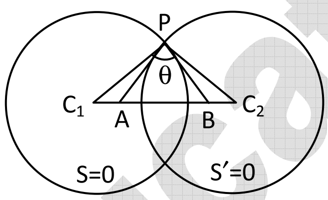
SYSTEM OF CIRCLES

Theorem:

If d is the distance between the centers of two intersecting circles with radii r_1, r_2 and θ is the angle between the circles then $\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$.

Proof:

Let C_1, C_2 be the centre s of the two circles S = 0, S′ = 0 with radii r_1, r_2 respectively. Thus $C_1C_2 = d$. Let P be a point of intersection of the two circles. Let PB, PA be the tangents of the circles S = 0, S′ = 0 respectively at P.



Now $PC_1 = r_1$, $PC_2 = r_2$, $\angle APB = \theta$

Since PB is a tangent to the circle S = 0, $\angle C_1PB = \pi/2$

Since PA is a tangent to the circle S′ = 0, $\angle C_2PA = \pi/2$

Now $\angle C_1PC_2 = \angle C_1PB + \angle C_2PA - \angle APB = \pi/2 + \pi/2 - \theta = \pi - \theta$

From ΔC_1PC_2, by cosine rule,

$$C_1^2C_2^2 = PC_1^2 + PC_2^2 - 2PC_1 \cdot PC_2 \cos\angle C_1PC_2 \Rightarrow d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\pi - \theta) \Rightarrow d^2 = r_1^2 + r_2^2 + 2r_1r_2\cos\theta$$

$$\Rightarrow 2r_1r_2\cos\theta = d^2 - r_1^2 - r_2^2 \Rightarrow \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

Corollary:

If θ is the angle between the circles $x^2 + y^2 + 2gx + 2fy + c = 0$, $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ then $\cos\theta = \frac{c + c' - 2(gg' + ff')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$.

Proof:

Let C_1, C_2 be the centre s and r_1, r_2 be the radii of the circles S = 0, S′ = 0 respectively and $C_1C_2 = d$.

$\therefore$ $C_1 = (-g, -f)$, $C_2 = (-g', -f')$,

$r_1 = \sqrt{g^2 + f^2 - c}$, $r_2 = \sqrt{g'^2 + f'^2 - c'}$

Now $\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2} = \frac{(g - g')^2 + (f - f')^2 - (g^2 + f^2 - c) - (g'^2 + f'^2 - c')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$

$= \frac{g^2 + g'^2 - 2gg' + f^2 + f'^2 - 2ff' - g^2 - f^2 + c - g'^2 - f'^2 + c'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$

$= \frac{c + c' - 2(gg' + ff')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$

Note: Let d be the distance between the centers of two intersecting circles with radii r_1, r_2. The two circles cut orthogonally if $d^2 = r_1^2 + r_2^2$.

Note: The condition that the two circles

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ may cut each other orthogonally is $2gg' + 2ff' = c + c'$.

Proof: Let C_1, C_2 be the centers and r_1, r_2 be the radii of the circles S = 0, S′ = 0 respectively.

$\therefore$ $C_1 = (-g, -f)$, $C_2 = (-g', -f')$

$r_1 = \sqrt{g^2 + f^2 - c}$, $r_2 = \sqrt{g'^2 + f'^2 - c'}$

Let P be point of intersection of the circles.

The two circles cut orthogonally at P

$\Leftrightarrow \angle C_1PC_2 = 90^0$.

$$\Rightarrow C_1C_2^2 = C_1P^2 + C_2P^2 \Leftrightarrow (g - g')^2 + (f - f')^2 = r_1^2 + r_2^2$$

$$\Leftrightarrow g^2 + g'^2 - 2gg' + f^2 + f'^2 - 2ff' = g^2 + f^2 - c + g'^2 + f'^2 + c'$$

$$\Leftrightarrow -(2gg' + 2ff') = -(c + c') \Rightarrow 2gg' + 2ff' = c + c'$$

Note:

1. The equation of the common chord of the intersecting circles $s = 0$ and $s^1 = 0$ is $s - s^1 = 0$.

2. The equation of the common tangent of the touching circles $s = 0$ and $s^1 = 0$ is $s - s^1 = 0$

3. If the circle s=0 and the line L =0 are intersecting then the equation of the circle passing through the points of intersection of s=0 and L=0 is $S + \lambda L = 0$.

4. The equation of the circle passing through the point of intersection of S=0 and $S^1 = 0$ is

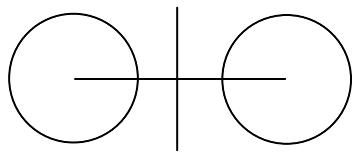
$S + \lambda S^1 = 0$.

Theorem: The equation of the radical axis of the circles S= 0, S′ = 0 is S – S′ = 0.

Theorem: The radical axis of two circles is perpendicular to their line of centers.

Proof:

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ be the given circles.



The equation of the radical axis is S – S′ = 0

$$\Rightarrow 2(g - g')x + 2(f - f')y + (c - c') = 0$$

$\Rightarrow a_1x + b_1y + c_1 = 0$ where

$a_1 = 2(g - g')$, $b_1 = 2(f - f')$, $c_1 = c - c'$

The centers of the circles are (–g,–f), (–g′,–f′)

The equation to the line of centers is:

$(x + g)(f - f') = (y + f)(g - g')$

$\Rightarrow (f - f')x - (g - g')y - gf' + fg' = 0$

$\Rightarrow a_2x + b_2y + c_2 = 0$ where

$a_2 = f - f'$, $b_2 = -(g - g')$, $c_2 = fg' - gf'$

Now $a_1a_2+b_1b_2=2(g - g')(f-f')-2(f-f')(g - g') = 0$.

