1+\frac{{}^nC_n}{{}^nC_{n-1}}\right)C_0C_1C_2\ldots C_{n-1}$$

$$=\left(1+\frac{n}{1}\right)\left(1+\frac{n-1}{2}\right)\ldots\left(1+\frac{1}{n}\right)\ C_n\cdot C_1\cdot C_2\cdot\ldots C_{n-1}[C_0=C_n]$$

$$=\left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right)......\left(\frac{1+n}{n}\right)C_1\cdot C_2\cdot ...C_{n-1}\cdot C_n$$

$$=\frac{(1+n)^n}{n!}C_1C_2...C_n$$

$$\therefore (C_0+C_1)(C_1+C_2)(C_2+C_3)...(C_{n-1}+C_n)=\frac{(n+1)^n}{n!}\cdot C_0\cdot C_1\cdot C_2\cdot ...C_n$$

9. Find the term independent of x in $(1+3x)^n\left(1+\frac{1}{3x}\right)^n$**.**

Sol. $(1+3x)^n\left(1+\frac{1}{3x}\right)^n = (1+3x)^n\left(\frac{3x+1}{3x}\right)^n$

$$=\left(\frac{1}{3x}\right)^n(1+3x)^{2n}=\frac{1}{3^n\cdot x^n}\sum_{r=0}^{2n}({}^{2n}C_r)(3x)^r$$

The term independent of x in

$(1+3x)^n\left(1+\frac{1}{3x}\right)^n$ is $\frac{1}{3^n}({}^{2n}C_n)3^n={}^{2n}C_n$

10. If $(1+3x-2x^2)^{10}=a_0+a_1x+a_2x^2+...+a_{20}x^{20}$ **then prove that**

i) $a_0+a_1+a_2+...+a_{20}=2^{10}$

ii) $a_0-a_1+a_2-a_3+...+a_{20}=4^{10}$

Sol. $(1+3x-2x^2)^{10}=a_0+a_1x+a_2x^2+...+a_{20}x^{20}$

i) Put x = 1

$(1+3-2)^{10}=a_0+a_1+a_2+...+a_{20}$

$\therefore a_0+a_1+a_2+...+a_{20}=2^{10}$

ii) Put x = –1

$(1-3-2)^{10}=a_0-a_1+a_2+...+a_{20}$

$\therefore a_0-a_1+a_2-a_3+...+a_{20}=(-4)^{10}=4^{10}$

11. If R, n are positive integers, n is odd, 0 < F < 1 and if $(5\sqrt{5}+11)^n = R+F$, then prove that

i) R is an even integer and

ii) (R + F)F = 4^n.

Sol. i) Since R, n are positive integers, 0 < F < 1 and $(5\sqrt{5}+11)^n = R+F$

Let $(5\sqrt{5}-11)^n = f$

Now, $11 < 5\sqrt{5} < 12 \Rightarrow 0 < 5\sqrt{5}-11 < 1$

$\Rightarrow 0 < (5\sqrt{5}-11)^n < 1 \Rightarrow 0 < f < 1 \quad \Rightarrow 0 > -f > -1 \therefore -1 < -f < 0$

R + F – f = $(5\sqrt{5}+11)^n - (5\sqrt{5}-11)^n$

$$= \left[{}^nC_0(5\sqrt{5})^n + {}^nC_1(5\sqrt{5})^{n-1}(11) + {}^nC_2(5\sqrt{5})^{n-2}(11)^2 + \ldots + {}^nC_n(11)^n\right] - \left[{}^nC_0(5\sqrt{5})^n - {}^nC_1(5\sqrt{5})^{n-1}(11) + {}^nC_2(5\sqrt{5})^{n-2}(11)^2 + \ldots + {}^nC_n(-11)^n\right]$$

$$= 2\left[{}^nC_1(5\sqrt{5})^{n-1}(11) + {}^nC_3(5\sqrt{5})^{n-3}(11)^2 + \ldots\right]$$

= 2k where k is an integer.

$\therefore$ R + F – f is an even integer.

$\Rightarrow$ F – f is an integer since R is an integer.

But 0 < F < 1 and –1 < –f < 0 $\Rightarrow$ –1 < F – f < 1

$\therefore$ F – f = 0 $\Rightarrow$ F = f

$\therefore$ R is an even integer.

ii) (R + F)F = (R + F)f, $\because$ F = f

$= (5\sqrt{5}+11)^n(5\sqrt{5}-11)^n$

$= \left[(5\sqrt{5}+11)(5\sqrt{5}-11)\right]^n = (125-121)^n = 4^n$

$\therefore$ (R + F)F = 4^n.

12. If I, n are positive integers, 0 < f < 1 and if $(7+4\sqrt{3})^n = I+f$, then show that (i) I is an odd integer and (ii) (I + f)(I – f) = 1.

Sol. Given I, n are positive integers and

$(7+4\sqrt{3})^n = I+f$, 0 < f < 1

Let $7-4\sqrt{3} = F$

Now $6 < 4\sqrt{3} < 7 \Rightarrow -6 > -4\sqrt{3} > -7$

$\Rightarrow 1 > 7-4\sqrt{3} > 0 \Rightarrow 0 < (7-4\sqrt{3})^n < 1$

$\therefore\ 0 < F < 1$

$1 + f + F = (7+4\sqrt{3})^n(7-4\sqrt{3})^n$

$$= \left[\begin{array}{r} {}^nC_0 7^n + {}^nC_1 7^{n-1}(4\sqrt{3}) + {}^nC_2 7^{n-2}(4\sqrt{3})^2 \\ +\ldots+{}^nC_n(4\sqrt{3})^n \end{array}\right]$$

$$= \left[\begin{array}{r} {}^nC_0 7^n - {}^nC_1 7^{n-1}(4\sqrt{3}) + {}^nC_2 7^{n-2}(4\sqrt{3})^2 \\ +\ldots+{}^nC_n(-4\sqrt{3})^n \end{array}\right]$$

$$= 2\left[{}^nC_0 7^n + {}^nC_2 7^{n-2}(4\sqrt{3})^2 \ldots\ldots\right]$$

= 2k where k is an integer.