Very Short Answer Questions

1. Find 'k' if the following pair of circles are orthogonal.

i) $x^2+y^2 + 2by-k = 0$, $x^2+y^2+2ax+8=0$

Sol. Given circles are $x^2+y^2 + 2by-k = 0$, $x^2+y^2+2ax+8=0$

From above equations $g_1 = 0$; $f_1 = b$; $c_1 = -k$

$g_2 = a$; $f_1 = 0$; $c_1 = 8$

Since the circles are orthogonal,

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2(0)(a) + 2(b)(0) = -k + 8$

$0 = -k + 8$

$K = 8$

ii) $x^2 + y^2 + 6x - 8y + 12 = 0$;

$x^2 + y^2 - 4x + 6y + k = 0$

Ans. $\boxed{k = -24}$

2 Find the angle between the circles given by the equations

i) $x^2 + y^2 - 12x - 6y + 41 = 0$; $x^2 + y^2 - 4x + 6y - 59 = 0$

Sol. $x^2 + y^2 - 12x - 6y + 41 = 0$

Centre $C_1 = (6, 3)$ radius $r_1 = \{36 + 9 - 41\}^{1/2} = 2$

$x^2 + y^2 - 4x + 6y - 59 = 0$, then centre $C_2 = (-2,-3)$

Radius $r_2 = \{4 + 9 - 59\}^{1/2} = \{72\}^{1/2} = 6\sqrt{2}$

$C_1C_2 = d = \sqrt{(6+2)^2 + (3+3)^2} = \sqrt{64 + 36} = 10$

Let θ be the angle between the circles, then $\cos\theta = \frac{d^2 - r_1^2 - r_1^2}{2r_1r_2} = \frac{100 - 4 - 72}{2 \times 2\sqrt{72}} = \frac{24}{4 \times 6\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\theta = 45°$

ii) $x^2 + y^2 + 6x - 10y - 135 = 0$; $x^2 + y^2 - 4x + 14y - 116 = 0$.

Ans. $\theta = \frac{2\pi}{3}$

3. Show that the angle between the circles $x^2 + y^2 = a^3$ and $x^2 + y^2 = ax$ is $\frac{3\pi}{2}$

Sol. Equations of the circles are

$S = x^2 + y^2 - a^2 = 0,$

$S^1 = x^2 + y^2 - ax - ay = 0$

$C_1\ (0, 0),\ C_2\left(\frac{a}{2}, \frac{a}{2}\right)$

$\therefore C_1C_2^2 = \left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{a}{2}\right)^2$

$\frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2} = d^2$

$r_1 = a,\quad r_2 = \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \sqrt{\frac{a^2}{4}} = \frac{a}{\sqrt{2}}$

$\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2} = \frac{\frac{a^2}{2} - a^2 - \frac{a^2}{2}}{2.a\frac{a}{\sqrt{2}}} = \frac{-a^2}{\sqrt{2}.a^2} = \frac{-1}{\sqrt{2}} = \cos\frac{3\pi}{4}$ Therefore, $\theta = \frac{3\pi}{4}$

4. Show that circles given by the following equations intersect each other orthogonally.

i). $x^2 + y^2 + 6x - 8y + 12 = 0$; $x^2 + y^2 - 4x + 6y + k = 0$

Sol. Given circles are

$x^2 + y^2 + 6x - 8y + 12 = 0$; $x^2 + y^2 - 4x + 6y + k = 0$from above circles,

$g = -1,\ f = -1,\ c = -7$, $g^1 = \frac{-4}{3}, f^1 = \frac{29}{6}$; $c^1 = 0$. Therefore, $c + c^1 = -7 + 0$

$2gg^1 + 2ff^1 = -2(-1)\left(\frac{-4}{3}\right) + 2(-1)\frac{29}{6} = \frac{8}{3} - \frac{29}{3} = \frac{-21}{3} = -7$

Therefore, $2gg^1 + 2ff^1 = c + c^1$

Hence the given circles cut each other orthogonally.

Hence both the circles cut orthogonally.

ii) $x^2 + y^2 + 2lx + 4 = 0;\ x^2 + y^2 + 2my - g = 0$

Sol. Given circles $x^2 + y^2 + 2lx + 4 = 0;\ x^2 + y^2 + 2my - g = 0$ from these equations,

$g_1 = -l;\ f_1 = 0,\ c_1 = g,\ g_2 = 0,\ f_2 = m,\ c_2 = -g$

Now $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2(-l)(0) + 2(0)(m) = g - g$

$0 = 0$ ∴Two circles are orthogonal.

5. Find the equation of the radical axis of the following circles.

i) $x^2 + y^2 - 3x - 4y + 5 = 0,\ 3(x^2 + y^2) - 7x + 8y + 11 = 0$

Sol. let $S \equiv x^2 + y^2 - 3x - 4y + 5 = 0$

$$S' = x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3} = 0$$

Radical axis is $S - S' = 0$

$$(x^2 + y^2 - 3x - 4y + 5) - \left(x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3}\right) = 0$$

$$-\frac{2}{3}x - \frac{20}{3}y + \frac{4}{3} = 0 \Rightarrow x + 10y - 2 = 0$$

ii) $x^2 + y^2 + 2x + 4y + 1 = 0,\ x^2 + y^2 + 4x + y = 0.$ **Ans.** $2x - 3y + 1 = 0$

iii) $x^2 + y^2 + 4x + 6y - 7 = 0, 4(x^2 + y^2) + 8x + 12y - 9 = 0.$ **Ans.** $8x + 12y - 19 = 0$

iv) $x^2 + y^2 - 2x - 4y - 1 = 0,\ x^2 + y^2 - 4x - 6y + 5 = 0$ **Ans.** $x + y - 3 = 0$

6. Find the equation of the common chord of the following pair of circles.

i) $x^2 + y^2 - 4x - 4y + 3 = 0,\ \ x^2 + y^2 - 5x - 6y + 4 = 0$

Sol. $S = x^2 + y^2 - 4x - 4y + 3 = 0$

$S^1 = x^2 + y^2 - 5x - 6y + 4 = 0$

Common chord is $S - S' = 0$

$(x^2 + y^2 - 4x - 4y + 3) - (x^2 + y^2 - 5x - 6y + 4) = 0$

$x + 2y - 1 = 0$

Short Answer Questions

1 **Find the equation of the circle which passes through the origin and intersects the circles given by the following equations orthogonally.**

i) $x^2 + y^2 – 4x + 6y + 10 = 0$, $x^2 + y^2 + 12y + 6 = 0$

Sol. Let equation of the circle be

$x^2+ y^2 + 2gx + 2fy + c = 0$ ----(1)

Above circle is passing through (0, 0)

$0+0+0+0+c = 0 \therefore c=0.$

Circle (1) is orthogonal to

$x^2+ y^2 – 4x + 67 + 10 = 0$ then

$2gg^1 + 2ff^1 = c + c^1$

$2g (-2) + 2f (3) = 0 + 10$

$-4g + 6f = 10$ ----- (2)

Circle (1) is orthogonal to

$x^2+ y2 + 12y + 6 = 0$

$\therefore 2g (0) + 2f (6) = 6 + 0$

$12f = 6$------ (3) $\Rightarrow f = \frac{1}{2}$

From (2) and (3),

$-4g + 6 \frac{1}{2} = 10$

$-4g = 10 – 3 \Rightarrow g = -\frac{7}{4}$

$\therefore$ Equation of circle is

$x^2 + y^2 - \frac{7}{2} x + y = 0 \Rightarrow 2x^2 + 2y^2 –7x + 2y = 0.$

2. Find the equation of the circle which passes through the point (0, -3) and intersects the circles given by the equations $x^2 + y^2 – 6x+3y+5 = 0$, $x^{2+} + y^2 – x – 7y = 0$ orthogonally.

Sol. Let circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$-------(1)

(1) is orthogonal to $x^2 + y^2 – 6x+3y+5 = 0$

$$\therefore 2g(-3) + 2f\left(\frac{+3}{2}\right) = c + 5$$

$–6g + 3f = c + 5$----------- (2)

(1) is orthogonal to $x^2 + y^2 – x – 7y = 0$

$$\therefore\ 2g\left(\frac{+1}{2}\right) + 2f\left(\frac{+7}{2}\right) = c$$

$–g –7f = c$ ---- (3)

Given (1) is passing through (0, –3)

$0 + 9 – 6f + c = 0$

(3) – (2)

$5g – 10f = – 5 \Rightarrow g – 2f = – 1$

(iii) + (iv)

$9 – g – 13f = 0 \Rightarrow g + 13f = 9$

$\underline{g – 2f = –1}$

$15f = 10$

$f = \frac{2}{3} \Rightarrow g = 2.\frac{2}{3} – 1 \Rightarrow g = + \frac{1}{3}$

$\Rightarrow 9 – 6.\frac{2}{3} + c = 0 \Rightarrow c = – 5$

Therefore, equations of the circles are

$x^2 + y^2 + \frac{4}{3}y + \frac{4}{3}\ x – 5 = 0$

(Or) $3x^2 + 3y^2 + 2x + 4y – 15 = 0$

(3) Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally?

Sol. Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$

S=0 is passing through (0, 0)

$\Rightarrow 0 + 0 + 2g.0 + 2f.0 + c = 0 \Rightarrow c = 0$

$x^2 + y^2 + 2gx + 2fy = 0$

Centre (-g,-f) is on $x + y = 4$

$\therefore -g - f = 4$------------ (1)

S=0 is orthogonal to

$x^2 + y^2 - 4x + 2y + 4 = 0$

$\Rightarrow -4g + 2f = 4 + 0$

$\Rightarrow f - 2g = 2$ -------(2)

Solving (1) and (2) we get

$-3g = 6 \Rightarrow g = -2$

$f = -2$

Equation of circle is $x^2 + y^2 - 4x - 4y = 0$

4. Find the equation of the circle which passes through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$

Sol. Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$

S=0 is passing through (2, 0), (0, 2),

$\Rightarrow 4 + 0 + 4g + c = 0$ ---- (1)

And $0 + 4 + 4f + c = 0$ ----- (2)

(1) –(2) $\Rightarrow$ f-g =0 $\Rightarrow g = f$

S=0 is orthogonal to $x^2 + y^2 + \frac{5}{2}x - \frac{6}{2}y + 2 = 0$

$\Rightarrow 2g\left(\frac{5}{4}\right) + 2f\left(-\frac{3}{2}\right) = 2 + c$

$\frac{5}{2}g - 3f = 2 + c$

But $g = f \Rightarrow \frac{5}{2}g - 3g = 2 + c$

$\Rightarrow -g = 4 + 2c$

Putting value of g in equation (1)

$-16-8c + c = -4 \Rightarrow c = -\frac{12}{7}$

$\Rightarrow -g\ 4 - \frac{24}{7} = +\frac{4}{7}$

Equation of the circle is

$x^2 + y^2 - \frac{8x}{7} + \frac{8y}{7} + \frac{12}{7} = 0$

$\Rightarrow 7(x^2 + y^2) - 8x - 8y - 12 = 0$

(5) Find the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 2y - 7 = 0$ and having centre at (2, 3)

Sol. Given circle is

$x^2 + y^2 - 4x + 2y - 7 = 0$ -----(1)

Let the required circle be

$S = x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f) = (2,3)$ given

$g = -2, f = -3$

Circles (1) and S=0 are cutting each other orthogonally.