$\therefore$ 1 + f + F is an even integer.

$\Rightarrow$ f + F is an integer since I is an integer.

But $0 < f < 1$ and $0 < F < 1 \Rightarrow f + F < 2$

$\therefore\ f + F = 1 \qquad \ldots(1)$

$\Rightarrow$ I + 1 is an even integer.

$\therefore$ I is an odd integer.

$(I + f)(I - f) = (I + f)F$, by (1)

$$= (7+4\sqrt{3})^n(7-4\sqrt{3})^n$$

$$= \left[(7+4\sqrt{3})(7-4\sqrt{3})\right]^n = (49-48)^n = 1$$

13. If n is a positive integer, prove that $\sum_{r=1}^{n} r^3\left(\frac{{}^nC_r}{{}^nC_{r-1}}\right)^2 = \frac{(n)(n+1)^2(n+2)}{12}$**.**

Sol. $\sum_{r=1}^{n} r^3\left(\frac{{}^nC_r}{{}^nC_{r-1}}\right)^2 = \sum_{r=1}^{n} r^3\left(\frac{n-r+1}{r}\right)^2$

$$= \sum_{r=1}^{n} r(n-r+1)^2 = \sum_{r=1}^{n} r[(n+1)^2 - 2(n+1)r + r^2]$$

$$= (n+1)^2\Sigma r - 2(n+1)\Sigma r^2 + \Sigma r^3$$

$$= (n+1)^2\frac{(n)(n+1)}{2}$$

$$-2(n+1)\frac{(n)(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4}$$

$$=\frac{(n+1)^2}{2}\left[n(n+1)-\frac{2n(2n+1)}{3}+\frac{n^2}{2}\right]$$

$$=\frac{(n+1)^2}{2}\left[\frac{6n^2+6n-8n^2-4n+3n^2}{6}\right]$$

$$=\frac{(n+1)^2}{2}\left[\frac{n^2+2n}{6}\right]=\frac{n(n+1)^2(n+2)}{12}$$

14. If x = $\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\frac{1\cdot 3\cdot 5\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\ldots$ **then prove that $9x^2 + 24x = 11$.**

Sol. Given x = $\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\frac{1\cdot 3\cdot 5\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\ldots$

$$=\frac{1\cdot 3}{1\cdot 2}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^2+\ldots$$

$$=1+\frac{1}{1}\cdot\frac{1}{3}+\frac{1\cdot 3}{1\cdot 2}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^2+\ldots-\left[1+\frac{1}{3}\right]$$

Here p = 1, q = 2, $\frac{x}{q}=\frac{1}{3}\Rightarrow x=\frac{2}{3}$

$$=(1-x)^{-p/q}-\frac{4}{3}=\left(1-\frac{2}{3}\right)^{-1/2}-\frac{4}{3}$$

$$=\left(\frac{1}{3}\right)^{-1/2}-\frac{4}{3}=\sqrt{3}-\frac{4}{3}$$

$$\Rightarrow 3x+4=3\sqrt{3}$$

Squaring on both sides

$$(3x+4)^2=(3\sqrt{3})^2\Rightarrow 9x^2+24x+16=27$$

$$\Rightarrow 9x^2+24x=11$$

15. (i) Find the coefficient of x^5 in $\frac{(1-3x)^2}{(3-x)^{3/2}}$.

Sol. $\frac{(1-3x)^2}{(3-x)^{3/2}} = \frac{(1-3x)^2}{\left[3\left(1-\frac{x}{3}\right)\right]^{3/2}} = \frac{(1-3x)^2}{3^{3/2}\left(1-\frac{x}{3}\right)^{3/2}}$

$$= \frac{1}{3^{3/2}}(1-3x)^2\left(1-\frac{x}{3}\right)^{-3/2}$$

$$= \frac{1}{\sqrt{27}}\left[(1-6x+9x^2)\left\{1+\frac{3}{2}\left(\frac{x}{3}\right)+\frac{\frac{3}{2}\cdot\frac{5}{2}}{1\cdot2}\left(\frac{x}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1\cdot2\cdot3}\left(\frac{x}{3}\right)^3+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{1\cdot2\cdot3\cdot4}\left(\frac{x}{3}\right)^4+......\right\}\right]$$

$$= \frac{1}{\sqrt{27}}\left[(1-6x+9x^2)\left(1+\frac{x}{2}+\frac{5}{24}x^2+\frac{35x^3}{16\times27}+\frac{35}{8\times16\times9}x^4\frac{77}{8\times32\times27}x^5+...\right)\right]$$

$\therefore$ The coefficient of x^5 in $\frac{(1-3x)^2}{(3-x)^{3/2}}$ is

$$= \frac{1}{\sqrt{27}}\left[\frac{77}{8\times32\times27}-\frac{6(35)}{8\times16\times9}+\frac{9(35)}{16\times27}\right]$$

$$= \frac{1}{\sqrt{27}}\left[\frac{77-1260+5040}{8\times32\times27}\right] = \frac{3857}{\sqrt{27}\times8\times32\times27}$$

ii) Find the coefficient of x^8 in $\frac{(1+x)^2}{\left(1-\frac{2}{3}x\right)^3}$.

Sol. $\frac{(1+x)^2}{\left(1-\frac{2}{3}x\right)^3} = (1+x)^2\left(1-\frac{2}{3}x\right)^{-3}$

$$= (1+2x+x^2)\left[1+3\left(\frac{2x}{3}\right)+\frac{(3)(4)}{1\cdot2}\left(\frac{2x}{3}\right)^2+\frac{3\cdot4\cdot5}{1\cdot2\cdot3}\left(\frac{2x}{3}\right)^3+\frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4}\left(\frac{2x}{3}\right)^4+\frac{3\cdot4\cdot5\cdot6\cdot7}{1\cdot2\cdot3\cdot4\cdot5}\left(\frac{2x}{3}\right)^5\right.$$

$$\left.+\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4\cdot5\cdot6}\left(\frac{2x}{3}\right)^6+\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\left(\frac{2x}{3}\right)^7+\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}\left(\frac{2x}{3}\right)^8+...\right]$$

$\therefore$ Coefficient of x^8 in $\dfrac{(1+x)^2}{\left(1-\dfrac{2}{3}x\right)^3}$ is

$$=45\left(\frac{2}{3}\right)^8+2\times 36\left(\frac{2}{3}\right)^7+28\left(\frac{2}{3}\right)^6$$

$$=\left(\frac{2}{3}\right)^6\left[45\times\frac{4}{9}+72\times\frac{2}{3}+28\right]$$

$$=\left(\frac{2}{3}\right)^6(20+48+28)=\frac{96\times 2^6}{3^6}=\frac{2048}{243}$$

iii) Find the coefficient of x^7 in $\dfrac{(2+3x)^3}{(1-3x)^4}$.

Sol. $\dfrac{(2+3x)^3}{(1-3x)^4}=(2+3x)^3(1-3x)^{-4}$

$$=(8+36x+54x^2+27x^3)$$
$$[1+{}^4C_1(3x)+{}^5C_2(3x)^2+{}^6C_3(3x)^3+{}^7C_4(3x)^4+{}^8C_5(3x)^5+{}^9C_6(3x)^6+\ldots]$$

$\therefore$ Coefficient of x^7 in $\dfrac{(2+3x)^3}{(1-3x)^4}$ is

$$=8\cdot\left({}^{10}C_7\cdot 3^7\right)+36\cdot\left({}^9C_6(3)^6\right)+54\left({}^8C_5(3^5)\right)+27\left({}^7C_4(3^4)\right)$$
$$=8\left({}^{10}C_3 3^7\right)+36\left({}^9C_3 3^6\right)+54\left({}^8C_3 3^5\right)+27\left({}^7C_3 3^4\right)$$

16. Find the coefficient of x^3 in the expansion of $\dfrac{(1-5x)^3(1+3x^2)^{3/2}}{(3+4x)^{1/3}}$.

Sol. $\dfrac{(1-5x)^3(1+3x^2)^{3/2}}{(3+4x)^{1/3}} = \dfrac{(1-5x)^3(1+3x^2)^{3/2}}{\left[3\left(1+\frac{4}{3}\right)\right]^{1/3}}$

$$= \frac{1}{3^{1/3}}(1-5x)^3(1+3x^2)^{3/2}\left(1+\frac{4}{3}\right)^{-1/3}$$

$$= \frac{1}{3^{1/3}}[1-15x+75x^2-125x^3]$$

$$\left[1+\frac{3}{2}(3x^2)+\frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)}{1\cdot 2}(3x^2)^2+\ldots\right]$$

$$\left[1+\left(\frac{-1}{3}\right)\frac{4x}{3}+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{1\cdot 2}\left(\frac{4x}{3}\right)^2+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{1\cdot 2\cdot 3}\left(\frac{4x}{3}\right)^3+\ldots\right]$$

$$= \frac{1}{3^{1/3}}(1-15x+75x^2-125x^3)\left(1+\frac{9}{2}x^2+\ldots\right)$$

$$\left(1-\frac{4x}{9}+\frac{32}{81}x^2-\frac{896}{2187}x^3+\ldots\right)$$

$$= \frac{1}{3^{1/3}}\left[1-15x+75x^2-125x^3+\frac{9}{2}x^2-\frac{135}{2}x^3+\ldots\right]$$

$$\left[1-\frac{4x}{9}+\frac{32}{81}x^2-\frac{896}{2187}x^3+\ldots\right]$$

$$= \frac{1}{3^{1/3}}\left[1-15x+\frac{159}{2}x^2-\frac{385}{2}x^3+\ldots\right]$$

$$\left[1-\frac{4x}{9}+\frac{32}{81}x^2-\frac{896}{2187}x^3+\ldots\right]$$

$\therefore$ Coefficient of x^3 in $\dfrac{(1-5x)^3(1+3x^2)^{3/2}}{(3+4x)^{1/3}}$ is $= \dfrac{1}{3^{1/3}}\left[-\dfrac{385}{2}-\dfrac{159}{2}\times\dfrac{4}{9}-15\times\dfrac{32}{81}-\dfrac{896}{2187}\right]$

$$=\frac{1}{3^{1/3}}\left[-\frac{385}{2}-\left\{\frac{77274+12960+896}{2187}\right\}\right]$$

$$=\frac{1}{3^{1/3}}\left[\frac{-841995-182260}{4374}\right]=-\frac{1024255}{\sqrt[3]{3}(4274)}$$

17. If x = $\frac{5}{(2!)\cdot 3}+\frac{5\cdot 7}{(3!)\cdot 3^2}+\frac{5\cdot 7\cdot 9}{(4!)3^3}+\ldots$, then find the value of x^2 + 4x.

Sol. $x=\frac{5}{(2!)\cdot 3}+\frac{5\cdot 7}{(3!)\cdot 3^2}+\frac{5\cdot 7\cdot 9}{(4!)3^3}+\ldots$

$$=\frac{3\cdot 5}{2!3^2}+\frac{3\cdot 5\cdot 7}{3!3^3}+\frac{3\cdot 5\cdot 7\cdot 9}{4!3^4}+\ldots$$

$$=\frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+\frac{3\cdot 5\cdot 7}{3!}\left(\frac{1}{3}\right)^3+\frac{3\cdot 5\cdot 7\cdot 9}{4!}\left(\frac{1}{3}\right)^4\ldots.=1+\frac{3}{1}\left(\frac{1}{3}\right)+x$$

$$=1+\frac{3}{1}\left(\frac{1}{3}\right)+\frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+\frac{3\cdot 5\cdot 7}{3!}\left(\frac{1}{3}\right)^3+\ldots$$

$$\Rightarrow 2+x=1+\frac{3}{1}\left(\frac{1}{3}\right)+\frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+\ldots$$

Comparing x + 2 with $(1-y)^{-p/q}$

$$=1+\frac{p}{1}\left(\frac{y}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{y}{q}\right)^2+\ldots$$