$\Rightarrow 2gg^1 + 2ff^1 = c + c^1$

$2(-2)(-2) + 2(-3)(1) = -7 + c$

$\Rightarrow 8 - 6 = -7 + c \Rightarrow +2 = -7 + c$

$c = 7 + 2 = 9 \Rightarrow c = 9$

Hence the required circle is,

$x^2 + y^2 - 4x - 6y + 9 = 0$

6. Find the equation of the common tangent of the following circles at their point of contact.

i) $x^2 + y^2 + 10x - 2y + 22 = 0$, $x^2 + y^2 + 2x - 8y + 8 = 0$.

Sol. $S = x^2 + y^2 + 10x - 2y + 22 = 0$

Centre A = (-5, 1), radius $r_1 = 2$

$S' = x^2 + y^2 + 2x - 8y + 8 = 0$.

Centre B = (-1, 4) radius $r_2 = 3$

$AB = \sqrt{16+9} = 5$

Therefore AB =5 = 3+2 = r_1+r_2.

Given circles touch each other externally.

When circles touch each other, their common tangent is S – S′ = 0

$\therefore (x^2 + y^2 + 10x - 2y + 22) - (x^2 + y^2 + 2x - 8y + 8) = 0$

$8x + 6y + 14 = 0$ (or) $4x + 3y + 7 = 0$

ii) $x^2 + y^2 - 8y - 4 = 0, x^2 + y^2 - 2x - 4y = 0.$

Ans. $x - 2y - 2 = 0$

7. Show that the circles $x^2 + y^2 - 8x - 2y + 8 = 0$ and $x^2 + y^2 - 2x + 6y + 6 = 0$ touch each other and find the point of contact?

Sol. $S = x^2 + y^2 - 8x - 2y + 8 = 0$

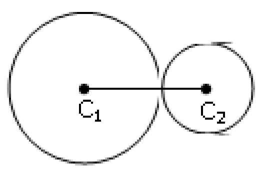
$C_1 = (4, 1)\,;\, r_1 = \sqrt{16+1-8} = 3$

$S^1 = x^2 + y^2 - 2x + 6y + 6 = 0$

$C_2 = (1, -3),\ r_2 = \sqrt{1+9-6} = 2$

$C_1C_2 = \sqrt{(4-1)^2 + (1+3)^2} = 5$

$r_1 + r_2 = C_1 + C_2$ they touch each other externally



The point of contact divides the centre of circles in the ratio $r_1 : r_2$ internally.

Point of contact is

$$= \left(\frac{3(1)+2(4)}{3+2}, \quad \frac{3(-3)+2(1)}{3+2}\right) = (11/5, -7/5)$$

$\therefore$ Point of contact is $\left(\frac{11}{5}, \frac{-7}{5}\right)$.

8. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other then show that $f'g = fg'$.

Sol. $S = x^2 + y^2 + 2gx + 2fy = 0$

Centre $C_1 = (-g, -f)$, radius $r_1 = \sqrt{g^2 + f^2}$

$S^1 = x^2 + y^2 + 2g'x + 2f'y = 0$

$C_2 = (-g', -f')$, $r_2 = \sqrt{g'^2 + f'^2}$

Given circles are touching circles,

$\therefore\ C_1C_2 = r_1 + r_2$

$\Rightarrow\ (C_1C_2)^2 = (r_1 + r_2)^2$

$$(g' - g)^2 + (f' - f)^2 = g^2 + f^2 + g'^2 + f'^2 + 2\sqrt{g^2 + f^2}\sqrt{g'^2 + f'^2}$$

$$-2(gg' + ff') = 2\{g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2\}^{1/2}$$

$$\Rightarrow (gg' + ff')^2 = g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2$$

$$g^2g'^2 + f^2f'^2 + 2gg'ff' = g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2$$

$$\Rightarrow 2gg'ff' = g^2f'^2 + f^2g'^2$$

$$\Rightarrow g^2f'^2 + f^2g'^2 - 2gg'ff' = 0$$

$$\Rightarrow (gf' - fg')^2 = 0 \Rightarrow gf' = fg'$$

9. Find the radical centre of the following circles

i) $x^2 + y^2 - 4x - 6y + 5 = 0$, $x^2 + y^2 - 2x - 4y - 1 = 0$, $x^2 + y^2 - 6x - 2y = 0$

Sol. Given circles

$S = x^2 + y^2 - 4x - 6y + 5 = 0$

$S' = x^2 + y^2 - 2x - 4y - 1 = 0$

$S'' = x^2 + y^2 - 6x - 2y = 0$

Radical axis 0f S =0 **And** S'=0 is S-S' =0

$\Rightarrow\ -2x - 2y + 6 = 0$

$\Rightarrow\ x + y - 3 = 0 \qquad \ldots (1)$

R.A. of S'=0 and S" =0 is S' –S" =0

$\Rightarrow$ 4x – 2y – 1 = 0 … (2)