Here p = 3, q = 2, $\frac{y}{q}=\frac{1}{3}\Rightarrow y=\frac{q}{3}=\frac{2}{3}$

$\therefore x+2=(1-y)^{-p/q}=\left(1-\frac{2}{3}\right)^{-3/2}=\left(\frac{1}{3}\right)^{-3/2}=(3)^{3/2}=\sqrt{27}$ Squaring on both sides

$$x^2+4x+4=27 \quad \Rightarrow x^2+4x=23$$

18. Find the sum of the infinite series $\frac{7}{5}\left(1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\frac{1}{10^4}+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\frac{1}{10^6}+...\right)$**.**

Sol. $1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\frac{1}{10^4}+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\frac{1}{10^6}+...$

$$=1+\frac{1}{1!}\left(\frac{1}{100}\right)+\frac{1\cdot3}{2!}\left(\frac{1}{100}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{1}{100}\right)^3+...$$

Comparing with $(1-x)^{-p/q}$

$$=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 \quad p = 1, p+q=3, q= 2$$

$$\frac{x}{q}=\frac{1}{100}\Rightarrow x=\frac{q}{100}=\frac{2}{100}=0.02$$

$$\therefore 1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\cdot\frac{1}{10^4}+...=(1-x)^{-p/q}$$

$$=(1-0.02)^{-1/2}=(0.98)^{-1/2}=\left(\frac{49}{50}\right)^{-1/2}=\left(\frac{50}{49}\right)^{1/2}=\frac{5\sqrt{2}}{7}$$

$$\therefore \frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\frac{1}{10^4}+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\frac{1}{10^6}+...\right]$$

$$=\frac{7}{5}\frac{5\sqrt{2}}{7}=\sqrt{2}$$

19. Show that

$$1+\frac{x}{2}+\frac{x(x-1)}{2\cdot4}+\frac{x(x-1)(x-2)}{2\cdot4\cdot6}+...$$
$$=1+\frac{x}{3}+\frac{x(x+1)}{3\cdot6}+\frac{x(x+1)(x+2)}{3\cdot6\cdot9}+...$$

Sol. L.H.S. = $1+\frac{x}{2}+\frac{x(x-1)}{2\cdot4}+\frac{x(x-1)(x-2)}{2\cdot4\cdot6}+...$

$$=1+x\left(\frac{1}{2}\right)\frac{(x)(x-1)}{1\cdot 2}\left(\frac{1}{2}\right)^2+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}\left(\frac{1}{2}\right)^3+\ldots$$

Comparing with $(1+x)^n=1+{}^nC_1\cdot x+{}^nC_2x^2+\ldots\ldots.$

$$=1+\frac{n}{1!}\cdot x+\frac{n(n-1)}{1\cdot 2}x^2+\ldots.$$

Here $x=\frac{1}{2}, n=x=\left(1+\frac{1}{2}\right)^x=\left(\frac{3}{2}\right)^x$

R.H.S. $=1+\frac{x}{3}+\frac{x(x+1)}{3\cdot 6}+\frac{x(x+1)(x+2)}{3\cdot 6\cdot 9}+\ldots$

$$=1+\frac{x}{1}\left(\frac{x}{3}\right)+\frac{(x)(x+1)}{1\cdot 2}\left(\frac{1}{3}\right)^2+\frac{(x)(x+1)(x+2)}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3+\ldots$$

Comparing with $(1-x)^{-n}$

$$=1+n(x)+\frac{n(n+1)}{1\cdot 2}x^2+\ldots\ldots$$

We get $x=\frac{1}{3}, n=x$

$$=\left(1-\frac{1}{3}\right)^{-x}=\left(\frac{2}{3}\right)^{-x}=\left(\frac{3}{2}\right)^x$$

$\therefore$ L.H.S. = R.H.S.

20. Suppose that n is a natural number and I, F are respectively the integral part and fractional part of $(7+4\sqrt{3})^n$, then show that

(i) I is an odd integer, (ii) (I + F)(I – F) = 1.

Sol. Given that $(7+4\sqrt{3})^n$ = I + F where I is an integer and 0 < F < 1.

Write f = $(7-4\sqrt{3})^n$

Now 36 < 48 < 49

$6<\sqrt{48}<7$

i.e. $-7 < -\sqrt{48} < -6$

i.e. $0 < 7-4\sqrt{3} < 1$

i.e. $0 < (7-4\sqrt{3})^n < 1$

$\therefore\ 0 < f < 1$

Now I + F + f = $(7+4\sqrt{3})^n + (7-4\sqrt{3})^n =$

$$= \left({}^nC_0 \cdot 7^n + {}^nC_1(7)^{n-1}(4\sqrt{3}) + {}^nC_2(7)^{n-2}(4\sqrt{3})^2 + \ldots\right) + \left({}^nC_0 \cdot 7^n - {}^nC_1(7)^{n-1}(4\sqrt{3}) + {}^nC_2(7)^{n-2}(4\sqrt{3})^2 - \ldots\right)$$

$$= 2\left[7^n + {}^nC_2 7^{n-2}(4\sqrt{3})^2 + {}^nC_4 7^{n-4}(4\sqrt{3})^4 + \ldots\right]$$

= 2k, where k is a positive integer …(1)

Thus I + F + f is an even integer.

Since I is an integer, we get that F + f is an integer. Also since 0 < F < 1 and 0 < f < 1

$\Rightarrow 0 < F + f < 2$

$\because$ F + f is an integer

We get F + f = 1

(i.e.) I – F = f …(2)

(i) From (1) I + F + f = 2k

$\Rightarrow$ f = 2k – 1, an odd integer.

(ii) (I + F)(I – F) = (I + F)f

$(7+4\sqrt{3})^n(7-4\sqrt{3})^n = (49-49)^n = 1$

21. Find the coefficient of x^6 in $(3 + 2x + x^2)^6$.

Sol. $(3+2x+x^2)=[(3+2x)+x^2)]^6$

$$={}^6C_0(3+2x)^6+{}^6C_1(3+2x)^5(x^2)+{}^6C_2(3+2x)^4(x^2)^2+{}^6C_3(3+2x)^3(x^2)^3+\ldots$$
$$=(3+2x)^6+6(3+2x)^5(x^2)+15x^4(3+2x)^4x^4+20x^6(3+2x)^3+\ldots$$

$$=\left[\sum_{r=0}^{6}{}^6C_r\cdot 3^{6-r}(2x)^r\right]+6x^2\left[\sum_{r=0}^{5}{}^5C_r\cdot 3^{5-r}(2x)^r\right]+15x^4\left[\sum_{r=0}^{4}{}^4C_r\cdot 3^{4-r}(2x)^r\right]+20x^6\left[\sum_{r=0}^{3}{}^3C_r\cdot 3^{3-r}(2x)^r\right]+\ldots$$

$\therefore$ The coefficient of x^6 in $(3 + 2x + x^2)^6$ is

$$={}^6C_6\cdot 3^0\cdot 2^6+6({}^5C_4\cdot 3^1\cdot 2^4)+15({}^4C_2\cdot 3^2\cdot 2^2)+20({}^3C_0\cdot 3^3\cdot 2^0)$$
$$=64+1440+3240+540=5284$$

22. If n is a positive integer, then prove that $C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$**.**

Sol. Write $S=C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}$ then

$$S={}^nC_0+\frac{1}{2}\cdot{}^nC_1+\frac{1}{3}\cdot{}^nC_2+\ldots+\frac{1}{n+1}\cdot{}^nC_n$$

$$\therefore (n+1)S=\frac{n+1}{1}\cdot{}^nC_0+\frac{n+1}{2}\cdot{}^nC_1+\frac{n+1}{3}\cdot{}^nC_2+\ldots+\frac{n+1}{n+1}\cdot{}^nC_n$$

$$\therefore (n+1)S={}^{(n+1)}C_1+{}^{(n+1)}C_2+{}^{(n+1)}C_3+\ldots+{}^{(n+1)}C_{n+1}$$

$\left(\text{since } \frac{n+1}{r+1}\cdot{}^nC_r={}^{n+1}C_{r+1}\right)$

$$=2^{n+1}-1$$

Hence $S=\frac{2^{n+1}-1}{n+1}$

$$\therefore\ C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$$

23. If n is a positive integer and x is any non-zero real number, then prove that

$$C_0 + C_1\frac{x}{2} + C_2\cdot\frac{x^2}{3} + C_3\cdot\frac{x^3}{4} + \ldots + C_n\cdot\frac{x^n}{n+1}$$
$$= \frac{(1+x)^{n+1}-1}{(n+1)x}$$

Sol. $C_0 + C_1\frac{x}{2} + C_2\cdot\frac{x^2}{3} + C_3\cdot\frac{x^3}{4} + \ldots + C_n\cdot\frac{x^n}{n+1}$

$$= {}^nC_0 + \frac{1}{2}\,{}^nC_1 x + \frac{1}{3}\,{}^nC_2 x^2 + \ldots + \frac{1}{n+1}\,{}^nC_n x^n$$

$$= 1 + \frac{n}{1!}\frac{x}{2} + \frac{n(n-1)}{2!}\frac{x^2}{3} + \ldots\ldots$$

$$= 1 + \frac{n}{2!}x^1 + \frac{n(n-1)}{3!}x^2 + \ldots\ldots$$

$$= \frac{1}{(n+1)x}\left[\frac{(n+1)x^1}{1!} + \frac{(n+1)n}{2!}x^2 + \frac{(n+1)n(n-1)}{3!}x^3 + \ldots\right]$$

$$= \frac{1}{(n+1)x}\left[{}^{(n+1)}C_1 x + {}^{(n+1)}C_2 x^2 + {}^{(n+1)}C_3 x^3 + \ldots\right]$$

$$= \frac{1}{(n+1)x}\left[1 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + \ldots + {}^{n+1}C_{n+1} x^{n+1} - 1\right]$$

$$= \frac{1}{(n+1)x}\left[(1+x)^{n+1} - 1\right]$$

$$C_0 + C_1\frac{x}{2} + C_2\cdot\frac{x^2}{3} + C_3\cdot\frac{x^3}{4} + \ldots + C_n\cdot\frac{x^n}{n+1} = \frac{(1+x)^{n+1}-1}{(n+1)x}$$

24. Prove that $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n C_n^2 = \begin{cases} (-1)^{n/2}\,{}^nC_{n/2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$

Sol. Take $(1-x)^n\left(1+\frac{1}{x}\right)^n$

$$= (C_0 - C_1x + C_2x^2 - C_3x^3 + \ldots + (-1)^n\cdot C_nx^n)\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}\right) \quad \ldots(1)$$

The term independent of x in R.H.S. of (1) is $= C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2$

Now we can find the term independent of in the L.H.S. of (1).

L.H.S. of (1) = $(1-x)^n \left(1+\frac{1}{x}\right)^n$

$$= (1-x)^n \left(\frac{1+x}{x}\right)^n = \frac{(1-x^2)^n}{x^n}$$

$$= \sum_{r=0}^{n} {}^nC_r(-x^2)^r \qquad ...(2)$$

Suppose n is an even integer, say n = 2k.

Then from (2),

$$(1-x)^n \left(1+\frac{1}{x}\right)^n = \frac{\sum_{r=0}^{n} {}^nC_r(-x^2)^r}{x^n}$$

$$= \frac{\sum_{r=0}^{2k} {}^{2k}C_r(-x^2)^r}{x^{2k}} = \sum_{r=0}^{2k} {}^{2k}C_r(-1)^r x^{2r-2k} ...(3)$$

To set term independent of x in (3), put

$2r - 2k = 0 \Rightarrow r = k$

Hence the term index. of x in

$(1-x)^n \left(1+\frac{1}{x}\right)^n$ is ${}^{2k}C_k(-1)^k = {}^nC_{(n/2)}(-1)^{n/2}$

When n is odd:

Observe that the expansion in the numerator of (2) contains only even powers of x.

∴ If n is odd, then there is no constant term in (2) (i.e.) the term independent of x in $(1-x)^n \left(1+\frac{1}{x}\right)^n$ is zero.