Solving (1) and (2),

$x = 7/6,\ y = \frac{11}{6}$

Radical centre is (7/6, 11/6).

ii) $x^2 + y^2 + 4x - 7 = 0,\ 2x^2 + 2y^2 + 3x + 5y - 9 = 0,\ x^2 + y^2 + y = 0.$

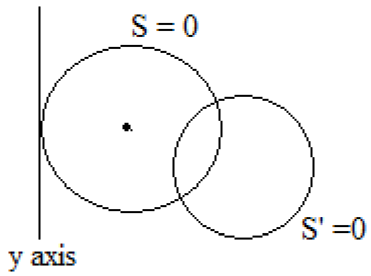
Ans. P (2, 1)

Long Answer Questions

1 **Find the equation of the circle which intersects the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ orthogonally and passes through the point (3, 0) and touches the Y–axis.**

Sol. Let (h, k) be the centre of the circle.

Since the circle is touching the y axis, therefore radius is |h|



Therefore equation of the circle is

$(x - h)^2 + (y - k)^2 = h^2$

$S = x^2 - 2hx + y^2 - 2ky + k^2 = 0$

S=0 is Passing through (3, 0),

$\Rightarrow 9 - 6h + k^2 = 0$ — (i)

S=0 is Orthogonal to $x^2 + y^2 - 6x + 4y - 3 = 0$

$\Rightarrow 2(-h)(-3) + 2(-k)(2) = -3 + k^2$

$\Rightarrow 6h - 4k = -3 + k^2$

$\Rightarrow 6h - 4k + 3 - k^2 = 0$----- (2)

(1) + (2) $\Rightarrow 12 - 4k = 0 \Rightarrow k = 3$

$\Rightarrow h = 3,$

Equation of circle be

$y^2 + x^2 - 6x - 6y + 9 = 0$

(2) Find the equation of the circle which cuts the circles $x^2 + y^2 – 4x – 6y + 11 = 0$, $x^2 + y^2 – 10x – 4y + 21 = 0$ orthogonally and has the diameter along the straight line $2x + 3y = 7$.

Sol. Let circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$

S=0 is Orthogonal to $x^2 + y^2 – 4x – 6y + 11 = 0$, $x^2 + y^2 – 10x – 4y + 21 = 0$

$\Rightarrow$ $2g(–2) + 2f(–3) = 11 + c$ —— (1)

$\Rightarrow$ $2g(–5) + 2f(–2) = 21 + c$ —— (2)

(1) –(2) $\Rightarrow –6g + 2f = 10$—— (3)

Centre (-g,-f) is on $2x + 3y = 7$,

$\therefore –2g –3f = 7$ —— (4)

Solving (3) and (4)

$f = –1, g = -2$,

Sub. These values in (1), then c = 3

Equation of circle $x^2 + y^2 – 4x –2y +3 = 0$

3) If P,Q are conjugate points with respect to a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then prove that the circle PQ as diameter cuts the circles S = 0 orthogonally.

Sol. Equation of the circle is

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

Let $P = (x_1,y_1)$, $Q (x_2,y_2)$ be the conjugate

Points w.r.t. the circle S=0.

Since $P = (x_1,y_1)$, $Q (x_2,y_2)$ are conjugate points, $S_{12} = 0$.

$\Rightarrow x_1x_2 + y_1y_2 + g(x_1+x_2) + f(y_1+y_2) + c = 0$

$\Rightarrow x_1x_2 + y_1y_2 + c = - g(x_1+x_2) - f(y_1+y_2)$----(1)

Equation of the circle on PQ as diameter is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

$\Rightarrow S^1 = x^2 + y^2 – (x_1+x_2)x – (y_1+y_2)y + (x_1x_2+y_1y_2) = 0$.

Given S=0 and $S^1 = 0$ are orthogonal,

$2g_1g_2 + 2f_1f_2$

$= 2g\left[-\left(\frac{x_1+x_2}{2}\right)\right] + 2f\left[-\left(\frac{y_1+y_2}{2}\right)\right]$

$= -g(x_1+x_2) - f(y_1+y_2) = x_1x_2 + y_1y_2 + c$

And $C + c^1 = x_1x_2 + y_1y_2 + c$

$\therefore 2g_1 g_2 + 2f_1 f_2 = C + c^1$

Hence circles are orthogonal to each other.

4) If the equation of two circles whose radii are a, a^1 be S = 0, S^1=0, then show that the circles $\frac{S}{a} \pm \frac{S'}{a'} = 0$ will interest orthogonally.

5) Find the equation of the circle which intersects each of the following circles orthogonally $x^2 + y^2 + 2x + 4y + 1 = 0$; $x^2 + y^2 - 2x + 6y - 3 = 0$; $2x^2 + 2y^2 + 6x + 8y - 3 = 0$

Sol. Let equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

This circle is orthogonal to

$x^2 + y^2 + 2x + 4y + 1 = 0$; $x^2 + y^2 - 2x + 6y - 3 = 0$; $x^2 + y^2 + 3x + 4y - 3/2 = 0$

$2g(1) + 2f(2) = c + 1$ -(i)

$2g\left(\frac{3}{2}\right) + 2f(2) = c - \frac{3}{2}$ - (ii)

$2g(-1) + 2f(3) = c - 3$ - (iii)

(iii) – (i)

$-5g + 2f = \frac{-3}{2}$ (or) $-10g + 4f = -3$ – (iv)

(iii) – (i)

$-4g + 2f = -4$

$F - 2g = -2$

Solving (iv) and (v) we get

$F = -7, g = -5/2, c = -34$

$\therefore$ Equation of circle be

$x^2 + y^2 - 5x - 14y - 34 = 0$

ii) $x^2 + y^2 + 4x + 2y + 1 = 0$;

$2x^2 + 2y^2 + 8x + 6y - 3 = 0$;

$x^2 + y^2 + 6x - 2y - 3 = 0$.