∴ From (1), we get

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2 = \begin{cases} (-1)^{n/2}\, {}^nC_{n/2}, & \text{if } n \text{ is even} \\ 0 & \text{, if n is odd} \end{cases}$$

25. Find the coefficient of x^{12} in $\frac{1+3x}{(1-4x)^4}$.

Sol. $\frac{1+3x}{(1-4x)^4} = (1+3x)(1-4x)^{-4} = (1+3x)\left[\sum_{r=0}^{\infty} {}^{(n+r-1)}C_r \cdot X^r\right]$

Here X = 4x, n = 4

$$= (1+3x)\left[\sum_{r=0}^{\infty} {}^{(4+r-1)}C_r \cdot (4x)^r\right]$$

$$= (1+3x)\left[\sum_{r=0}^{\infty} {}^{(r+3)}C_r \cdot (4)^r (x)^r\right]$$

$\therefore$ The coefficient of x^{12} in $\frac{1+3x}{(1-4x)^4}$ is

$$= (1)\cdot {}^{(12+3)}C_{12} \cdot 4^{12} + 3\cdot {}^{(11+3)}C_3 \cdot 4^{11}$$

$$= {}^{15}C_3 \cdot 4^{12} + 3\cdot {}^{14}C_3 \cdot 4^{11}$$

$$= 455 \times 4^{12} + (1092)4^{11} = 728 \times 4^{12}$$

26. Find coefficient of x^6 in the expansion of $(1 - 3x)^{-2/5}$.

Sol. General term of $(1 - x)^{-p/q}$ is

$$T_{r+1} = \frac{(p)(p+q)(p+2q)+...+[p+(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here X = 3x, p = 2, q = 5, r = 6, $\frac{X}{q} = \frac{3x}{5}$

$$T_{6+1} = \frac{(2)(2+5)(2+2.5)...[2+(6-1)5]}{6!}\left(\frac{3x}{5}\right)^6$$

$$T_7 = \frac{(2)(7)(12)...(27)}{6!}\left(\frac{3x}{5}\right)^6$$

$\therefore$ Coefficient of x^6 in $(1-3x)^{-2/5}$ is $=\frac{(2)(7)(12)...(27)}{6!}\left(\frac{3}{5}\right)^6$

27. Find the sum of the infinite series $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\left(\frac{1}{2}\right)^2+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\left(\frac{1}{2}\right)^3+...\infty$

Sol. Let $S=1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\left(\frac{1}{2}\right)^2+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\left(\frac{1}{2}\right)^3+...$

$$=1+\frac{2}{1}\cdot\frac{1}{6}+\frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2+\frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3+...$$

$$\because 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+...=(1-x)^{-p/q}$$

Here $p=2,\ q=3,\frac{x}{q}=\frac{1}{6}\Rightarrow x=\frac{3}{6}=\frac{1}{2}$

$$=(1-x)^{-p/q}=\left(1-\frac{1}{2}\right)^{-2/3}=2^{2/3}=\sqrt[3]{4}$$

28. Find the sum of the series $\frac{3\cdot5}{5\cdot10}+\frac{3\cdot5\cdot7}{5\cdot10\cdot15}+\frac{3\cdot5\cdot7\cdot9}{5\cdot10\cdot15\cdot20}+...\infty$

Sol. Let $S=\frac{3\cdot5}{5\cdot10}+\frac{3\cdot5\cdot7}{5\cdot10\cdot15}+\frac{3\cdot5\cdot7\cdot9}{5\cdot10\cdot15\cdot20}+...$

$$\frac{3\cdot5}{1\cdot2}\left(\frac{1}{5}\right)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^3+\frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{5}\right)^4+...$$

Add $1+3\cdot\frac{1}{5}$ on both sides

$$1+\frac{3}{5}+S=1+\frac{3}{1}\left(\frac{1}{5}\right)+\frac{3\cdot5}{1\cdot2}\left(\frac{1}{5}\right)^2+.....$$

$$=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+.....$$

Here $p=3, q=2, \frac{x}{q}=\frac{1}{5}\Rightarrow x=\frac{2}{5}$

$$=(1-x)^{-p/q}$$

$$=\left(1-\frac{2}{5}\right)^{-3/2}=\left(\frac{5}{3}\right)^{3/2}=\frac{5\sqrt{5}}{3\sqrt{3}}$$

$$\Rightarrow \frac{8}{5}+S=\frac{5\sqrt{3}}{3\sqrt{3}} \Rightarrow S=\frac{5\sqrt{3}}{3\sqrt{3}}-\frac{8}{5}$$

29. If x = $\frac{1}{5}+\frac{1\cdot3}{5\cdot10}+\frac{1\cdot3\cdot5}{5\cdot10\cdot15}+...\infty$, find $3x^2$ + 6x.

Sol. Given that

$$x=\frac{1}{5}+\frac{1\cdot3}{5\cdot10}+\frac{1\cdot3\cdot5}{5\cdot10\cdot15}+......$$

$$=\frac{1}{5}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{5}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^3+......$$

$$\Rightarrow 1+x=1+1\cdot\frac{1}{5}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{5}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^3+...$$

$$=1+\frac{p}{1!}\frac{1}{5}+\frac{p(p+q)}{2!}\left(\frac{1}{5}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{1}{5}\right)^3=(1-x)^{-p/q}$$

Here $p=1, q=2, \frac{x}{q}=\frac{1}{5} \Rightarrow x=\frac{2}{5}$

$$=\left(1-\frac{2}{5}\right)^{-1/2}=\left(\frac{3}{5}\right)^{-1/2}=\sqrt{\frac{5}{3}}$$

$$\Rightarrow 1+x=\sqrt{\frac{5}{3}} \Rightarrow 3(1+x)^2=5$$

$$\Rightarrow 3x^2+6x+3=5 \Rightarrow 3x^2+6x=2$$

30. Find an approximate value of $\sqrt[6]{63}$ correct to 4 decimal places.

Sol. $\sqrt[6]{63} = (63)^{1/6} = (64-1)^{1/6}$

$$= (64)^{1/6}\left(1-\frac{1}{64}\right)^{1/6}$$

$$= 2\left[1-(0.5)^6\right]^{1/6}$$

$$= 2\left[i-\frac{\left(\frac{1}{6}\right)(0.5)^6}{1!}+\frac{\left(\frac{1}{6}\right)\left(\frac{1}{6}-1\right)}{2!}(0.5)^{12}+\ldots\right]$$

$$= 2[1-0.0026041] = 2[0.9973959]$$

$$= 1.9947918 = 1.9948 \text{ (correct to 4 decimals)}$$

31. If |x| is so small that x^2 and higher powers of x may be neglected, then find an approximate values of $\dfrac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}$.