Ans. $x^2 + y^2 - 5y - 14x - 34 = 0$

6) If the Straight line 2x + 3y = 1 intersects the circle $x^2 + y^2 = 4$ at the points A and B. Find the equation of the circle having AB as diameter.

Sol. Circle is $S = x^2 + y^2 = 4$

Equation of the line is L = 2x + 3y = 1 Equation of circle passing through S=0 and L=0 is $S+\lambda L=0$

$\Rightarrow (x^2 + y^2 - 4) + \lambda(2x + 3y - 1) = 0$

$\Rightarrow x^2 + y^2 + 2\lambda x + 3\lambda y - 4 - \lambda = 0$

$\Rightarrow$ Center $\left(-\lambda, \frac{-3\lambda}{2}\right)$

Centre lies on $2x + 3y - 1 = 0$

$\Rightarrow 2(-\lambda) + 3\frac{-3\lambda}{2} - 1 = 0$

$\Rightarrow \lambda = \frac{-2}{13}$

$\therefore$ Equation of circle be

$13(x^2 + y^2) - 4 \times 13 - 2(2x + 3y - 1) = 0$

$13(x^2 + y^2) - 4x - 6y - 50 = 0.$

7) If x + y = 3 is the equation of the chord AB of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, Find the equation of the circle having AB as diameter.

Ans. $x^2 + y^2 - 6x + 4 = 0$

8) Find the equation of the circle passing through the intersection of the circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ and having its center on the line $\frac{x}{a} - \frac{y}{b} = 2$.

Sol. $S = x^2 + y^2 = 2ax$

$S1 = x^2 + y^2 = 2by$

Equation of circle passes through the point of intersected of S = 0 and $S^1 = 0$ can be

Written as $S + \lambda S^1 = 0$

$\Rightarrow x^2 + y^2 - 2ax + \lambda(x^2 + y^2 - 2by) = 0$

$\Rightarrow x^2(1+\lambda) + y^2(1+\lambda) + x(-2a) - (2b\lambda)y = 0$

$\Rightarrow x^2 + y^2 - \frac{2ax}{1+\lambda} - \frac{2b\lambda y}{1+\lambda} = 0$

Centre C = $\left[\frac{+a}{1+\lambda}, \frac{+b\lambda}{1+\lambda}\right]$

Centre is a point on $\frac{x}{a}-\frac{y}{b}=2$

$\Rightarrow \frac{+a}{a(1+\lambda)}-\frac{b\lambda}{(1+\lambda)b}=2 \Rightarrow 1-\lambda=2(1+\lambda)$

$\Rightarrow \lambda = -1/3$

Equation of circle be

$3x^2 + 3y^2 - 6ax - x^2 - y^2 + 2by = 0$

$\Rightarrow 2x^2 + 2y^2 - 6ax + 2by = 0$

$\Rightarrow x^2 + y^2 - 3ax + by = 0.$

9. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ is the diameter of the second circle also and find its length.

Sol. $S = x^2 + y^2 - 6x - 4y + 9 = 0$, $S' = x^2 + y^2 - 8x - 6y + 23 = 0$

Common chord is S-S" =0

$\Rightarrow (x^2 + y^2 - 6x - 4y + 9) - (x^2 + y^2 - 8x - 6y + 23) = 0$

$\Rightarrow 2x + 2y - 14 = 0$

$\Rightarrow x + y - 7 = 0 \ldots$ (i)

Centre of circle (4, 3)

Substituting (4, 3) in $x + y - 7 = 0$,

We get $4+3-7 = 0 \Rightarrow 0=0.$

(i) is a diameter of S' =0.

Radius is $\sqrt{4^2 + 3^2 - 23} = \sqrt{2} \Rightarrow$ diameter $=2\sqrt{2}$

10. Find the equation and the length of the common chord of the following circles.

i) $x^2 + y^2 + 2x + 2y + 1 = 0$, $x^2 + y^2 + 4x + 3y + 2 = 0$

Sol. $S = x^2 + y^2 + 2x + 2y + 1 = 0$,

$S' = x^2 + y^2 + 4x + 3y + 2 = 0$

Equation of common chord is

$S - S' = 0$

$(x^2 + y^2 + 2x+2y+1) - (x^2 +y^2 +4x +3y+2) = 0$

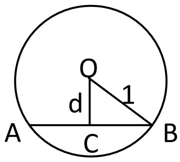
$-2x - y - 1 = 0 \Rightarrow 2x + y + 1 = 0$

Centre of S =0 is (–1, –1)

Radius $= \sqrt{1+1-1} = 1$

Length of $\perp$ from centre (–1,–1) to the chord is

$$d = \left|\frac{2(-1)+(-1)+1}{\sqrt{2^2+1^2}}\right| = \frac{2}{\sqrt{5}}$$



AB=2BC

$$\text{length of the chord} = 2\sqrt{r^2 - d^2} = 2\sqrt{1-\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

ii) $x^2 + y^2 - 5x - 6y + 4 = 0,\ x^2 + y^2 - 2x - 2 = 0$

Ans. $2\sqrt{\frac{14}{5}}$

11. Prove that the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is the diameter of the later circle (or) former bisects the circumference of later) if $2g'(g - g') + 2f'(f - f') = c - c'$.