Sol. $\dfrac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}$

$$= \left(1+\frac{3x}{2}\right)^{-4}\left[8\left(1+\frac{9}{8}x\right)\right]^{1/3}(1+2x)^{-2}$$

$$= \left(1+\frac{3x}{2}\right)^{-4}\cdot 8^{1/3}\left(1+\frac{9}{8}x\right)^{1/3}(1+2x)^{-2}$$

$$= 2\left[1-\frac{4}{1}\left(\frac{3x}{2}\right)\right]\left[1+\frac{1}{3}\left(\frac{9x}{8}\right)\right][1+(-2)(2x)]$$

$\because x^2$ and higher powers of x are neglecting

$$= 2(1-6x)\left(1+\frac{3x}{8}\right)(1-4x)$$

$$= 2\left(1-6x+\frac{3x}{8}\right)(1-4x)$$

($\because x^2$ and higher powers of x are neglecting)

$$=2\left(1-\frac{45}{8}x\right)(1-4x)=2\left(1-4x-\frac{45}{8}x\right)$$

$\because x^2$ and higher powers of x are neglecting

$$=2\left(1-\frac{77}{8}x\right)$$

$$\therefore \frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}=2\left(1-\frac{77}{8}x\right)$$

32. If |x| is so small that x^4 and higher powers of x may be neglected, then find the approximate value of $\sqrt[4]{x^2+81}-\sqrt[4]{x^2+16}$.

Sol. $\sqrt[4]{x^2+81}-\sqrt[4]{x^2+16}$

$$=(81+x^2)^{1/4}-(16+x^2)^{1/4}$$

$$=(81+x^2)^{1/4}-(16+x^2)^{1/4}$$

$$=\left[81\left(1+\frac{x^2}{81}\right)\right]^{1/4}-\left[16\left(1+\frac{x^2}{16}\right)\right]^{1/4}$$

$$=3\left(1+\frac{x^2}{81}\right)^{1/5}-2\left(1+\frac{x^2}{16}\right)^{1/4}$$

$$=3\left(1+\frac{1}{4}\cdot\frac{x^2}{81}\right)-2\left(1+\frac{1}{4}\cdot\frac{x^2}{16}\right)$$

$$=3+\frac{3}{4}\cdot\frac{x^2}{81}-2-\frac{2}{4}\frac{x^2}{16}=1+\left(\frac{1}{108}-\frac{1}{32}\right)x^2$$

$$=1-\frac{19}{864}x^2 \text{ (After neglecting } x^4 \text{ and higher powers of x)}$$

$$\therefore \sqrt[4]{x^2+81}-\sqrt[4]{x^2+16}=1-\frac{19}{864}x^2$$

33. Suppose that x and y are positive and x is very small when compared to y. Then find the approximate value of $\left(\frac{y}{y+x}\right)^{3/4}-\left(\frac{y}{y+x}\right)^{4/5}$.

Sol. $\left(\frac{y}{y+x}\right)^{3/4}-\left(\frac{y}{y+x}\right)^{4/5}$

$$=\left(\frac{y}{y\left(1+\frac{x}{y}\right)}\right)^{3/4}-\left(\frac{y}{y\left(1+\frac{x}{y}\right)}\right)^{4/5}$$

$$=\left(1+\frac{x}{y}\right)^{-3/4}-\left(1+\frac{x}{y}\right)^{-4/5}$$

$$=\left\{1+\left(\frac{-3}{4}\right)\left(\frac{x}{y}\right)+\frac{\left(-\frac{3}{4}\right)\left(\frac{-3}{4}-1\right)}{1\cdot 2}\left(\frac{x}{y}\right)^2+\ldots\right\}$$

$$-\left\{1+\left(\frac{-4}{5}\right)\left(\frac{x}{y}\right)+\frac{\left(\frac{-4}{5}\right)\left(\frac{-4}{5}-1\right)}{1\cdot 2}\left(\frac{x}{y}\right)^2+\ldots\right\}$$

(By neglecting $(x/y)^3$ and higher powers of x/y

$$=\left[1-\frac{3}{4}\left(\frac{x}{y}\right)-\frac{21}{32}\left(\frac{x}{y}\right)^2\right]-\left[1-\frac{4}{5}\left(\frac{x}{y}\right)+\frac{18}{25}\left(\frac{x}{y}\right)^2\right]$$

$$=\left(\frac{4}{5}-\frac{3}{4}\right)\frac{x}{y}-\left(\frac{21}{32}+\frac{18}{25}\right)\left(\frac{x}{y}\right)^2$$

$$=\frac{1}{20}\left(\frac{x}{y}\right)-\frac{1101}{800}\left(\frac{x}{y}\right)^2$$

34. Expand $5\sqrt{5}$ in increasing power of $\frac{4}{5}$.

Sol. $5\sqrt{5} = 5^{3/2} = \left(\frac{1}{5}\right)^{-3/2}$

$$= \left(1 - \frac{4}{5}\right)^{-3/2}$$

$$= 1 + \frac{\left(\frac{3}{2}\right)}{1!}\left(\frac{4}{5}\right) + \frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\left(\frac{4}{5}\right)^2 + \ldots + \frac{\frac{3}{2}\cdot\frac{5}{2}\ldots\left(\frac{3}{2}+r-1\right)}{r!}\left(\frac{4}{5}\right)^r + \ldots\infty$$

$$= 1 + \frac{3}{1!2}\frac{4}{5} + \frac{3\cdot5}{2!2^2}\left(\frac{4}{5}\right)^2 + \ldots + \frac{3\cdot5\ldots(2r-1)}{r!2^r}\left(\frac{4}{5}\right)^r + \ldots$$