Sol. $S = x^2 + y^2 + 2gx + 2fy + c = 0$

$S^1 = x^2 + y^2 + 2g'x + 2f'y + c' = 0$

Radical axis is $S - S' = 0$

$(x^2 + y^2 + 2gx + 2fy + c) - (x^2 + y^2 + 2g'x + 2f'y + c') = 0$

$2(g - g')x + 2(f - f')y + c - c' = 0 \quad \ldots(i)$

Centre of second circle is $(-g', -f')$

Radius = $\sqrt{g'^2 + f'^2 - c'}$

Now (–g′, –f′) should lie on (i)

$\therefore$ –2g (g – g′) – 2f′(f – f′) + c – c′ = 0

Or 2g (g – g′) + 2f′(f – f′) = c – c′

12. Show that the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if $1/a^2 + 1/b^2 = 1/c$.

Sol. $S = x^2 + y^2 + 2ax + c = 0$

$S^1 = x^2 + y^2 + 2by + c = 0$

The centre of the circles C_1 (–a, 0) and C_2 (0, –b) respectively.

Radius $r_1 = \sqrt{a^2 - c}$, $r_2 = \sqrt{b^2 - c}$

Given circles are touching circles,

$\Rightarrow C_1C_2 = r_1 + r_2$

$\Rightarrow (C_1C_2)^2 = (r_1 + r_2)^2$

$\Rightarrow (a^2 + b^2) = a^2 - c + b^2 - c + 2\sqrt{a^2 - c}\sqrt{b^2 - c}$

$\Rightarrow c = \sqrt{a^2 - c}\sqrt{b^2 - c}$

$\Rightarrow c^2 = (a^2 - c)(b^2 - c)$

$\Rightarrow c^2 = -c(a^2 + b^2) + a^2b^2 + c^2$

$\Rightarrow c(a^2 + b^2) = a^2b^2 \Rightarrow \frac{1}{c} = \frac{1}{a^2} + \frac{1}{b^2}$.

13. Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ touch each other. Find the coordinates of the point of contact. Is the contact external or internal?

Sol. $S = x^2 + y^2 - 2x = 0$

Centre $C_1 = (1, 0)$, Radius = $r_1 = \sqrt{1+0} = 1$

$S' = x^2 + y^2 + 6x - 6y + 2 = 0$

Centre $C_2 = (-3, 3)$, $r_2 = \sqrt{9+9-2} = 4$

$$C_1C_2 = \sqrt{(1+3)^2 + (0-3)^2} = \sqrt{16+9} = \sqrt{25} = 5 \quad r_1 + r_2 = 1+4 = 5$$

As $C_1C_2 = r_1 + r_2$ the two circles touch each other externally, the point of contact P divides line of centres internally in the ratio

$r_1 : r_2 = 1 : 4$

Hence point of contact

$$P = \left(\frac{1(-3)+4(1)}{1+4}, \frac{1(3)+4(0)}{1+4}\right) = \left(\frac{1}{5}, \frac{3}{5}\right).$$

14. Find the equation of circle which cuts the following circles orthogonally.

i) $x^2 + y^2 + 2x + 4y + 1 = 0$, $2x^2 + 2y^2 + 6x + 8y - 3 = 0$, $x^2 + y^2 - 2x + 6y - 3 = 0$.

Sol.

$S \equiv x^2 + y^2 + 2x + 4y + 1 = 0$

$S^1 \equiv 2x^2 + 2y^2 + 6x + 8y - 3 = 0$

$S^{11} \equiv x^2 + y^2 - 2x + 6y - 3 = 0$

Radical axis of $S = 0$, $S^1 = 0$ is $S - S^1 = 0$

$$-x + \frac{5}{2} = 0 \Rightarrow x = \frac{5}{2} \quad \text{---- (1)}$$

Radical axis of $S = 0$, $S^{11} = 0$ is $S - S^{11} = 0$

$4x - 2y + 4 = 0 \Rightarrow 2x - y + 2 = 0$

$$x = \frac{5}{2} \Rightarrow 5 - y + 2 = 0 \Rightarrow y = 7 \text{ ---- (2)}$$

Solving (1) and (2),

Radical centre is P(5/2, 7)

PT = Length of the tangent from P to S = 0

$$= \sqrt{\frac{25}{4} + 49 + 5 + 28 + 1} = \sqrt{\frac{25}{4} + 83} = \sqrt{\frac{25+332}{4}} = \frac{\sqrt{357}}{2}$$

Equation of the circles cutting the given circles orthogonally is

$$\left(x-\frac{5}{2}\right)^2+(y-7)^2=\frac{357}{4}$$

$$\Rightarrow x^2-5x+\frac{25}{4}+y^2-14y+49=\frac{357}{4}$$

$$\Rightarrow x^2+y^2-5x-14y+\frac{25}{4}+49-\frac{357}{4}=0$$

$$\Rightarrow x^2+y^2-5x-14y+\frac{25+196-357}{4}=0$$

$$\Rightarrow x^2+y^2-5x-14y+\frac{136}{4}=0$$

$$\Rightarrow x^2+y^2-5x-14y-34=0$$

ii) $x^2 + y^2 + 2x + 17y + 4 = 0, x^2 + y^2 + 7x + 6y + 11 = 0, x^2 + y^2 - x + 22y + 3 = 0$

Ans. $x^2 + y^2 - 6x - 4y - 44 = 0$

iii) $x^2 + y^2 + 4x + 2y + 1 = 0,\ 2(x^2 + y^2) + 8x + 6y - 3 = 0, x^2 + y^2 + 6x - 2y - 3 = 0.$

Ans. $x^2 + y^2 - 14x - 5y - 34 = 0.$

15. Show that the circles $S \equiv x^2 + y^2 - 2x - 4y - 20 = 0$ and $S' \equiv x^2 + y^2 + 6x + 2y - 90 = 0$ touch each other internally. Find their point of contact and the equation of common tangent.

Sol. $S \equiv x^2 + y^2 - 2x - 4y - 20 = 0$ …(1) and

$S' \equiv x^2 + y^2 + 6x + 2y - 90 = 0$ …(2)

Let C_1, C_2 be the centres and r_1, r_2 be the radii of the given circles (1) and (2).

Then $C_1 = (1, 2)$, $C_2 = (-3, -1)$, $r_1 = 5$, $r_2 = 10$

C_1C_2 = distance between the centers = 5

$|r_1 - r_2| = |5 - 10| = 5 = C_1C_2$

$\therefore$ The given two circles touch internally. In this case, the common tangent is nothing but the radical axis. Therefore its equation is

$S - S' = 0.$

i.e. $4x + 3y - 35 = 0$

Now we find the point of contact. The point of contact divides $\overline{C_1C_2}$ in the ratio 5: 10

i.e. 1: 2 (Externally)

$\therefore$ Point of contact = $\left(\frac{(1)(-3)-2(1)}{1-2}, \frac{(1)(-1)-2(2)}{1-2}\right) = (5, 5)$

16. If the straight line represented by xcosα + ysinα = p

Intersect the circle $x^2 + y^2 = a^2$ at the points A and B then show that the equation of the circle with $\overline{AB}$ as diameter is $(x^2 + y^2 - a^2) - 2p(x\cos\alpha + y\sin\alpha - p) = 0$.

Sol. The equation of the circle passing through the points A and B is:

$(x^2 + y^2 - a^2) - \lambda(x\cos\alpha + y\sin\alpha - p) = 0$ …(iii)

The centre of this circle is:

$\left(-\frac{\lambda\cos\alpha}{2}, -\frac{\lambda\sin\alpha}{2}\right)$

If the circle given by (3) has $\overline{AB}$ as diameter then the centre of it must lie on (1)

$\therefore -\frac{\lambda\cos\alpha}{2}(\cos\alpha) - \frac{\lambda\sin\alpha}{2}(\sin\alpha) = p$

i.e. $-\frac{\lambda}{2}(\cos^2\alpha + \sin^2\alpha) = p$ i.e. $\lambda = -2p$

Hence the equation of the required circle is

$(x^2 + y^2 - a^2) - 2p(x\cos\alpha + y\sin\alpha - p) = 0$.

Some Important Problems for Practice

1. **If the angle between the circles $x^2 + y^2 - 12x - 6y + 41 = 0$ and $x^2 + y^2 + kx + 6y - 59 = 0$ is $45°$, find k.**

Ans. ± 4

2. **Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles. $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$.**

Ans. $3(x^2 + y^2) - 14x + 23y - 15 = 0$.

3. **Find the equation of the circle which is orthogonal to each of the following three circles.**

$x^2 + y^2 + 2x + 17y + 4 = 0, x^2 + y^2 + 7x + 6y + 11 = 0, x^2 + y^2 - x + 22y + 3 = 0$

Ans. $x^2 + y^2 - 6x - 4y - 44 = 0$.

4. **Find the equation of the circle passing through the points of intersection of the circles.**

$x^2 + y^2 - 8x - 6y + 21 = 0$, $x^2 + y^2 - 2x - 15 = 0$ and (1, 2).

Ans. $3(x^2 + y^2) - 18x - 12y + 27 = 0$.

5. **Let us find the equation the radical axis of $S \equiv x^2 + y^2 - 5x + 6y + 12 = 0$ and**

$S' = x^2 + y^2 + 6x - 4y - 14 = 0$.

Ans. $11x - 10y - 26 = 0$.

6. **Let us find the equation of the radical axis of $2x^2 + 2y^2 + 3x + 6y - 5 = 0$ and**

$3x^2 + 3y^2 - 7x + 8y - 11 = 0$.

Ans. $23x + 2y + 7 = 0$

7. **Let us find the radical centre of the circles**

$x^2 + y^2 - 2x + 6y = 0$, $x^2 + y^2 - 4x - 2y + 6 = 0$ and $x^2 + y^2 - 12x + 2y + 3 = 0$.

Ans. (0, 3/4)

8. Find the equation and length of the common chord of the two circles

$S \equiv x^2 + y^2 + 3x + 5y + 4 = 0$ and $S' = x^2 + y^2 + 5x + 3y + 4 = 0$.

Ans. 4 units

9. Find the equation of the circle whose diameter is the common chord of the circles

$S \equiv x^2 + y^2 + 2x + 3y + 1 = 0$ and $S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0$.

Ans. $2(x^2 + y^2) + 2x + 6y + 1 = 0$

10. Find the equation of a circle which cuts each of the following circles orthogonally $x^2 + y^2 + 3x + 2y + 1 = 0$; $x^2 + y^2 - x + 6y + 5 = 0$; $x^2 + y^2 + 5x - 8y + 15 = 0$.