35. Find the sum of the infinitive terms

$$\frac{5}{6\cdot12} + \frac{5\cdot8}{6\cdot12\cdot18} + \frac{5\cdot8\cdot11}{6\cdot12\cdot18\cdot24} + \ldots\infty$$

Sol. Let S = $\frac{5}{6\cdot12} + \frac{5\cdot8}{6\cdot12\cdot18} + \frac{5\cdot8\cdot11}{6\cdot12\cdot18\cdot24} + \ldots$

$$\Rightarrow 2S = \frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3 + \frac{2\cdot5\cdot8\cdot11}{1\cdot2\cdot3\cdot4}\left(\frac{1}{6}\right)^4 + \ldots\ldots$$

$$\Rightarrow 1 + \frac{2}{1}\left(\frac{1}{6}\right) + 2S = 1 + \frac{2}{1}\left(\frac{1}{6}\right) + \frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3 + \ldots\ldots$$

$$\Rightarrow \frac{4}{3} + 2S = 1 + \frac{2}{1}\left(\frac{1}{6}\right) + \frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3 + \ldots\ldots$$

Comparing $\frac{4}{3} + 2S$ with $(1-x)^{-p/q}$

$$= 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2 + \ldots$$

Here p = 2, q = 3, $\frac{x}{q} = \frac{1}{6} \Rightarrow x = \frac{q}{6} = \frac{3}{6} = \frac{1}{2}$

$$\therefore \frac{4}{3}+2S=(1-x)^{-p/q}=\left(1-\frac{1}{2}\right)^{-2/3}$$

$$=\left(\frac{1}{2}\right)^{-2/3}=(2)^{2/3}=\sqrt[3]{4}$$

$$\therefore 2S=\sqrt[3]{4}-\frac{4}{3}\Rightarrow S=\frac{\sqrt[3]{4}}{2}-\frac{2}{3}=\frac{1}{\sqrt[3]{2}}-\frac{2}{3}$$

$$\therefore \frac{5}{6\cdot 12}+\frac{5\cdot 8}{6\cdot 12\cdot 18}+\frac{5\cdot 8\cdot 11}{6\cdot 12\cdot 18\cdot 24}+\ldots=\frac{1}{\sqrt[3]{2}}-\frac{2}{3}$$

36. If the coefficients of x^9, x^{10}, x^{11} in the expansion of $(1+x)^n$ are in A.P. then prove that $n^2-41n+398=0$.

Sol: Coefficient of x^r in the expansion $(1 – x)^n$ is ${}^nC_r$.

Given coefficients of x^9, x^{10}, x^{11} in the expansion of $(1 – x)^n$ are in A.P., then

$$2({}^nC_{10})={}^nC_9+{}^nC_{11}$$

$$\Rightarrow 2\frac{n!}{(n-10)!10!}=\frac{n!}{(n-9)!9!}+\frac{n!}{(n-11)!+11!}$$

$$\Rightarrow \frac{2}{10(n-10)}=\frac{1}{(n-9)(n-10)}+\frac{1}{11\times 10}$$

$$\Rightarrow \frac{2}{(n-10)10}=\frac{110+(n-9)(n-10)}{110(n-9)(n-10)}$$

$$\Rightarrow 22(n-9)=110+n^2-19n+90$$

$$\Rightarrow n^2-41n+398=0$$

37. **Find the number of irrational terms in the expansion of** $(5^{1/6}+2^{1/8})^{100}$**.**

Sol: Number of terms in the expansion of $(5^{1/6}+2^{1/8})^{100}$ are 101.

General term in the expansion of $(x + y)^n$ is

$T_{r+1} = {}^nC_r x^{n-r} \cdot y^r$.

$\therefore$ General term in the expansion of $(5^{1/6}+2^{1/8})^{100}$ is

$$T_{r+1} = {}^{100}C_r \cdot (5^{1/6})^{100-r} \cdot (2^{1/8})^r$$

$$= {}^{100}C_r \cdot 5^{\frac{100-r}{6}} \cdot 2^{\frac{r}{8}}$$

For T_{r+1} to be a rational.

Clearly 'r' is a multiple of 8 and 100 – r is a multiple of 6.

$\therefore$ r = 16, 40, 64, 88.

Number of rational terms are 4.

$\therefore$ Number of irrational terms are 101 – 4 = 97.

38. **If** $t = \frac{4}{5}+\frac{4.6}{5.10}+\frac{4.6.8}{5.10.15}+...\infty$**, then prove than 9t = 16.**

Sol: Given

$$t = \frac{4}{5}+\frac{4.6}{5.10}+\frac{4.6.8}{5.10.15}+...\infty$$

$$\Rightarrow 1+t = 1+\frac{4}{5}+\frac{4.6}{5.10}+\frac{4.6.8}{5.10.15}+...\infty$$

$$\Rightarrow 1+t = 1+\frac{4}{1!}\left(\frac{1}{5}\right)+\frac{4.6}{2!}\left(\frac{1}{5}\right)^2+\frac{4.6.8}{3!}\left(\frac{1}{5}\right)^3+...\infty \quad ...(1)$$

We know that

$$1+\frac{p}{1!}\left(\frac{x}{p}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{p}\right)^2+$$

$$\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{p}\right)^3+...\infty = (1-x)^{-p/q}$$

Here p = 4, p + q = 6, $\frac{x}{q} = \frac{1}{5}$

$\Rightarrow q = 2 \Rightarrow x = \frac{2}{5}$

$\therefore 1 + t = \left(1 - \frac{2}{5}\right)^{\frac{-4}{2}}$

$\Rightarrow 1 + t = \left(\frac{3}{5}\right)^{-2}$

$\Rightarrow 1 + t = \left(\frac{5}{3}\right)^{2} = \frac{25}{9}$

$\Rightarrow 9t = 16$.